Symbolical algebra and the quadrics containing a rational curve.

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Introduction. In regard to the algebra of binary forms and the theory of rational curves there exists a wide literature, with which I am not well acquainted.* Very possibly the simple remark made in this note is found elsewhere. But the note is a grateful echo of recent delightful colloquy † with persons of good will.

If, in the familiar way, the lines of our ordinary space of three dimensions are represented each by a point of a quadric in space of five dimensions, the line coordinates of a line being the homogeneous coordinates of the point which represents it, then a ruled surface of ordinary space is represented by a curve lying on the quadric. But, in particular, if the ruled surface be a developable, the curve on the quadric in the space of five dimensions has the further property that its tangent line at any point lies entirely on this quadric, instead of merely touching it; in this case a generator of the original developable is a line of a flat pencil of lines, in a tangent plane of the developable, in which the generator itself occurs as a double ray. A particular application of this which is familiar gives the differential equation of a line of curvature of any surface in ordinary space, by expressing that the normals of the surface at the points of the line of curvature form a developable. This differential equation is $dl\,dl' + dm\,dm' + dn\,dn' = 0$, where l, m, n, l', m', n' are the line coordinates of the normal (given, in terms of the homogeneous coordinates of two points (x, y, z, t), (x', y', z', t') of

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* *Cf.* SEGRE, *Enzykl. Math. III*, *Mehrdimensionale Räume*, No. 27 (1912; published 1921). He refers to Clifford, *Classification of Loci*, 1878 (*Papers*, p. 311); P. H. SCHOUTE, *Proc. Akad. Amsterdam* I (1899), p. 313; BRUSOTTI, *Ann. di. Mat.* IX (1904), p. 311, and *Rend. Ist. Lomb.*, XLII (1909), p. 144.

† SKEAT *Concise Etymol. Dict.* gives, under *Colloquy*, "see Loquacious"; under which however are given *Soliloquy*, and *Obloquy*. *Floreat Congressus Mathematicus.*

the line, by $l = tx' - t'x$, $l' = yz' - y'z$, etc.); this equation expresses that the curve, on the quadric $ll' + mm' + nn' = 0$, representing the normals, has its tangent lines on this quadric.

A well known developable is that formed by the tangent lines of a rational cubic curve of ordinary space. This curve is the locus of points of coordinates $(t^3, t^2, t, 1)$, for varying t. The chord joining any two such points, (t_1) and (t_2), if $s = \frac{1}{2}(t_1 + t_2)$, $p = t_1 t_2$, has for its line coordinates l, m, n, l', m', n' the ratios of the six numbers

$$4s^2 - p,\ 2s,\ 1,\ -p,\ 2sp,\ -p^2;$$

if we put

$$x_0 = n,\ x_1 = \tfrac{1}{2}m,\ x_2 = -l',\ x_3 = \tfrac{1}{2}m',\ x_4 = -n',\ x_5 = \tfrac{1}{3}l,$$

then $x_0, x_1, \ldots, x_5$ are in the ratios of

$$1,\ s,\ p,\ sp,\ p^2,\ \tfrac{1}{3}(4s^2 - p);$$

in particular, for a tangent line of the cubic curve we have $p = s^2$, and $x_0, x_1, \ldots, x_5$ are in the ratios of $1, s, s^2, s^3, s^4, s^2$. The identity $ll' + mm' + nn' = 0$ then leads to

$$x_0 x_4 - 4x_1 x_3 + 3x_2 x_5 = 0,$$

and all the tangents of the cubic curve lie in the linear complex given by $x_5 = x_2$. If now we regard $x_0, x_1, \ldots, x_5$ as homogeneous coordinates in a space of five dimensions, the developable formed by the tangents of the cubic curve is represented by the rational quartic curve given by

$$x_0 : x_1 : x_2 : x_3 : x_4 = 1 : s : s^2 : s^3 : s^4,$$

lying in the four-fold space given by $x_5 = x_2$, on the quadric whose equation is

$$x_0 x_4 - 4x_1 x_3 + 3x_2^2 = 0,$$

and the tangent lines of the rational quartic curve also lie on this quadric. In fact any general point of such a tangent has homogeneous coordinates

$$1,\ s + m,\ s^2 + 2ms,\ s^3 + 3ms^2,\ s^4 + 4ms^3,$$

and it is at once verified that this point lies on this quadric for all values of s and m.*

The quartic curve lies in fact on six linearly independent quadrics in the space of four dimensions, for it evidently lies on any quadric whose equation is of the form

$$x_{4-r}x_r - x_{4-s}x_s = 0, \qquad (r, s = 0, \ldots, 4),$$

and thus it lies on any quadric whose equation is a linear function of these. It can be verified however that the only quadric which also contains the tangents of the quartic curve is that considered above, namely $x_0x_4 - x_2^2 - 4(x_1x_3 - x_2^2) = 0$.

Further, let the polar form obtained from the left side of this quadric, namely

$$x_0x_4' + x_0'x_4 - 4(x_1x_3' + x_1'x_3) + 6x_2x_2'$$

be denoted by (x, x'). It is clear that if we substitute $(1, s, s^2, s^3, s^4)$ for $(x_0, x_1, x_2, x_3, x_4)$, and $(1, s', s'^2, s'^3, s'^4)$ for $(x_0', x_1', x_2', x_3', x_4')$, then

$$(x, x') = (s - s')^4.$$

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* It is remarked by Clifford (*loc. cit.*) that for this rational quartic curve, the four points whose osculating threefolds pass through an arbitrary point of the fourfold space lie on the polar threefold of the point in regard to the quadric; with a similar result for the rational curve of even order in its own normal space of the same order. That the tangents of the quartic curve lie on the quadric is recognised by Brusotti (*loc. cit.*) The tangents also lie on the cubic locus expressed by the vanishing of the determinant of which the three rows consist respectively of the elements x_0, x_1, x_2; x_1, x_2, x_3; x_2, x_3, x_4. Segre (*loc. cit.*), following Schoute (*loc. cit.*), remarks that if we take the locus of the $(l-1)$ folds joining l points of the rational curve of order n in n - fold space, where $l \leqq \frac{1}{2}n$, this locus, which is of dimension $2l - 1$, and of order $\binom{n-l+1}{l}$, satisfies the equations

$$\left\| \begin{matrix} x_0 & x_1 & \cdots & x_{n-l} \\ x_1 & x_2 & \cdots & x_{n-l+1} \\ \cdot & \cdot & \cdots & \cdot \\ x_l & x_{l+1} & \cdots & x_n \end{matrix} \right\| = 0.$$

For $n = 3$, this gives the equations of the cubic curve; for $n = 4$, beside the equations of the curve, it gives the equation of the cubic threefold, just referred to, containing all the chords of the curve and in particular its tangents; and so on.

Thus, if $(x^{(1)})$, $(x^{(2)})$, $(x^{(3)})$, $x^{(4)})$ be any four points of the rational quartic curve we have

$$[(x^{(2)}, x^{(3)})(x^{(1)}, x^{(4)})]^{\frac{1}{2}} + [(x^{(3)}, x^{(1)})(x^{(2)}, x^{(4)})]^{\frac{1}{2}} + [(x^{(1)}, x^{(2)})(x^{(3)}, x^{(4)})]^{\frac{1}{2}} = 0.$$

A relation to which this is practically equivalent was remarked* by Professor A. C. Dixon, *Quart. J. of Maths.*, XXIII (1889), p. 352, as connecting the line coordinates of any four tangents of a cubic curve in ordinary space; in his formula, in place of $(x^{(1)}, x^{(2)})$, for example, there occurs the moment of two lines, of line coordinates $(l_1, m_1, \dots)(l_2, m_2, \dots)$, namely

$$l_1 l_2' + m_1 m_2' + n_1 n_2' + l_1' l_2 + m_1' m_2 + n_1' n_2.$$

This moment, as we know, is covariant for linear transformation of the coordinates of our ordinary space, which retain the relation, $ll' + mm' + nn' = 0$, connecting the coordinates of a line. Equally the polar form for any quadric is covariant for transformation preserving the equation of the quadric. For instance, to take one case, the form

$$\{x, x'\} = x_0 x_4' + x_0' x_4 - 2x_2 x_2'$$

is covariant when $x_0 x_4 - x_2^2$ is preserved; and we evidently have, in the same way,

$$[\{x^{(2)}, x^{(3)}\}\{x^{(1)}, x^{(4)}\}]^{\frac{1}{2}} + [\{x^{(3)}, x^{(1)}\}\{x^{(2)}, x^{(4)}\}]^{\frac{1}{2}} + [\{x^{(1)}, x^{(2)}\}\{x^{(3)}, x^{(4)}\}]^{\frac{1}{2}} = 0,$$

this being interpretable as the equation of a conic touching the three sides of a triangle.

Another case of such an irrational equation of three terms is that connecting the line coordinates of any three generators of the same system of a quadric surface in ordinary space with the line coordinates of any tangent line of the quadric. Using ϖ_{12} for the moment of two lines, above referred to, if 1, 2, 3 refer to three generators of the same system and 4 refers to any tangent line of the quadric, the relation † is

$$[\varpi_{23}\varpi_{14}]^{\frac{1}{2}} + [\varpi_{31}\varpi_{24}]^{\frac{1}{2}} + [\varpi_{12}\varpi_{34}]^{\frac{1}{2}} = 0.$$

* See P. W. Wood, *Cambridge Math. Tracts*, No. 14, p. 21. Professor Dixon's discovery is equivalent to saying that four tangents of a cubic curve in space cut, upon their two transversals, ranges whose cross ratios are $-p^3 q$, and $-pq^3$, where p, q are two numbers which are harmonic conjugates with respect to the two imaginary cube roots of unity.

† I should like to mention that this relation was remarked to me by Professor Turnbull.

Any tangent line of a quadric is in a flat pencil of lines of which two members are the generating lines of the two systems of the quadric at the point of contact of the tangent line; also ϖ_{14} is linear in the coordinates of the tangent line (4). Thus ϖ_{14} is the sum of two parts, of which the part referring to the generating line of the system other than that of the line (1) is identically zero, because these generators intersect. The irrational relation may therefore be regarded as referring to four generators of a quadric surface, of the same system. Interpreting the lines of ordinary space by points of a quadric in space of four dimensions, the relation becomes one connecting four points of a conic, and is that above remarked.

The polar form employed in the irrational equations is suggested by the form of the moment of two lines. But when we consider the general rational curve of order n, in space of n dimensions, of which any point has coordinates of the forms

$$(x_0, x_1, \ldots, x_n) = (1, t, t^2, \ldots, t^n),$$

the equation

$$x_0 \theta^n - \binom{n}{1} x_1 \theta^{n-1} + \binom{n}{2} x_2 \theta^{n-2} - \ldots + (-)^n x_n = 0,$$

wherein $(x_0, \ldots, x_n)$ are current coordinates, is that of the $(n-1)$-fold which meets the curve in n coincident points at (θ); for example when $n = 3$ this is the equation of the osculating plane of the cubic. If then we put

$$(x, x') = x_0 x_n' - \binom{n}{1} x_1 x'_{n-1} + \binom{n}{2} x_2 x'_{n-2} - \ldots + (-1)^n x_n x_0',$$

which, when (x), (x') are the points (t), (θ) of the curve, reduces to $(\theta - t)^n$, we see that, for any four points of the curve, we have

$$[(x^{(2)}, x^{(3)})(x^{(1)}, x^{(4)})]^{\frac{1}{n}} + [(x^{(3)}, x^{(1)})(x^{(2)}, x^{(4)})]^{\frac{1}{n}} + [(x^{(1)}, x^{(2)})(x^{(3)}, x^{(4)})]^{\frac{1}{n}} = 0,$$

whether n be odd or even, though it is only in the latter case that (x, x') is the polar form for a quadric. This relation leads to that given by Clifford (*loc. cit.*) for a general rational curve of order n in space of $(n - 1)$ dimensions.

Reference may be made in this connexion to an interpretation of the relation

$$\varpi_{23}\varpi_{14} + \varpi_{31}\varpi_{24} + \varpi_{12}\varpi_{34} = 0$$

connecting the mutual moments of four lines, given by Mr P. W. Wood in No. 14 of the *Cambridge Math. Tracts* (1913), p. 78. Represented as a relation connecting four points of the quadric in four dimensions, since four points determine a space of three dimensions, it expresses a condition for four points of an ordinary quadric. If we define the Hessian point of three points upon a conic as the intersection of the three lines each joining one of the points to the pole of the other two, the condition is that the tangent plane of the quadric at one of the four points should pass through the Hessian point of the other three; and, as the relation is symmetrical, if this is true for one of the four points it is true for the others. If the quadric, referred to the four points, be

$$2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt = 0,$$

the condition is easily seen to be $uf + vg + wh = 0$.

Enunciation. Consider, in space of n dimensions, homogeneous coordinates denoted by $a_0, a_1, a_2, \ldots, a_n$. Also, consider in this space the rational normal curve of order n of which any point is given in terms of a parameter θ by the equations

$$\frac{a_0}{1} = \frac{a_1}{\theta} = \frac{a_2}{\theta^2} = \ldots = \frac{a_n}{\theta^n}.$$

This curve lies upon $\frac{1}{2}n(n-1)$ linearly independent quadrics. From these a certain number of quadrics can be formed which contain, not only the curve, but also all its tangent lines. From these again, when n is large enough, a certain number of quadrics can be formed which contain, not only the curve and its tangent lines, but also all its osculating planes. From these again, when n is large enough, can be formed a certain number of quadrics which determine all the threefold spaces determined by four consecutive points of the curve. And so on.

Now consider the expression

$$P_, = (ab)^{2h}\, a_1^{\,n-r-h}\, a_2^{\,r-h}\, b_1^{\,n-s-h}\, b_2^{\,s-h},$$

where h is 1, or 2, ..., or $\frac{1}{2}n$ (or $\frac{1}{2}(n-1)$ when n is odd), r is h, or $h+1$, ..., or $(n-h)$, and s is h, or $h+1$, ..., or $(n-h)$. Here a_1, a_2 and b_1, b_2 are equivalent symbols, as used in invariantive algebra, to be interpreted finally in terms of the real coordinates $a_0, a_1, \ldots, a_n$ by replacing both $a_1^{n-p} a_2^p$ and $b_1^{n-p} b_2^p$ by a_p; and (ab) denotes, as usual, $a_1 b_2 - a_2 b_1$. For a given h, the expression P is one of those occurring in the expansion of

$$(ab)^{2h} (a_1 x_1 + a_2 x_2)^{n-2h} (b_1 y_1 + b_2 y_2)^{n-2h},$$

which it is usual to denote by

$$(ab)^{2h} a_x^{n-2h} b_y^{n-2h}.$$

When replaced by its real value the expression P is a quadric in $a_0, a_1, \ldots, a_n$.

We say then that this quadric P contains not only the rational curve referred to, but all the spaces determined by h (or fewer) consecutive points of the curve, at every point of this; say, all its osculating $(h-1)$-folds, and therefore all its osculating $(k-1)$ folds for $k < h$. But further also that all the quadrics containing the curve and its osculating $(h-1)$-folds are linear functions of the $\frac{1}{2}(n-2h+1)(n-2h+2)$ quadrics P which so arise for a given h.

In particular, when n is even, there is one unique quadric, which contains the curve, its tangents, its osculating planes, ..., and finally its osculating $(\frac{1}{2}n-1)$-folds, namely the quadric symbolically given by $(ab)^n$; while when n is odd, there are three quadrics containing the curve, its tangents, ..., finally its osculating $\frac{1}{2}(n-3)$-folds, namely those symbolically denoted by

$$(ab)^{n-1}a_1b_1, \quad (ab)^{n-1}a_1b_2, \quad (ab)^{n-1}a_2b_2.$$

It is known, but may be recalled for the sake of clearness, that the linear spaces of highest dimension entirely lying on a quadric in space of n dimensions are of dimension $\frac{1}{2}n-1$, or $\frac{1}{2}(n-1)$, according as n is even or odd. In the former case these spaces form a single system, in the latter case they form two systems. For instance, a quadric in four dimensions contains a single system of lines, but a quadric in five dimensions contains two systems of planes (as well as lines lying thereon). Cf. Bertini, *Geometria degli iperspazi* (Pisa, 1923), Chap. VI.

Proof. First remark that if a quadric, in $(n+1)$ homogeneous variables, which, for a moment, we denote, in the usual way, by $x_0, x_1, \ldots, x_n$, contain, for all values of θ, the two points for which $x_0, x_1, \ldots, x_n$ have the respective sets of values

$$\begin{array}{l} 1,\ \theta,\ \theta^2,\ \ldots,\ \theta^n, \\ 0,\ 1,\ 2\theta,\ \ldots,\ n\theta^{n-1}, \end{array}$$

then this quadric contains the tangent lines of the curve which is the locus, as θ varies, of the former point; and, more generally, if the quadric contain, for all values of θ, the h points with respective coordinates

$$\begin{array}{llll} 1,\ \theta,\ \theta^2,\ \ldots, & \theta^m, & \ldots, & \theta^n \\ 0,\ 1,\ 2\theta,\ \ldots, & m\theta^{m-1}, & \ldots, & n\theta^{n-1} \\ 0,\ 0,\ \ 1,\ \ldots, & \frac{1}{2}m(m-1)\,\theta^{m-2}, & \ldots, & \frac{1}{2}n(n-1)\,\theta^{n-2} \\ \cdot\ \ \cdot\ \ \cdot & \cdot & \cdot & \cdot \\ 0,\ 0,\ \ 0,\ \ldots, & \binom{m}{h-1}\theta^{m-h+1}, & \ldots, & \binom{n}{h-1}\theta^{n-h+1}, \end{array}$$

then this quadric contains not only the curve given by the first point, as θ varies, but also its tangent line, its osculating plane, ..., and finally its osculating $(h-1)$-fold, determined by h "consecutive" points, at every point of the curve.

This is a familiar fact. If the quadric be denoted symbolically by $c_x^2 = 0$, and the sets of coordinates in these h rows be denoted, respectively, with a slight inconsistency of notation, by $(x), (x_1), \ldots, (x_{h-1})$, we may formulate the proof of this lemma in an elementary way thus: The equation $c_x^2 = 0$, when satisfied identically for all values of θ, leads, by differentiation in regard to θ, to $c_x c_{x_1} = 0$. If we also have, for all values of θ, $c^2_{x_1} = 0$, and we differentiate $c_x c_{x_1} = 0$ to obtain $c^2_{x_1} + c_x c_{x_2} = 0$, we can infer $c_x c_{x_2} = 0$; while, from $c^2_{x_1} = 0$, we have $c_{x_1} c_{x_2} = 0$. Thus we may write

$$\begin{array}{lll} c_x^2, & c_x c_{x_1} & \\ c^2_{x_1}, & c_x c_{x_2}, & c_{x_1} c_{x_2}. \end{array}$$

If now we also have $c^2_{x_2} = 0$, for all values of θ, and differentiate

$$c_x c_{x_2} = 0,\ c_{x_1} c_{x_2} = 0,\ c^2_{x_2} = 0$$

in regard to θ, we obtain results which we can denote by writing

$$c^2_{x_2}, \qquad c_x c_{x_3}, \qquad c_{x_1} c_{x_3}, \qquad c_{x_2} c_{x_3};$$

the process can be continued, finally, assuming $c^2_{x_{h-1}} = 0$, we have results which we can denote by writing

$$c^2_{x_{h-1}}, \; c_x c_{x_h}, \; c_{x_1} c_{x_h}, \; c_{x_2} c_{x_h}, \; \ldots, \; c_{x_{h-1}} c_{x_h}$$

Thus, if we take any point on the planar $(h - 1)$ fold determined by (x), (x_1), ..., (x_{h-1}), of which point the general coordinate will be of the form

$$\xi, = x + \lambda_1 x_1 + \ldots + \lambda_{h-1} x_{h-1},$$

in which λ_1, ..., λ_{h-1} are arbitrary but the same for all the $(n + 1)$ coordinates, we at once find by substitution that

$$c_\xi^2 = 0$$

which is what we desired to prove.

Indeed, we also have

$$c_\xi c_{x_h} = 0,$$

which expresses that the tangent $(n - 1)$-fold of the quadric, at the point of coordinates (x_h), contains the osculating $(h - 1)$-fold of the curve (x).

This lemma being clear, we now seek the quadric containing, for all values of θ, the h points whose coordinates are those put down above, namely

$$1, \; \theta, \; \ldots, \; \theta^m, \; \ldots, \; \theta^n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0, \; 0, \; \ldots, \binom{m}{h-1} \theta^{m-h+1}, \; \ldots, \binom{n}{h-1} \theta^{n-h-1}.$$

But we return to the notation $a_0 a_1$, ..., a_n, for the coordinates of a point.

The general quadric in these coordinates contains $\frac{1}{2}(n+1)(n+2)$ terms. These are in fact the distinct terms in the symbolical product $a_x^n b_y^n$, in which a, b are equivalent symbols, namely the coefficients of the $(n+1)$ terms $x_1^{n-m} x_2^m y_1^{n-m} y_2^m$, for $m = 0, 1, \ldots, n$, together with the $\frac{1}{2}[(n+1)^2-(n+1)]$ terms $a_p\,a_q$, in which p, q are different, arising from

$$a_1^{n-p}\,a_2^p\,x_1^{n-p}\,x_2^p\,.\,b_1^{n-q}\,b_2^q\,y_1^{n-q}\,y_2^q \\ \qquad\qquad + a_1^{n-q}\,a_2^q\,x_1^{n-q}\,x_2^q\,b_1^{n-p}\,b_2^p\,y_1^{n-p}\,y_2^p,$$

which, since a, b are equivalent, is the same as

$$a_1^{n-p}\,a_2^p\,.\,b_1^{n-q}\,b_2^q\,(x_1^{n-p}\,x_2^p\,y_1^{n-q}\,y_2^q + x_1^{n-q}\,x_2^q\,y_1^{n-p}\,y_2^p).$$

We do not write $a_x^n\,b_x^n$, since, for instance, if n were 3, both the terms $a_1^3\,b_2^3$ and $a_1^2\,a_2\,.\,b_1\,b_2^2$, that is $a_0\,a_3$ and $a_1\,a_2$, would otherwise come with the same multiplier, $x_1^3\,x_2^3$ being the same as $x_1^2\,x_2\,.\,x_1\,x_2^2$.

To find the quadrics containing the point $(1, \theta, \theta^2, \ldots, \theta^n)$, for all values of θ, we may substitute these coordinates in the general quadric, and equate to zero the coefficients of the various powers of θ, say the terms of the same *weight*. The result of the substitution is a polynomial in θ of order $2n$, so that the $\frac{1}{2}(n+1)(n+2)$ coefficient are subject to $2n+1$ linear conditions, and there remain $\frac{1}{2}(n+1)(n+2)-(2n+1)$, or $\frac{1}{2}n(n-1)$ linearly independent quadrics containing the curve of points $(1, \theta, \theta^2, \ldots, \theta^n)$.

Consider now the terms of the symbolical expression

$$(ab)^2\,a_x^{n-2}\,b_y^{n-2}\,;$$

these consist of $(n-1)$ terms such as the product of

$$x_1^{n-r-1}\,x_2^{r-1}\,y_1^{n-r-1}\,y_2^{r-1}$$

with

$$(ab)^2\,a_1^{n-r-1}\,a_2^{r-1}\,b_1^{n-r-1}\,b_2^{r-1}, \qquad\qquad r = 1_0\,2_0\,\ldots\,(n-1),$$

which, in real form, is

$$2\,(a_{r-1}\,a_{r+1} - a_r^2),$$

together with $\frac{1}{2}[(n-1)^2-(n-1)]$ terms which, allowing for the fact that a, b are equivalent symbols, are all of the form

$$(ab)^2\,a_1^{n-r-1}\,a_2^{r-1}\,b_1^{n-s-1}\,b_2^{s-1} \\ \qquad [x_1^{n-r-1}\,x_2^{r-1}\,y_1^{n-s-1}\,y_2^{s-1} + x_1^{n-s-1}\,x_2^{s-1}\,y_1^{n-r-1}\,y_2^{r-1}]\,;$$

here the factor not containing x, y, has the real value

$$a_{r-1}\,a_{s+1} + a_{s-1}\,a_{r+1} - 2a_r\,a_s \qquad\qquad r, s = 1, 2, \ldots, (n-1),$$

including the former for $r = s$. We thus find in all

$$\tfrac{1}{2}\left[(n-1)^2 + (n-1)\right], \text{ or } \tfrac{1}{2}n(n-1),$$

quadrics, evidently linearly independent; and their form shews that they all contain the curve of points $(1, \theta, \theta^2, \ldots, \theta^n)$.

Thus all the quadrics containing the curve can be expressed in terms of such as this last, whose expression is symmetrical in regard to r and s. And to find the expression of any quadric containing the curve, in terms of such as this, is quite easy; for, instance, take the quadric $a_{\lambda-r}\,a_r - a_{\lambda-s}\,a_s = 0$, which evidently contains the curve, and is a type in terms of which all such quadrics can be expressed To express this in terms of the quadrics we have obtained it is only necessary to express

$$\alpha^{\lambda-r}\beta^r + \beta^{\lambda-r}\alpha^r - \alpha^{\lambda-s}\beta^s - \beta^{\lambda-s}\alpha^s$$

in the form $(\alpha - \beta)^2\pi$, where π is a polynomal in α and β, and then to replace α, β respectively by a_2/a_1 and b_2/b_1, as is clearly possible.

Thus all the quadrics containing the curve are those found from

$$(ab)^2\, a_x^{\,n-2}\, b_y^{\,n-2}$$

by dealing with this in the manner explained.

Now consider the quadrics, chosen from among these, which contain, not only the curve $(1, \theta, \theta^2, \ldots, \theta^n)$, but also the points $(0, 1, 2\theta, \ldots, n\theta^{n-1})$, for all values of θ. As before we may substitute the last written values for the coordinates in the general quadric of the type $(ab)^2\, a_x^{\,n-2}\, b_y^{\,n-2}$, and equate to zero the coefficients of the various powers of θ. The substitution in the particular quadric

$$a_{r-1}\,a_{s+1} + a_{s-1}\,a_{r+1} - 2a_r\,a_s$$

leads to $-2\,\theta^{r+s-2}$; thus the highest power of θ, after the substitution in the general quadric which contains the curve, which arises when $r = s = n - 1$, is θ^{2n-4}. The coefficients therein must thus be subject to $2n - 3$ conditions. Thus there are

$$\tfrac{1}{2}n(n-1) - (2n-3), \text{ or } \tfrac{1}{2}(n-2)(n-3)$$

linearly independent quadrics containing the rational curve and all its tangent lines.

Take now the symbolical expression

$$(ab)^4 a_x^{n-4} b_y^{n-4},$$

of which a general term, as in the previous case, is

$$(ab)^4 a_1^{n-r-2} a_2^{r-2} b_1^{n-s-2} b_2^{s-2}$$
$$(x_1^{n-r-2} x_2^{r-2} y_1^{n-s-2} y_2^{s-2} + x_1^{n-s-2} x_2^{s-2} y_1^{n-r-2} y_2^{r-2})$$

wherein the real form of the factor independent of x, y is

$$a_{r-2} a_{s+2} - 4a_{r-1} a_{s+1} + 6a_r a_s - 4a_{r+1} a_{s-1} + a_{r+2} a_{s-2}.$$

The number of such terms, r and s having the values $2, 3, \ldots, (n-2)$, is

$$n - 3 + \tfrac{1}{2}[(n-3)^2 - (n-3)], \text{ or } \tfrac{1}{2}(n-2)(n-3),$$

and these are clearly linearly independent. Moreover, the substitution, in this form, of $m\theta^{m-1}$ for a_m leads to zero; and the reason for this is evidently that the expression

$$\frac{\partial^2}{\partial\alpha\partial\beta}[\alpha^{r-2}\beta^{s-2} - 4\alpha^{r-1}\beta^{s+1} + 6\alpha^r\beta^s - 4\alpha^{r+1}\beta^{s-1} + \alpha^{r+2}\beta^{s-2}]$$

vanishes for $\alpha = \beta$ In fact the expression within the square brackets divides by $(\alpha - \beta)^4$, and, after differentiation, divides by $(\alpha - \beta)^2$. In other words, if we operate upon $(ab)^4 a_x^{n-4} b_y^{n-4}$ with

$$\frac{\partial^2}{\partial a_2 \partial b_2}$$

we obtain an expression dividing by $(ab)^2$.

Thus we infer that all the quadrics containing the rational curve and its tangents are those obtained from

$$(ab)^4 a_x^{n-4} b_y^{n-4}.$$

The next step is precisely similar. If in the general quadric last obtained we substitute, for the coordinates, respectively

$$0, 0, 1, 3\theta, \ldots, \tfrac{1}{2}m(m-1)\theta^{m-2}, \ldots, \tfrac{1}{2}n(n-1)\theta^{n-2},$$

there will result a polynomial in θ in which the highest term is that derived from

$$a_{r-2} a_{s+2} - 4a_{r-1} a_{s+1} + 6a_r a_s - 4a_{r+1} a_{s-1} + a_{r+2} a_{s-2}$$

for $r = s = n - 2$. For general values of r, s the substitution in this quadric leads* to $6\theta^{r+s-4}$; the highest power of θ is thus θ^{2n-8}, and this step imposes $(2n-7)$ conditions upon the coefficients in the preceding quadrics; indeed, at each step four less conditions must be imposed than for the preceding step. The number of resulting quadrics is thus

$$\tfrac{1}{2}(n-2)(n-3) - (2n-7), \text{ or } \tfrac{1}{2}(n-4)(n-5).$$

This is precisely the same as the number of independent terms obtained from

$$(ab)^6\, a_x^{\,n-6}\, b_y^{\,n-6};$$

and these terms all satisfy the necessary conditions as before, because this expression, when operated on with $\partial^2 / \partial a_2 \partial b_2$, gives an expression dividing by $(ab)^4$.

The process can be continued. In order to obtain the quadrics containing the osculating $(h-1)$-folds of the curve we must at the last step impose $2n - 4h + 5$ linear conditions, and these quadrics contain $\tfrac{1}{2}(n-2h+1)(n-2h+2)$ terms, which are given by the expression

$$(ab)^{2h}\, a_x^{\,n-2h}\, b_y^{\,n-2h}.$$

In particular, when n is even, h may be as great as $\tfrac{1}{2}n$, and there is one resulting quadric; but when n is odd, h cannot be greater than $\tfrac{1}{2}(n-1)$, and there are three resulting quadrics, as has been stated.

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* In general we have

$$\binom{r-h}{h}\binom{s+h}{h} - \binom{2h}{1}\binom{r-h+1}{h}\binom{s+h-1}{h} + \binom{2h}{2}\binom{r-h+2}{h}\binom{s+h-2}{h} - \binom{2h}{3}\binom{r-h+3}{h}\binom{s+h-3}{h} + \dots = \binom{2h}{h},$$

where $h \leqq r \leqq 2h$, $h \leqq s \leqq 2h$, as we may see by considering

$$\frac{\partial^{2h}}{\partial\alpha^h\,\partial\beta^h}\left[\alpha^{r-h}\beta^{s-h}(\beta-\alpha)^{2h}\right].$$