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SYMPLECTIC REFLECTION ALGEBRAS IN POSITIVE CHARACTERISTIC

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Abstract Basic properties of symplectic reflection algebras over an algebraically closed field k of positive characteristic are laid out. These algebras are always finite modules over their centres, in contrast to the situation in characteristic 0. For the subclass of rational Cherednik algebras, we determine the PI-degree and the Goldie rank, and show that the Azumaya and smooth loci of the centre coincide.

Keywords: symplectic reflection algebra; rational Cherednik algebra; fields of positive characteristic

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1. Introduction

1.1. Throughout this paper, k will be an algebraically closed field of characteristic p, where p is an odd prime. Symplectic reflection algebras over $\mathbb{C}$ were introduced in [**11**]. The same definition makes sense over k, and indeed symplectic reflection algebras over k have been studied in [**3**, **17**]. Let (V,ω) be a finite-dimensional symplectic vector space over k with $S(V)$ its symmetric algebra, and let Γ be a finite subgroup of $\mathrm{Sp}(V)$ with $\operatorname{char} k \nmid |\Gamma|$. The symplectic reflection algebra $H = H_{t,\boldsymbol{c}}(V,\omega,\Gamma)$ is a deformation of the skew group algebra $S(V) * \Gamma$; the precise definition is in § 2.1. One can limit the study at once to the case where Γ is generated by its set S of symplectic reflections: $s \in S$ if and only if $\dim_k \mathrm{Im}(\mathrm{Id} - s) = 2$. In the definition, $t \in k$ and $\boldsymbol{c} : S \to k$ is a Γ-invariant function; $S(V) * \Gamma$ (respectively, $\mathcal{D}(V) * \Gamma$) corresponds to the case where $\boldsymbol{c}$ is the zero map and $t = 0$ (respectively, $t = 1$).

1.2. Recall that Weyl algebras in positive characteristic are finite modules over their centres (see Lemma 5.1). In a parallel fashion, all symplectic reflection algebras over k are finite modules over their centres by a result of Etingof [**3**, Appendix 10]. In contrast, over $\mathbb{C}$, when $t \neq 0$, $Z(H_{t,\boldsymbol{c}}) = \mathbb{C}$ [**7**, Proposition 7.2(2)]. When $t = 0$ the theory over k

appears to be essentially the same as over $\mathbb{C}$. Thus, the focus here will be on the case where t is non-zero; in fact, after rescaling, we may then assume that $t = 1$.

1.3. In § 2 minor adjustments to the characteristic 0 arguments suffice to show that $H = H_{1,\mathbf{c}}$ is a prime Noetherian k-algebra with excellent homological properties; namely, it is Auslander-regular and Cohen–Macaulay.

Let $e \in k\Gamma \subseteq H$ be the symmetrizing idempotent

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma.$$

We show in § 3 that the symmetrizing algebra eHe is a Noetherian domain, a maximal order with (again) good homological properties. Just as in characteristic 0, the Satake homomorphism defines an isomorphism between $Z(H)$ and $Z(eHe)$. The algebras H and eHe are connected by a Morita context, whose details are laid out in §§ 3.2 and 3.8.

1.4. In §§ 4 and 5 we specialize to the case where H is a rational Cherednik algebra, and prove our main results. That is, Γ acts on a vector space $\mathfrak{h}$ of dimension n, with Γ generated by pseudo-reflections for this action, and $V = \mathfrak{h} \oplus \mathfrak{h}^*$ with the canonical Γ-invariant symplectic form, denoted by ω.

Theorem 1.1. *Let $k, \mathfrak{h}, n, \Gamma, \omega$ and H be as above. Denote the centre of H by $Z(H)$.*

(i) *H is a free module of rank $p^{2n}|\Gamma|^3$ over the central subalgebra*

$$Z_0 := (S(\mathfrak{h})^{\mathrm{p}})^{\Gamma} \otimes (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma},$$

where $(\cdot)^{\mathrm{p}}$ denotes the Frobenius map.

(ii) *The Goldie rank of H is $|\Gamma|$.*

(iii) *The PI-degree of H is $p^n|\Gamma|$.*

(iv) *Every irreducible H-module of maximal k-dimension $p^n|\Gamma|$ is a regular $k\Gamma$-module of rank p^n.*

(v) *The Azumaya locus of H is equal to the smooth locus of* $\operatorname{Maxspec}(Z(H))$*: for a maximal ideal $\mathfrak{m}$ of $Z(H)$, $\mathfrak{m}$ annihilates a simple H-module of the maximal possible k-dimension $p^n|\Gamma|$ if and only if $Z(H)_{\mathfrak{m}}$ has finite global dimension.*

In § 6, we gather together some questions and conjectures arising from this work.

2. Definitions and background

2.1. Recall the hypothesis on k from § 1.1. When we want to emphasize that a particular result is valid over all algebraically closed fields K such that

$$2|\Gamma| \text{ is a unit in } K, \tag{2.1}$$

we will always denote the ground field by K. All fields will be assumed to satisfy (2.1). The ingredients needed for the construction of a symplectic reflection algebra are a finite-dimensional symplectic K-vector space (V,ω) and a finite group Γ of symplectic automorphisms of V. An element $\gamma \in \Gamma$ is called a *symplectic reflection* (on V) if the rank of $\mathrm{Id}-\gamma$ is 2. Let S denote the set of symplectic reflections in Γ and let $\boldsymbol{c} : S \to K : s \mapsto c_s$ be a Γ-invariant map. Let $t \in K$. The *symplectic reflection algebra* $H_{t,\boldsymbol{c}}$ [**11**, p. 245] is the deformation of the skew group algebra $S(V)*\Gamma$ obtained by replacing the commutativity relations $xy - yx = 0$ defining the polynomial algebra $S(V)$ by new relations

$$xy - yx = t\omega(x,y)1_\Gamma - \sum_{s\in S} c_s\omega_s(x,y)s \tag{2.2}$$

for all $x, y \in V$. Here, w_s is the skew-symmetric form on V which has $\ker(\mathrm{Id}-s)$ in its radical, and coincides with ω on $\mathrm{Im}(\mathrm{Id}-s)$. Thus, writing $T(V)$ for the tensor algebra of V, $H_{t,\boldsymbol{c}}$ is the factor of the skew group algebra $T(V)*\Gamma$ by the ideal generated by the elements (2.2). Throughout this paper we will denote the dimension of V by $2n$ and assume for convenience that $\Gamma = \langle S\rangle$.

2.2. Poincaré–Birkhoff–Witt Theorem and homological consequences

It is clear from the definition that if $H_{t,\boldsymbol{c}} = H_{t,\boldsymbol{c}}(V,\omega,\Gamma)$ is a symplectic reflection algebra, then $H_{t,\boldsymbol{c}}$ is an $\mathbb{N}$-filtered algebra with filtration $\mathcal{F}_0 := K\Gamma$, $\mathcal{F}_1 := K\Gamma + K\Gamma V$ and $\mathcal{F}_i := \mathcal{F}_1^i$ for $i \geqslant 1$. It is also immediate from the relations (2.2) that there is an algebra epimorphism

$$\pi : S(V)*\Gamma \to \mathrm{gr}_{\mathcal{F}}(H_{t,\boldsymbol{c}});$$

the starting point for the study of symplectic reflection algebras is [**11**, Theorem 1.3], the Poincaré–Birkhoff–Witt (PBW) Theorem, which asserts* that

$$\pi \text{ is an isomorphism.} \tag{2.3}$$

Equivalently, the identity map on V and Γ extends to a vector space isomorphism between $H_{t,\boldsymbol{c}}$ and $S(V)\otimes\Gamma$. Some of the consequences of this fact are gathered in the theorem below, for which we need to recall a few definitions. A Noetherian algebra A is *Auslander–Gorenstein* if A has finite left injective dimension d, and for every finitely generated left A-module M, every non-negative integer i and every submodule N of $\mathrm{Ext}^i_A(M,A)$ we have $\mathrm{Ext}^j_A(N,A) = 0$ for all $j < i$; the same conditions hold with left replaced by right throughout. By a theorem of Zaks [**29**], if a Noetherian algebra has finite right and left injective dimensions, then these are equal. If A is Auslander–Gorenstein and has finite global dimension, then A is called *Auslander-regular* (and then necessarily $\mathrm{gl\,dim}(A) = d$). The *grade* $j_A(M) \in \mathbb{N}\cup\{+\infty\}$ of a non-zero finitely generated A-module M is the least integer j such that $\mathrm{Ext}^j_A(M,A)$ is non-zero. If A has finite Gel'fand–Kirillov dimension, then A is *Cohen–Macaulay* if

$$j_A(M) + \mathrm{GK}-\dim(M) = \mathrm{GK}-\dim(A) \tag{2.4}$$

* In [**11**] it is assumed that K has characteristic 0, but in fact the same proof works over any field satisfying (2.1), so we shall make frequent use of it in this paper.

for every non-zero finitely generated left or right A-module M. We recall that if A has a positive filtration such that $\mathrm{gr}(A)$ is Auslander–Gorenstein, and M is a finitely generated A-module endowed with a standard filtration, then

$$j_A(M) = j_{\mathrm{gr}(A)}(\mathrm{gr}(M))$$

by [**4**, proof of Theorem 3.9 (1)], so that, since

$$\mathrm{GK}-\dim(M) = \mathrm{GK}-\dim_{\mathrm{gr}(A)}(\mathrm{gr}(M))$$

by [**19**, Proposition 8.6.5],

$$\text{if } \mathrm{gr}(A) \text{ is Cohen–Macaulay, then so is } A. \tag{2.5}$$

The following basic facts are well known, and were for the most part proved for algebras over $\mathbb{C}$ in [**11**]; the proofs over a general field k are identical, depending on filtered-graded techniques and the corresponding statements for $S(V) * \Gamma$, once one knows from (2.3) that $\mathrm{gr}_{\mathcal{F}}(H_{t,\boldsymbol{c}}) \cong S(V) * \Gamma$.

Theorem 2.1. *Let K be an arbitrary field, and let $H_{t,\boldsymbol{c}} = H_{t,\boldsymbol{c}}(V, \omega, \Gamma)$ be a symplectic reflection algebra, with $\dim_K(V) = 2n$. Then the following hold.*

(i) *$H_{t,\boldsymbol{c}}$ is a prime Noetherian algebra.*

(ii) *$H_{t,\boldsymbol{c}}$ has finite Gel'fand–Kirillov dimension,* $\mathrm{GK}-\dim(H_{t,\boldsymbol{c}}) = 2n$.

(iii) *$H_{t,\boldsymbol{c}}$ is Auslander-regular and Cohen–Macaulay, with* $\mathrm{gl}\dim(H_{t,\boldsymbol{c}}) \leqslant 2n$.

Proof. (i) See [**19**, Theorem 1.6.9] and [**9**, Example 6.6].

(ii) See [**19**, Proposition 8.6.5] and [**16**, Proposition 5.5].

(iii) $S(V)*\Gamma$ is Auslander-regular by [**28**]. This property lifts to $H_{t,\boldsymbol{c}}$ by [**4**, Theorem 3.9 and the remark following]. The Cohen–Macaulay property is dealt with in the discussion preceding the theorem. □

2.3. The centre

There is a fundamental dichotomy in the theory, determined by whether or not the parameter t is zero. Since $H_{t\boldsymbol{c}} \cong H_{\lambda t, \lambda \boldsymbol{c}}$ for $0 \neq \lambda \in k$, we need consider only the cases $t = 0$ and $t = 1$.

Theorem 2.2. *Let $H_{t,\boldsymbol{c}} = H_{t,\boldsymbol{c}}(V, \omega, \Gamma)$ be a symplectic reflection algebra over an arbitrary field K.*

(i) *(See [**11**, Theorem 3.1], [**7**, Proposition 7.2 (2)].) Suppose that K has characteristic 0. If $t = 0$, then $H_{0,\boldsymbol{c}}$ is a finite module over its centre $Z(H_{0,\boldsymbol{c}})$, which is Gorenstein. If $t = 1$, then $Z(H_{1,\boldsymbol{c}}) = K$.*

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(ii) *Suppose for the rest of the theorem that $K = k$.*

(a) $H_{t,\boldsymbol{c}}$ *is a finite module over its centre for all values of* t *[**3**, Appendix 10].*

(b) *With respect to the filtration* $\{F_i\}$ *of § 2.2, the associated graded algebra of* $Z = Z(H_{1,\boldsymbol{c}})$ *is* $(S(V)^{\mathrm{p}})^{\Gamma}$*, where* $(\cdot)^{\mathrm{p}}$ *denotes the Frobenius homomorphism [**3**, Appendix 10].*

(c) $Z(H_{t,\boldsymbol{c}})$ *is Gorenstein.*

Proof. The statements for $t = 0$ are given by [**11**, Theorem 3.3] for all ground fields. The only other claim which is not as stated in the cited references is (c) for non-zero t. For this, by [**4**, Theorem 3.9], it is sufficient to prove that $\mathrm{gr}(Z(H))$ is Gorenstein. By (b), bearing in mind that $(S(V)^{\mathrm{p}})^{\Gamma} = (S(V)^{\Gamma})^{\mathrm{p}}$, $\mathrm{gr}(Z(H))$ is isomorphic to $S(V)^{\Gamma}$. Since $\Gamma \subseteq SL(V)$, $S(V)^{\Gamma}$ is Gorenstein by Watanabe's Theorem [**26**, **27**]. □

Note that the uncertainty over the precise value of the global dimension of $H_{t,\boldsymbol{c}}$ in Theorem 2.1 (iii) disappears when its centre is big. For in this case every irreducible $H_{t,\boldsymbol{c}}$-module M is a finite-dimensional K-vector space, so that $\mathrm{GK}-\dim(M) = 0$. Therefore, by (2.4) and Theorem 2.1 (ii), $\mathrm{pr}\dim(M) = 2n$, and so we deduce the the following result.

Theorem 2.3. *Let* K *be an arbitrary field satisfying (2.1), and suppose that the symplectic reflection algebra* $H = H_{t,\boldsymbol{c}}(V, \omega, \Gamma)$ *is a finite module over its centre. Then*

$$\mathrm{gl}\dim(H) = \mathrm{Krull}\dim(H) = \mathrm{GK}-\dim(H) = \dim(V).$$

3. Interplay with the spherical subalgebra

For the rest of the paper our ground field will be k, and we focus on the case where k has positive characteristic and the theory deviates from that over characteristic 0; that is, we shall assume that the parameter t is non-zero, so that, after rescaling, we can assume that $t = 1$. So from now on H will denote a symplectic reflection k-algebra $H := H_{1,\boldsymbol{c}}(V, \omega, \Gamma)$.

3.1. Basic properties

Recall that, following [**11**], the *symmetrizing idempotent* e of H is

$$e = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma,$$

and eHe is the *spherical subalgebra* of H. (Note that it is *not* a subalgebra of H in the usual sense of the word, since it does not contain 1_H.) The filtration $\{\mathcal{F}_i\}$ of H induces a filtration $\{e\mathcal{F}e\}$ of eHe with the ith subspace $\mathcal{F}_i H \cap eHe = e\mathcal{F}_i He$. It follows that

$$\mathrm{gr}_{e\mathcal{F}e}(eHe) = e\,\mathrm{gr}_{\mathcal{F}}(H)e = eS(V) * \Gamma e \cong S(V)^{\Gamma}. \tag{3.1}$$

Standard filtered-graded methods can therefore be used to deduce the following theorem. For (v), we recall the definition of a maximal order. Suppose that the Noetherian algebra

A is prime, so that it has a simple Artinian quotient ring Q by Goldie's Theorem [**19**, Theorem 2.3.6]. There is an equivalence relation on orders in Q defined by $R \sim S$ if and only if there exist units a, b, c, d of Q with $aRb \subseteq S$ and $cSd \subseteq R$ [**19**, §§ 5.1.1, 3.1.9]. Then A is a *maximal order* if it is a maximal member of its equivalence class of orders in Q. A commutative Noetherian integral domain is a maximal order if and only if it is integrally closed [**19**, Lemma 5.3.2].

Theorem 3.1.

(i) *eHe is a finitely generated module over its centre eZe.*

(ii) *eHe is an affine Noetherian domain.*

(iii) *He is a finitely generated right eHe-module.*

(iv) *eHe is Auslander–Gorenstein and Cohen–Macaulay.*

(v) *eHe is a maximal order.*

Proof. (i) is clear from Theorem 2.2 (ii)(a). Note next that $S(V)^{\Gamma}$ is affine and $S(V)$ is a finitely generated $S(V)^{\Gamma}$-module by the Hilbert–Noether theorem, [**2**, Theorem 1.3.1]; thus, eHe is affine by (3.1), and since $\mathrm{gr}(He) \cong S(V)$ (iii) follows also. The rest of (ii) follows from (3.1) and [**19**, Theorems 1.6.9] and [**9**, Example 6.6].

To prove (iv) we first observe that $S(V)^{\Gamma}$ is Gorenstein by Watanabe's Theorem [**26**, Theorem 1], since $\Gamma \subseteq \mathrm{Sp}(V) \subseteq SL(V)$. The Auslander–Gorenstein condition thus follows from [**18**, Chapter II, Proposition 2.2.1]. Moreover, since $S(V)^{\Gamma}$ is Cohen–Macaulay, either by the Hochster–Eagon Theorem [**2**, Theorem 4.3.6] or since commutative Gorenstein rings are Cohen–Macaulay, it follows from (2.5) that eHe is Cohen–Macaulay. For (v) it is sufficient, in view of [**19**, Theorem 5.1.6] and (3.1), to note that the fixed ring $S(V)^{\Gamma}$ is integrally closed [**2**, Proposition 1.1.1]. $\square$

3.2. The Morita context

In this section we prove a version of [**11**, Theorem 1.5] for symplectic reflection algebras of positive characteristic. For the most part the proofs follow the proof of [**11**, Theorem 1.5], so we only indicate the differences. Part (ii) is an example of a *double centralizer property* in the spirit of [**15**].

Theorem 3.2.

(i) $\mathrm{End}_H(He) \cong eHe$.

(ii) $\mathrm{End}_{eHe}(He) \cong H$.

(iii) $\mathrm{Hom}_{eHe}(He_|, eHe_|) \cong eH$ *and* $\mathrm{Hom}_{eHe}({}_|eH, {}_|eHe) \cong He$. *In particular, He is a reflexive right eHe-module, and eH is left eHe-reflexive.*

Proof. We prove only parts (ii) and (iii).

Step 1 (construction of an ascending filtration on $\mathrm{End}_{eHe}(He)$). This is done exactly as in [**11**, p. 269].

Step 2 (the map $\eta : H \to \operatorname{End}_{eHe}(He)$). For $h \in H$, let $\eta(h)$ denote left multiplication of He by h, so it is clear that $\eta : H \to \operatorname{End}_{eHe}(He)$ is an algebra homomorphism. We have to prove that η is an isomorphism. Since η preserves the filtrations, it induces a homomorphism $\operatorname{gr}(\eta)$ of the corresponding graded algebras, and it is sufficient to prove that the latter is an isomorphism. Consider then the composition

$$\begin{aligned} S(V) * \Gamma \cong \operatorname{gr}(H) &\xrightarrow{\operatorname{gr}(\eta)} \operatorname{gr}(\operatorname{End}_{eHe}(He)) \\ &\xrightarrow{\;j\;} \operatorname{End}_{\operatorname{gr}(eHe)}(\operatorname{gr}(He)) \cong \operatorname{End}_{S(V)^\Gamma}(S(V)), \end{aligned} \tag{3.2}$$

where j is the obvious homomorphism. Injectivity follows because, after tensoring with the quotient field $Q(S(V)^\Gamma)$ of $S(V)^\Gamma$, the induced map $\psi := \operatorname{Id} \otimes_{S(V)^\Gamma} (j \circ \operatorname{gr}(\eta))$ is easily seen to be injective.

Step 3 (surjectivity of η). We prove this by demonstrating surjectivity of $j \circ \operatorname{gr}(\eta)$, and claim first that

$$\psi \text{ is surjective.}$$

This follows because ψ is an injective map from $Q(S(V)) * \Gamma$ to $\operatorname{End}_{Q(S(V)^\Gamma)}(Q(S(V)))$, and these two $Q(S(V)^\Gamma)$-vector spaces have the same dimension, namely $|\Gamma|^2$. Hence, $j \circ \operatorname{gr}(\eta)(S(V) * \Gamma) \subseteq \operatorname{End}_{S(V)^\Gamma}(S(V))$, and these two algebras have the same simple Artinian quotient ring, namely $\operatorname{End}_{Q(S(V)^\Gamma)}(Q(S(V)))$. But note that, since $S(V)$ is a finitely generated $S(V)^\Gamma$-module, so too is $\operatorname{End}_{S(V)^\Gamma}(S(V))$.

A fortiori, $\operatorname{End}_{S(V)^\Gamma}(S(V))$ is a finitely generated module over its subalgebra $j \circ \operatorname{gr}(\eta)(S(V) * \Gamma)$. However, this last algebra is a maximal order, by [**20**, Theorem 4.6], so the inclusion of algebras must be an equality as required.

(iii) This follows from (ii) just as in [**11**]; that is, one confirms using (ii) that the map from eH to $\operatorname{Hom}_{eHe}(He, eHe)$ induced by left multiplication is an isomorphism of right eHe-modules. The second statement follows symmetrically. □

Corollary 3.3. *H is a maximal order.*

Proof. Suppose that T is an order in $Q(H)$ with $H \subseteq T$ and T equivalent to H. By [**19**, Lemma 3.1.10], there is a non-zero ideal I of H with either $IT \subseteq H$ or $TI \subseteq H$. Suppose the former holds. Then $0 \neq eIe \lhd eHe$, since H is prime, with

$$(eIe)(eTe) \subseteq eITe \subseteq eHe.$$

Thus, eTe is an order in $eQ(H)e = Q(eHe)$, which contains eHe and is equivalent to it. So, by Theorem 3.1 (v),

$$eTe = eHe. \tag{3.3}$$

Note that, if instead $TI \subseteq H$, we can still arrive at (3.3). Bearing in mind the identifications of Theorem 3.2 (iii), (3.3) shows that $Te \subseteq (eH)^*$, and so, by Theorem 3.2 (iii),

$$Te = He.$$

In other words, $T \subseteq \operatorname{End}_{eHe}(He)$. From Theorem 3.2 (ii) we deduce that $T = H$, as required. □

Remarks 3.4.

(i) The proof of the corollary works over fields of characteristic 0, and is independent of the value of the parameter t; in these other cases the result does not seem to have been previously recorded.

(ii) One would naturally expect to prove the above corollary by lifting the result using filtered-graded methods from the corresponding result for the skew group algebra $S(V) * \Gamma \cong \mathrm{gr}_{\mathcal{F}}(H)$. However, the relevant lifting theorem in the literature, Chamarie's Theorem [**19**, Theorem 5.1.6], requires the algebras involved to be domains. While this defect can presumably be rectified, it seems more efficient to proceed as above.

Below and in § 3.4 we need the concept of a *localizable* prime ideal P of a Noetherian ring R: this means that the set $\mathcal{C}(P)$ of elements of R whose images *modulo* P are not zero divisors forms a (right and left) Ore set in R. When this happens, one can invert the elements of $\mathcal{C}(P)$ to obtain the local ring R_P, a partial quotient ring of the factor ring of R by the ideal $I = \{r \in R : cr = 0 \text{ or } rc = 0 \text{ for some } c \in \mathcal{C}(P)\}$. When R is a finite module over its centre Z, a prime ideal P of R is localizable when (and, in fact, only when) it is the *unique* prime of R lying over $P \cap Z$, and one sees easily that in this case we can form R_P by inverting the elements of $Z \setminus (P \cap Z)$. (For background on these ideas, see, for example, [**12**].)

Standard Morita theory [**19**, Proposition 3.5.6] applied to Theorem 3.2 tells us that H is Morita equivalent to eHe exactly when $HeH = H$. More precisely, the size of HeH indicates how close H and eHe are to being Morita equivalent. In this connection, we thus have the following lemma.

Lemma 3.5. *Let e be the symmetrizing idempotent of H. Let $\mathfrak{p}$ be a prime ideal of $Z := Z(H)$ with $HeH \cap Z \subseteq \mathfrak{p}$. Then $\mathfrak{p}$ has height at least 2.*

Proof. Suppose the result is false: so there is a prime $\mathfrak{p}$ of Z of height 1, with

$$HeH \cap Z \subseteq \mathfrak{p}. \tag{3.4}$$

Thus, we can localize at $\mathfrak{p}$ by inverting the elements $Z \setminus \mathfrak{p}$, to obtain the ring $H_{\mathfrak{p}}$. Since H is a prime maximal order and is a finite module over its centre, by Theorems 2.1 (i) and 2.3 (ii), and Corollary 3.3, all its primes of height one are localizable, by [**22**, Propositions II.2.2 and II.2.6, and Theorème IV.2.15]. Equivalently, by [**6**, Theorem III.9.2] or [**23**, Theorem 7], there is a *unique* height one prime of H lying over $\mathfrak{p}$, let us call it P. Thus, $H_{\mathfrak{p}}$ is a local ring with Jacobson radical $PH_{\mathfrak{p}}$. Now (3.4) ensures that $H_{\mathfrak{p}}eH_{\mathfrak{p}}$ is a proper ideal of $H_{\mathfrak{p}}$, so that $e \in PH_{\mathfrak{p}}$, the Jacobson radical of $H_{\mathfrak{p}}$. Hence, $1 - e$ is a unit in $H_{\mathfrak{p}}$, a contradiction. So the result is proved. □

Corollary 3.6. *Let $\mathfrak{p}$ be any prime ideal of Z such that $HeH \cap Z$ is not contained in $\mathfrak{p}$. Then $H_{\mathfrak{p}}$ is Morita equivalent to $eH_{\mathfrak{p}}e$. In particular, this is true for every prime ideal of Z of height 1.*

3.3. The Satake homomorphism

The proof of the next result also follows the corresponding argument used to prove [**11**, Theorem 3.1], but note the important difference that, when $t = 0$, eHe is commutative.

Theorem 3.7. *The map $\theta : H \to eHe : u \mapsto eue$ is an algebra homomorphism when restricted to the centre Z of H, and maps Z isomorphically to the centre of eHe.*

Sketch of the proof. It is obvious that the restriction of θ to Z is an algebra homomorphism. We define an inverse map $\xi : Z(eHe) \to Z$ as follows. Let $eae \in Z(eHe)$, so that right multiplication of He by eae defines a right eHe-module endomorphism r_{eae} of He. By Theorem 3.2 (ii) this endomorphism must be induced by left multiplication of He by an element $\xi(eae)$ of H; moreover, since r_{eae} commutes with the left multiplications of He by the elements of H, $\xi(eae) \in Z$. It is now clear that ξ is an algebra homomorphism, and it is easy to check that it is inverse to $\theta|_Z$. $\square$

3.4. The Morita context revisited

An important theme in the study of those symplectic reflection algebras H which are finite modules over their centres Z has been the determination of the groups and parameter values (if any) for which the centre is smooth [**1**, **11**, **13**, **21**].

Recall [**6**, Theorem III.1.6] that a prime k-algebra A, finitely generated as a module over its affine centre Z, is *Azumaya* over Z if (and only if) all the irreducible A-modules have the same vector space dimension (which is then necessarily equal to the PI-degree of A). As we will see in § 5.3, the smoothness of $Z(H)$ is closely related to the question of when H is Azumaya over Z. Regarding the latter question, we have the following result.

Theorem 3.8. *Let P be a prime ideal of H, and let $\mathfrak{p}$ be the prime $P \cap Z$ of $Z := Z(H)$. Consider the following statements.*

(i) *$\mathfrak{p}$ is in the Azumaya locus.*

(ii) *There exists a positive integer s such that*

$$H_{\mathfrak{p}} \cong M_s(eH_{\mathfrak{p}}e)$$

and $eH_{\mathfrak{p}}e$ is a local ring with Jacobson radical $e\mathfrak{p}H_{\mathfrak{p}}e$.

(iii) *P is a localizable ideal of H.*

(iv) *P is the unique prime ideal of H lying over $\mathfrak{p}$.*

(v) *$HeH \cap Z \not\subseteq \mathfrak{p}$ and ePe is the unique prime ideal of eHe lying over $e\mathfrak{p}e$.*

(vi) *There exists a positive integer s such that*

$$H_{\mathfrak{p}} \cong M_s(eH_{\mathfrak{p}}e),$$

and $eH_{\mathfrak{p}}e$ is a local ring.

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Then (i) $\iff$ (ii) $\implies$ (iii) $\iff$ (iv) $\iff$ (v) $\iff$ (vi).

Proof. (i) $\implies$ (iii). This is clear, since, if $\mathfrak{p}$ is Azumaya in H, then inverting $Z \setminus \mathfrak{p}$ yields the local ring H_P.

(iii) $\iff$ (iv). This is Müller's Theorem (see [**23**, Theorem 7] or [**6**, Theorem III.9.2]).

(iv) $\implies$ (v). Assume (iv), which by the above is equivalent to (iii). If $HeH \cap Z \subseteq \mathfrak{p}$, then $H_P e H_P$ is a proper ideal of $H_P = H[Z \setminus \mathfrak{p}]^{-1}$, and this is impossible just as in the proof of Lemma 3.2. This proves the first part of (v); the second part follows from Corollary 3.6, which shows that $eH_P e = eH_{\mathfrak{p}}e$ is local with Jacobson radical $ePH_P e$.

(v) $\implies$ (vi). Assume (v). By Corollary 3.6 $H_{\mathfrak{p}}$ is Morita equivalent to $eH_{\mathfrak{p}}e$, and the latter ring is local by (v). Now $H_{\mathfrak{p}}e$ is a finitely generated projective right module over the local domain $eH_{\mathfrak{p}}e$, by the Morita theorems and Theorem 2.1 (ii) and (iii). Hence, $H_{\mathfrak{p}}e|_{eH_{\mathfrak{p}}e}$ is free of finite rank s, and its endomorphism ring $H_{\mathfrak{p}}$ is thus isomorphic to $M_s(eH_{\mathfrak{p}}e)$.

(vi) $\implies$ (iii). Assume (vi). Then $H_{\mathfrak{p}}$ is a local ring in which $PH_{\mathfrak{p}}$ is a maximal ideal. Thus, $PH_{\mathfrak{p}}$ must be the Jacobson radical of $H_{\mathfrak{p}}$, so P is localizable, proving (iii).

(ii) $\implies$ (i). If (ii) holds, then $H_{\mathfrak{p}}$ is a local ring whose Jacobson radical is generated by its intersection with the centre. Thus, $H_{\mathfrak{p}}$ is Azumaya, by [**6**, Theorem III.1.6].

(i) $\implies$ (ii). Assume (i). Then, as we have seen, (vi) holds, and, writing $J(R)$ for the Jacobson radical of a ring R,

$$J(H_{\mathfrak{p}}) \cong J(M_s(eH_{\mathfrak{p}}e)) = M_s(J(eH_{\mathfrak{p}}e)) = M_s(e\mathfrak{p}H_{\mathfrak{p}}e),$$

since $\mathfrak{p}$ is Azumaya. □

Remarks 3.9.

(i) We expect that (i)–(vi) in the above theorem should be equivalent. This is true for all symplectic reflection algebras (over any field) at $t = 0$, by [**11**, Theorem 1.7]. But the proof depends crucially on the commutativity of eHe, in particular [**11**, Lemma 2.24]. When k has positive characteristic and $t = 1$ we cannot prove the equivalence of (i)–(vi), even in the case of Cherednik algebras; the difficulty lies in our inadequate understanding of the relation of the maximal order eHe to its centre.

(ii) We expect the integer s of Theorem 3.8 (vi) to be $|\Gamma|$; we prove this in Theorem 4.7 when H is a Cherednik algebra (see also Remark 4.8).

4. Cherednik algebras: structure

4.1. For the rest of the paper we shall assume that H is a *rational Cherednik algebra* over an algebraically closed field k of positive characteristic p. We fix a finite-dimensional k-vector space $\mathfrak{h}$ and a finite group Γ of automorphisms of $\mathfrak{h}$. We shall assume throughout

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that Γ is generated by pseudo-reflections for its action on $\mathfrak{h}$. Then Γ acts on $V := \mathfrak{h} \oplus \mathfrak{h}^*$, and this space admits a canonical Γ-invariant symplectic form ω, defined by $\omega((u,f),(x,g)) := g(u) - f(x)$ for $u, x \in \mathfrak{h}$ and $f, g \in \mathfrak{h}^*$. The set S of pseudo-reflections in Γ is then the set of symplectic reflections for Γ acting on V, so, as in § 2.1, we can choose $t \in k$ and a Γ-invariant function $\boldsymbol{c} : S \to k$, and define the symplectic reflection algebra $H_{t,\boldsymbol{c}} := H_{t,\boldsymbol{c}}(V, \omega, \Gamma)$. We shall continue to assume throughout that

$$t = 1.$$

4.2. The central invariant subalgebra

Let $H = H_{1,\boldsymbol{c}} := H_{1,\boldsymbol{c}}(\mathfrak{h} \oplus \mathfrak{h}^*, \omega, \Gamma)$ be a rational Cherednik algebra. Notice that the skew group algebras $S(\mathfrak{h}) * \Gamma$ and $S(\mathfrak{h}^*) * \Gamma$ are contained in H, as is clear from the defining relations of H and (2.3), the PBW Theorem. Hence, the centres of these algebras, $S(\mathfrak{h})^{\Gamma}$ and $S(\mathfrak{h}^*)^{\Gamma}$, are also in H. Less trivially, it is proved in [**11**, Proposition 4.15] using Dunkl operators that, when $t = 0$ and k has characteristic 0,

$$S(\mathfrak{h})^{\Gamma} \otimes S(\mathfrak{h}^*)^{\Gamma} \subseteq Z(H); \tag{4.1}$$

an alternative proof of the same result is given in [**13**, Proposition 3.6]. When $t = 1$ and the characteristic is positive, $H_{1,\mathbf{0}}$ is $D(\mathfrak{h}) * \Gamma$, the skew group algebra of the Weyl algebra, so its centre is easily calculated to be $(S(\mathfrak{h} \oplus \mathfrak{h}^*)^{\mathrm{p}})^{\Gamma}$. It is obvious, therefore, what the characteristic p analogue of (4.1) should be; we offer here a proof of the result which is completely elementary and can also be adapted to give an easy proof of the original characteristic 0 theorem.

Proposition 4.1. *Let $H = H_{1,\boldsymbol{c}}(\mathfrak{h} \oplus \mathfrak{h}^*, \omega, \Gamma)$ be a rational Cherednik algebra. Then*

$$Z_0 := (S(\mathfrak{h})^{\mathrm{p}})^{\Gamma} \otimes (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma} \subseteq Z(H).$$

Proof. Fix a pair of dual bases $\{x_i\}$ and $\{y_i\}$ of $\mathfrak{h}$ and $\mathfrak{h}^*$. Thus, the filtration $\{\mathcal{F}_i\}$ of § 2.2 is defined by setting $\deg(x_i) = \deg(y_i) = 1$ and $\deg(\gamma) = 0$ for $1 \leqslant i \leqslant n$ and $\gamma \in \Gamma$. Note that H is also a $\mathbb{Z}$-graded algebra, with $\deg(x_i) = 1$, $\deg(y_i) = -1$ and $\deg(\gamma) = 0$ for $1 \leqslant i \leqslant n$ and $\gamma \in \Gamma$. Let us denote the ith graded subspace by $\mathcal{M}_i(H)$, so $H = \oplus_{i \in \mathbb{Z}} \mathcal{M}_i(H)$ and, putting $\mathcal{M}_i(Z) := \mathcal{M}_i(H) \cap Z$,

$$Z := Z(H) = \oplus_{i \in \mathbb{Z}} \mathcal{M}_i(Z). \tag{4.2}$$

Let f be a homogeneous element of $(S(\mathfrak{h})^{\mathrm{p}})^{\Gamma}$ of degree m. We aim first to show that

$$f \in Z. \tag{4.3}$$

Let $\sigma_m^{\mathcal{F}} : \mathcal{F}_m Z \to \mathcal{F}_m Z / \mathcal{F}_{m-1} Z$ be the symbol map of degree m. Since $f \in S(\mathfrak{h})$, its $\mathcal{F}$-degree and $\mathcal{M}$-degree are the same, namely

$$\deg_{\mathcal{F}}(f) = \deg_{\mathcal{M}}(f) = m.$$

Note that $(S(V)^{\Gamma})^{\mathrm{p}} \cong (S(V)^{\mathrm{p}})^{\Gamma}$ by the Γ-equivariance of the Frobenius homomorphism. Thus, by Theorem 2.2 (ii)(b), there exists $z \in \mathcal{F}_m Z$ with $\sigma_m^{\mathcal{F}}(z) = f$. On the other hand, by (4.2), we can write

$$z = z_1 + z_2,$$

where $z_1 \in \mathcal{M}_m(Z)$ and $z_2 \in \oplus_{j \neq m} \mathcal{M}_j(Z)$. Now, by the PBW Theorem, (2.3), there is *no* cancellation between $\sigma_m^{\mathcal{F}}(z_1)$ and $\sigma_m^{\mathcal{F}}(z_2)$. That is,

$$f = \sigma_m^{\mathcal{F}}(z) = \sigma_m^{\mathcal{F}}(z_1) + \sigma_m^{\mathcal{F}}(z_2).$$

Since $\mathcal{M}\text{-}\deg(f) = m$,

$$z_1 = f + g,$$

where $g \in \mathcal{M}_m(H)$. We claim that $g = 0$. Suppose that $g \neq 0$. If there is a monomial $\boldsymbol{x}^I$ in $\{x_1, \dots, x_n\}$, with $I = (m_i) \in \mathbb{Z}^n_{\geqslant 0}$, and $\gamma \in \Gamma$, such that $u := \boldsymbol{x}^I \gamma$ occurs in the PBW expression for g with non-zero coefficient, then u would appear in $\sigma_m^{\mathcal{F}}(z_1)$ and could not be cancelled by any term from $\sigma_m^{\mathcal{F}}(z_2)$ since $\mathcal{M}\text{-}\deg(u) = m$. This would contradict the fact that $\sigma_m^{\mathcal{F}}(z) = f$. Hence, every basis term u in g has the form $\boldsymbol{x}^I \boldsymbol{y}^J \gamma$ with $J \neq (0, \dots, 0)$. But $g \in \mathcal{M}_m(H)$, so that $|I| - |J| = m$, where $|I| = \sum_i m_i$. Therefore, $|I| > m$, forcing $\mathcal{F} - \deg(u) > m$. However, this contradicts $z \in \mathcal{F}_m Z$, and so $g = 0$. Therefore, $z = z_1 + z_2 = f + z_2$. That is, $f = z_1 \in \mathcal{M}_m(Z)$, so (4.3) is proved.

Since $(S(\mathfrak{h})^{\mathrm{p}})^{\Gamma}$ is generated by homogeneous elements, it follows that $(S(\mathfrak{h})^{\mathrm{p}})^{\Gamma} \subseteq Z$; by swapping the grading on $\mathfrak{h}$ and $\mathfrak{h}^*$, we can show in the same way that $(S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma} \subseteq Z$. Finally, the PBW Theorem implies that the subalgebra of Z generated by these two invariant rings is $(S(\mathfrak{h})^{\mathrm{p}})^{\Gamma} \otimes (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma}$, completing the proof of the proposition. $\square$

Remark 4.2. Keep the notation of the proposition. By the Shepherd–Todd–Chevalley Theorem [**2**, Theorem 7.2.1], the central subalgebra Z_0 is a polynomial algebra in $2n$ indeterminates. Moreover, by classical invariant theory $S(\mathfrak{h})$ (respectively, $S(\mathfrak{h}^*)$) is a free $(S(\mathfrak{h})^{\mathrm{p}})^{\Gamma}$-module (respectively, $(S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma}$-module) of rank $p^n|\Gamma|$. Thus, by the PBW Theorem,

$$H \text{ is a free } Z_0 - \text{module of rank } p^{2n}|\Gamma|^3. \tag{4.4}$$

Hence, there is a bundle $\mathcal{B}$ of algebras of k-dimension $p^{2n}|\Gamma|^3$ over affine $2n$-space, and every irreducible H-module is a module for precisely one of the algebras in $\mathcal{B}$. Thus, it makes sense to study the representation theory of H by studying $\mathcal{B}$.

Example 4.3 (Kleinian singularities of type *A*). Let $r \in \mathbb{Z}$, $r > 1$, with r coprime to p, and let η be a primitive rth root of 1 in k. Let $\mathfrak{h} = kx$, $\mathfrak{h}^* = ky$ and let $\Gamma = \langle \gamma \rangle$ be the cyclic group of order r acting on $\mathfrak{h}$ by $\gamma \cdot x = \eta x$, so that $\gamma \cdot y = \eta^{-1} y$. Thus, for $\boldsymbol{c} = (c_1, \dots, c_{r-1}) \in k^{r-1}$, $H = H_{1,\boldsymbol{c}}(\mathfrak{h} \oplus \mathfrak{h}^*, \omega, \Gamma)$ is the algebra

$$k\left\langle x, y, \gamma : \gamma x = \eta x \gamma,\ \gamma y = \eta^{-1} y \gamma,\ [y, x] = 1 - \sum_{j=1}^{r-1} c_j \gamma^j \right\rangle. \tag{4.5}$$

One may check easily that

$$x^{pr} \in Z(H), \qquad y^{pr} \in Z(H).$$

It is convenient to have available also the basis of $k\Gamma$ afforded by the primitive idempotents

$$e_j := \frac{1}{r}\sum_{i=0}^{r-1} \eta^{ij}\gamma^i$$

for $j = 0, \dots, r-1$, with respect to which we write the commutator relation as

$$[y, x] = \sum_{j=0}^{r-1} f_j e_j$$

for $\boldsymbol{f} = (f_j) \in k^r$ with $\sum_j f_j = r$. The interested reader may write down the linear relations between the c_i and the f_j. Define

$$\tau = xy + \sum_{i=1}^{r-1}\biggl(i - \sum_{j=0}^{i-1} f_j\biggr)e_i \in H.$$

Then $[\tau, x] = x$ and $[\tau, y] = -y$, so that

$$h := \tau^p - \tau \in Z(H).$$

Define elements $\{\delta_m : 0 \leqslant m \leqslant r-1\}$ of k by

$$\delta_m := \sum_{j=1}^{r-1} c_j(1-\eta^{-j})^{-1}\eta^{mj} = -\frac{1}{r}\sum_{l=0}^{r-1}(\rho_{m,l+1})f_l,$$

where

$$\rho_{m,l+1} := \sum_{j=1}^{r-1}[\eta^{(m+l)j}/(\eta^j - 1)].$$

The following result is proved by direct calculation in [**8**, Chapter 3].

Proposition 4.4. *Let $H = H_{1,\boldsymbol{c}}$ be a symplectic reflection algebra for the Kleinian singularity of type A_{r-1}, as defined above. Keep the notation as above.*

(i) $Z(H) = k\langle x^{pr}, y^{pr}, \tau\rangle$.

(ii) $Z(H) \cong k[X, Y, Z : XY = \prod_{m=0}^{r-1}(Z + \delta_m^p - \delta_m)]$.

(iii) *$Z(H)$ is smooth if and only if $\delta_i - \delta_j \in \mathbb{Z}$ only when $i = j$.*

4.3. The Dunkl embedding

One reason why rational Cherednik algebras over $\mathbb{C}$ at $t = 1$ are easier to handle than arbitrary symplectic reflection algebras is that the $\mathbb{C}$-algebra $H_{\mathbb{C}} := H_{1,\boldsymbol{c}}(\mathfrak{h} \oplus \mathfrak{h}^*, \omega, \Gamma)$ embeds in the skew group algebra $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) * \Gamma$; indeed $H_{\mathbb{C}}$ is birationally equivalent to the skew group algebra $\mathcal{D}(\mathfrak{h}) * \Gamma$ over the Weyl algebra $\mathcal{D}(\mathfrak{h})$ [**11**, Proposition 4.5].

In fact, the same is true in positive characteristic, with essentially the same proof. We keep the notation as in § 4.1. In addition, for each pseudo-reflection $s \in S \subseteq \Gamma$ choose

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$\alpha_s \in \mathfrak{h}^*$, whose kernel is the hyperplane stabilized by s. If $s, s' \in S$ with $s' = s^\gamma$ for $\gamma \in \Gamma$, take $\alpha_{s'} = \alpha_s^\gamma$. Then $\delta := \prod_{s \in S} \alpha_s$ is a Γ-semi-invariant, so some power δ^ℓ of δ is Γ-invariant. Write $\mathfrak{h}^{\mathrm{reg}}$ for the regular points of $\mathfrak{h}$, so that $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})$, the algebra of differential operators on $\mathfrak{h}^{\mathrm{reg}}$, is just $\mathcal{D}(\mathfrak{h})[\delta^{-1}]$, the localization of the nth Weyl algebra $\mathcal{D}(\mathfrak{h}) = k[y_1, \dots, y_n, \partial/\partial y_1, \dots, \partial/\partial y_n]$ at the powers of δ. Observe that $\mathcal{D}(\mathfrak{h})[\delta^{-1}] = \mathcal{D}(\mathfrak{h})[\delta^{-\ell}]$; that is, we can pass to $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})$ by inverting a Γ-invariant Ore set.

Theorem 4.5. *Let H be a rational Cherednik algebra, with notation as above. There are elements $\tau_1, \dots, \tau_n$ of $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) * \Gamma$ such that the assignment $\gamma \mapsto \gamma$, $y_i \mapsto y_i$ and $x_i \mapsto \partial/\partial y_i + \tau_i$ for $\gamma \in \Gamma$ and $i = 1, \dots, n$ extends to an injective algebra homomorphism $\Theta_{\boldsymbol{c}} : H \to \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) * \Gamma$.*

Proof. The proof of [**11**, Proposition 4.5] works equally well in positive characteristic. □

Remark 4.6. In fact, the proof yields a stronger statement: $\Theta_{\boldsymbol{c}}$ becomes an isomorphism after inverting δ^ℓ; that is, $H[\delta^{-\ell}] \cong \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) * \Gamma$.

4.4. Goldie rank

Recall [**19**, Theorem 10.4.4] that the *Goldie* or *uniform rank* $\mathrm{udim}_R(M)$ of a module M over the Noetherian ring R is the largest number of non-zero modules whose direct sum embeds into M, or ∞ if no such supremum exists. If $\mathcal{C}$ is an Ore set of non-zero divisors in R and M is $\mathcal{C}$-torsion free, then it is easy to check [**19**, proof of Lemma 10.2.13] that

$$\mathrm{udim}_R(M) = \mathrm{udim}_{R\mathcal{C}^{-1}}(M \otimes_R R\mathcal{C}^{-1}). \tag{4.6}$$

Apply this in particular when R is a prime Noetherian ring, $\mathcal{C}$ is the set of *all* non-zero divisors in R, and $M = R$. In this case $\mathcal{C}$ is an Ore set and $R\mathcal{C}^{-1}$ is the simple Artinian quotient ring $Q(R)$ of R, by Goldie's Theorem [**19**, Theorem 10.4.10]. Thus, $Q(R) \cong M_s(D)$ for a division ring D by the Artin–Wedderburn Theorem, and (4.6) shows that the integer s is the Goldie rank of R.

Theorem 4.7. *Let $H = H_{1,\boldsymbol{c}}(\mathfrak{h}, \omega, \Gamma)$ be a rational Cherednik algebra.*

(i) $\mathrm{udim}(H) = |\Gamma|$.

(ii) *The integer s appearing in Theorem 3.8 is the Goldie rank of H, and so equals $|\Gamma|$.*

Proof. Fix a prime $\mathfrak{p}$ of $Z(H)$ which is in the Azumaya locus. Then the isomorphism of Theorem 3.8 (ii) holds, and since $eH_{\mathfrak{p}}e$ is a domain by Theorem 3.1 (ii), the first claim in (ii) follows. Thus, it remains to prove that $\mathrm{udim}(H) = |\Gamma|$. Since the Goldie rank of an algebra is unaltered by inverting an Ore set of non-zero divisors, by (4.6), in view of Theorem 4.3 we only need to show that

$$\mathrm{udim}(\mathcal{D}(\mathfrak{h}) * \Gamma) = |\Gamma|.$$

This follows from a special case of Moody's Theorem [**24**, Theorem 37.14]. □

Remark 4.8. In fact Theorem 4.7 (i) is true for all symplectic reflection algebras over all fields, as was proved (independently and later) by Gordon in [**14**, Corollary 6.3].

5. Cherednik algebras: representation theory

5.1. PI-degree and centre

When k has characteristic 0 and $t = 0$, the PI-degree of a symplectic reflection algebra $H_{0,\boldsymbol{c}}$ is $|\Gamma|$ [**11**, Theorem 1.7 (iv)]; indeed the irreducible $H_{0,\boldsymbol{c}}$-modules in the Azumaya locus are isomorphic as $k\Gamma$-modules to $k\Gamma$. The same conclusions remain valid when $t = 0$ and k has positive characteristic, with essentially the same proofs. It remains to consider the case $t = 1$ when k has positive characteristic; here we deal with the Cherednik algebras. For this we need the two following well-known facts.

Lemma 5.1.

(i) *The centre of the Weyl algebra*

$$\mathcal{D}(\mathfrak{h}) = k\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$$

is $k\langle x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p\rangle$*; indeed,* $\mathcal{D}(\mathfrak{h})$ *is Azumaya over* $k\langle x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p\rangle$.

(ii) *Let R be a domain and let G be a finite group of automorphisms of R which acts faithfully on $Z(R)$. Then the centre of the skew group ring $R * G$ is $Z(R)^G$.*

Proof. Part (i) is proved in [**25**] and (ii) is a straightforward exercise. □

Theorem 5.2. *Let $H = H_{1,\boldsymbol{c}}(\mathfrak{h} \oplus \mathfrak{h}^*, \omega, \Gamma)$ be a rational Cherednik algebra over an algebraically closed field k of positive characteristic p. Let $\mathfrak{h}$ have dimension n. Then*

$$\mathrm{PI} - \deg(H) = p^n|\Gamma|.$$

Proof. By Theorem 4.5 and Remark 4.6, H and $\mathcal{D}(\mathfrak{h}) * \Gamma$ become isomorphic after inverting certain central elements, so it is sufficient, by [**19**, Lemma 10.2.13], to prove that $\mathcal{D}(\mathfrak{h}) * \Gamma$ has PI-degree $p^n|\Gamma|$. From the lemma we deduce that

$$Z := Z(\mathcal{D}(\mathfrak{h}) * \Gamma) = k\langle x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p\rangle^\Gamma.$$

Now, inverting the non-zero elements of the centre in $\mathcal{D}(\mathfrak{h}) * \Gamma$ and calculating that

$$\dim_{Q(Z)}(Q(Z) \otimes_Z \mathcal{D}(\mathfrak{h}) * \Gamma) = p^{2n}|\Gamma|^2,$$

we obtain the desired conclusion. □

Presumably Theorem 5.2 is true for all symplectic reflection algebras with $t = 1$ over a characteristic p field; we leave this as an open question.

Corollary 5.3. *Let $H = H_{1,\boldsymbol{c}}(\mathfrak{h} \oplus \mathfrak{h}^*, \omega, \Gamma)$ be a rational Cherednik algebra over an algebraically closed field k of positive characteristic p. Then $Z(H)$ is a free module of rank $|\Gamma|$ over its polynomial subalgebra $Z_0 = S(\mathfrak{h}^p)^\Gamma \otimes S(\mathfrak{h}^{*p})^\Gamma$.*

Proof. Let $\mathfrak{h}$ have dimension n. By Theorem 2.2 (iv), $Z(H)$ is Gorenstein, and therefore Cohen–Macaulay. Thus, by [**10**, Corollary 18.17], it is free over its polynomial subalgebra Z_0. The rank is determined by comparing $\dim_{Q(Z_0)}(Q(Z_0) \otimes H)$ with $\dim_{Q(Z)}(Q(Z_0) \otimes H)$; the first of these is $p^{2n}|\Gamma|^3$ by (4.4), while the second is $p^{2n}|\Gamma|^2$ by Theorem 5.2. □

5.2. Γ-regularity of the generic irreducible modules

It follows from Theorem 5.2 that the maximal k-dimension of the irreducible H-modules is $p^n|\Gamma|$. By the structure theory of Noetherian PI-rings, this dimension is achieved precisely by those irreducible H-modules V for which $\mathfrak{m} := \mathrm{Ann}_{Z(H)}(V)$ has the property that $H/\mathfrak{m}H$ is simple Artinian, and in this case $H/\mathfrak{m}H \cong M_{p^n|\Gamma|}(k)$. The open set of such $\mathfrak{m}$ is precisely the Azumaya locus of § 3.4. Recall that [**11**, Theorem 1.7(vi)] shows that, for *any* symplectic reflection algebra over any field, at $t = 0$, the irreducible modules of maximal dimension are $k\Gamma$-regular of rank 1. Analogously, we can describe the $k\Gamma$-structure of the Azumaya irreducibles over Cherednik algebras for $t \neq 0$ when k has positive characteristic. We begin with an easy lemma, which essentially ensures that the desired result is true for $H_{1,\mathbf{0}}$.

Lemma 5.4. *Let k have characteristic $p > 0$ as usual, and let (V, ω) be a symplectic k-vector space with basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, with $\omega(x_i, y_j) = \delta_{ij}$, $\omega(x_i, x_j) = \omega(y_i, y_j) = 0$. Let Γ be a finite subgroup of $\mathrm{Sp}(V)$, of order prime to p. Write $\mathfrak{h}$ for the subspace $\sum_i kx_i$ of V, so that Γ acts by automorphisms on both $S(V)$ and $\mathcal{D}(\mathfrak{h})$. Let F be the quotient field of $(S(V)^\Gamma)^{\mathrm{p}}$. Then $F \otimes_{(S(V)^\Gamma)^{\mathrm{p}}} \mathcal{S}(V)$ is a free $F\Gamma$-module of rank p^{2n}.*

Proof. Note first that $(S(V)^\Gamma)^{\mathrm{p}}$ is a central Γ-invariant subalgebra of $\mathcal{D}(\mathfrak{h})$, so the statement of the lemma makes sense. Moreover, as $F\Gamma$-modules, there is no difference between

$$F \otimes_{(S(V)^\Gamma)^{\mathrm{p}}} \mathcal{D}(\mathfrak{h}) \quad \text{and} \quad F \otimes_{(S(V)^\Gamma)^{\mathrm{p}}} S(V),$$

so we can work with the latter. Then $F \otimes S(V)$ is simply the quotient field of $S(V)$, which is a free $Q(S(V)^\Gamma)\Gamma$-module of rank 1 by the Primitive Element Theorem of Galois theory. Since $\dim_F(Q(S(V)^\Gamma)) = p^{2n}$, the result follows. □

Theorem 5.5. *Let H be as in Theorem 5.2, and let the maximal ideal $\mathfrak{m}$ of $Z = Z(H)$ be in the Azumaya locus of H.*

(i) *$He/\mathfrak{m}He$ is a regular $k\Gamma$-module of rank p^{2n}.*

(ii) *The irreducible H-module U with $\mathrm{Ann}_Z(U) = \mathfrak{m}$ is a regular $k\Gamma$-module of rank p^n.*

Proof. (i) By Theorems 3.8 (ii) and 4.7 (ii), for every prime $\mathfrak{p}$ of Z in the Azumaya locus of H,

$$H_{\mathfrak{p}} \cong M_{|\Gamma|}(eH_{\mathfrak{p}}e). \tag{5.1}$$

Set $Q := Q(Z)$, the quotient field of Z, so by Theorem 5.2 we have

$$\dim_Q(Q \times_Z H) = p^{2n}|\Gamma|^2.$$

Thus, (5.1) implies that

$$\dim_Q(Q \otimes_Z He) = p^{2n}|\Gamma|. \tag{5.2}$$

Now let $\mathfrak{m}$ be a maximal ideal of Z in the Azumaya locus of H. The Azumaya property ensures that $H_{\mathfrak{m}}$ and hence also $H_{\mathfrak{m}}e$ are projective and thus free $Z_{\mathfrak{m}}$-modules. In particular, from (5.2),

$$H_{\mathfrak{m}}e \text{ is } Z_{\mathfrak{m}}\text{-free of rank } p^{2n}|\Gamma|. \tag{5.3}$$

Now let $\mathrm{Irr}(k\Gamma)$ be the set of isomorphism classes of irreducible $k\Gamma$-modules. We decompose He as the direct sum of its isotypic components as left $k\Gamma$-module:

$$He = \oplus_{E\in\mathrm{Irr}(k\Gamma)} \mathrm{Isot}_E(He). \tag{5.4}$$

Of course this is a sum of Z-modules as well as $k\Gamma$-modules, so applying $Q \otimes_Z \cdot$ to (5.4) yields

$$Q \otimes_Z He = \oplus_{E\in\mathrm{Irr}(k\Gamma)}(Q \otimes_Z \mathrm{Isot}_E(He)).$$

Thanks to the Dunkl embedding, Theorem 4.5 and Remark 4.6, H is birationally equivalent to the skew group algebra $\mathcal{D}(\mathfrak{h}) * \Gamma$, via a map which is the identity when restricted to $k\Gamma$. By this and the above lemma,

$$Q \otimes_Z He \text{ is } Q\Gamma\text{-regular of rank } p^{2n}. \tag{5.5}$$

By (5.3), the localized isotypic components

$$Z_{\mathfrak{m}} \otimes_Z \mathrm{Isot}_E(He) \cong \mathrm{Isot}_E(H_{\mathfrak{m}}e)$$

are $Z_{\mathfrak{m}}$-free for each $k\Gamma$-irreducible E; and so, in view of (5.5),

$$Z_{\mathfrak{m}} \otimes_Z \mathrm{Isot}_E(He) \text{ has } Z_{\mathfrak{m}}\text{-rank } p^{2n}(\dim_k(E))^2.$$

We deduce from this that, factoring $Z_{\mathfrak{m}}$ and the isotypic component by $\mathfrak{m}Z_{\mathfrak{m}}$,

$$\dim_k(\mathrm{Isot}_E(He/\mathfrak{m}He)) = p^{2n}(\dim_k(E))^2.$$

That is, the multiplicity of E in $He/\mathfrak{m}He$ is $p^{2n}\dim_k(E)$, proving (i).

(ii) By Theorem 5.2, $\dim_k(U) = p^n|\Gamma|$. Since U is the unique irreducible module for the simple Artinian ring $H/\mathfrak{m}H$, $He/\mathfrak{m}He$ is the sum of p^n copies of U. Therefore, it follows from (i) that U is $k\Gamma$-regular of rank p^n. □

5.3. Azumaya versus smooth locus

Theorem 5.6. *Let $H = H_{1,\mathbf{c}}(\mathfrak{h}, \omega, \Gamma)$ be a rational Cherednik algebra over an algebraically closed field k of positive characteristic p. Let Z be the centre of H, and write $\mathcal{A}_H$ for the Azumaya locus of Z in H and $\mathcal{S}_Z$ for the singular locus of* $\mathrm{Maxspec}(Z)$. *Then*

$$\mathcal{A}_H = \mathrm{Maxspec}(Z) \setminus \mathcal{S}_Z.$$

Proof. Since H is a finite Z-module and is Auslander-regular and Cohen–Macaulay, by Theorems 2.2 (ii) and 2.1 (iii), it follows from [**5**, Theorem 3.8] that it is sufficient to prove that H is Azumaya in codimension 1. That is, let $\mathfrak{p}$ be a prime ideal of Z of height 1.

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We must show that $H_{\mathfrak{p}}$ is Azumaya; equivalently, we must exhibit a maximal ideal $\mathfrak{m}$ of Z with $\mathfrak{m}$ Azumaya and $\mathfrak{p} \subseteq \mathfrak{m}$. Let Z_0 be the polynomial subalgebra $(S(\mathfrak{h})^{\mathrm{p}})^{\Gamma} \otimes (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma}$ of Z provided by Proposition 4.1. Thus, $\mathfrak{p}_0 := \mathfrak{p} \cap Z_0$ is a prime ideal of Z_0, and

$$\mathrm{height}(\mathfrak{p}_0) = 1, \tag{5.6}$$

by lying over [**10**, Proposition 4.15]. We claim that

$$\text{either} \quad \mathfrak{p}_0 \cap (S(\mathfrak{h})^{\mathrm{p}})^{\Gamma} = 0 \quad \text{or} \quad \mathfrak{p}_0 \cap (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma} = 0. \tag{5.7}$$

Suppose for a contradiction that both intersections are non-zero. Then

$$\mathfrak{q} := (\mathfrak{p}_0 \cap (S(\mathfrak{h})^{\mathrm{p}})^{\Gamma})Z_0 = (\mathfrak{p}_0 \cap (S(\mathfrak{h})^{\mathrm{p}})^{\Gamma}) \otimes (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma}$$

is a non-zero prime of Z_0 contained in $\mathfrak{p}_0$, and clearly $\mathfrak{q} \cap (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma} = 0$, so that $\mathfrak{q} \subsetneqq \mathfrak{p}_0$. But this contradicts (5.6), and hence (5.7) is true.

Let us suppose first that $\mathfrak{p}_0 \cap (S(\mathfrak{h}^*)^{\mathrm{p}})^{\Gamma} = 0$. Then, in particular, $\mathfrak{p}$ does *not* contain the central element $\delta^{p\ell}$ as defined in § 4.3, and hence there is a maximal ideal $\mathfrak{m}$ of Z with $\mathfrak{p} \subseteq \mathfrak{m}$, such that $\mathfrak{m}$ does not contain $\delta^{p\ell}$. We claim that $\mathfrak{m}$ is Azumaya; in view of [**6**, Theorem III.1.6] and Theorem 5.2, this amounts to showing that if W denotes an irreducible H-module killed by $\mathfrak{m}$, then

$$\dim_k(W) = p^n|\Gamma|. \tag{5.8}$$

Now δ acts as multiplication by a non-zero scalar on W; so, since $H[\delta^{-1}] \cong \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) * \Gamma$ by Theorem 4.5 and Remark 4.6, W admits actions of

(i) $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})$, and

(ii) $S(\mathfrak{h}^*)[\delta^{-1}] * \Gamma$.

From (i) and Lemma 5.1 (i) we deduce that

$$p^n \mid \dim_k(W). \tag{5.9}$$

Let U be any irreducible $S(\mathfrak{h}^*)[\delta^{-1}] * \Gamma$-module, so $\dim_k(U) < \infty$ and so there is a maximal ideal $\mathfrak{t}$ of $S(\mathfrak{h}^*)[\delta^{-1}]$ and $0 \neq u \in U$ with $\mathfrak{t}u = 0$. Set $U_1 := \mathrm{Ann}_U(\mathfrak{t})$, a non-zero $S(\mathfrak{h}^*)[\delta^{-1}]$-submodule of U. For each $\gamma \in \Gamma$, $\gamma U_1 = \mathrm{Ann}_U(\mathfrak{t}^{\gamma})$ is isomorphic as a vector space to U_1. Now $\mathfrak{t}$ has $|\Gamma|$ distinct Γ-conjugates, by definition of δ. Consider $U' := \sum_{\gamma \in \Gamma} \gamma U_1 \subseteq U$. Clearly, U' is a non-zero $S(\mathfrak{h}^*)[\delta^{-1}] * \Gamma$-submodule of U, and therefore $U' = U$. Moreover, the sum in the definition of U' has $|\Gamma|$ distinct terms, each killed by a distinct maximal ideal. So the sum is direct, and hence $\dim_k(U) = |\Gamma| \dim_k(U_1)$. In particular, $|\Gamma| \mid \dim_k(U)$; since W has a finite composition series as an $S(\mathfrak{h}^*)[\delta^{-1}] * \Gamma$-module,

$$|\Gamma| \mid \dim_k(W). \tag{5.10}$$

Combining (5.9) and (5.10), recalling that $p \nmid |\Gamma|$ by hypothesis, proves (5.8), and so the theorem follows. □

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6. Questions and conjectures

Throughout, $H = H_{1,\boldsymbol{c}}(V, \omega, \Gamma)$ is a symplectic reflection algebra over k, which is algebraically closed of characteristic $p > 0$. Let V have dimension $2n$.

It seems reasonable to expect that the value for the PI-degree of Cherednik algebras obtained in § 5.1 applies in general.

Question 6.1. Does H have PI-degree $p^n|\Gamma|$?

A more precise version of the above question is the following.

Question 6.2. Is every simple H-module of maximal dimension a regular $k\Gamma$-module of rank p^n?

It is of interest from the perspective of noncommutative resolutions of singularities to ask the following.

Question 6.3. For which H do there exist values of the parameter $\boldsymbol{c}$ for which $\mathrm{Maxspec}(Z(H))$ is smooth? When such values exist, determine them all.

The analogue of the first part of Question 6.3 in characteristic 0 at $t = 0$ has been answered completely, as a result of a considerable body of work (see [**1**, **11**, **13**, **21**]). A natural strategy to attack this problem in the Cherednik case is afforded by Theorem 5.6. To have this route available in the setting of an arbitrary symplectic reflection algebra, one needs therefore to answer the following.

Question 6.4. Does the Azumaya locus coincide with the smooth locus for an arbitrary symplectic reflection algebra?

Work on the finite-dimensional representation theory in characteristic 0 is aided considerably by the underlying Poisson structure; in view of [**7**, Theorems 4.2 and 7.8], there are only finitely many symplectic leaves in $\mathrm{Maxspec}(Z(H))$, and the representation theory is constant across leaves, in the sense that, if $\mathfrak{m}$ and $\mathfrak{n}$ belong to the same leaf, then $H/\mathfrak{m}H \cong H/\mathfrak{n}H$. This motivates the following question.

Question 6.5. Are there only finitely many isomorphism classes of factors $H/\mathfrak{m}H$ as $\mathfrak{m}$ ranges through $\mathrm{Maxspec}(Z(H))$?

The annoying gap in the equivalences of Theorem 3.8 is one indication that the symmetrizing subalgebra is not very well understood. We therefore ask the following.

Question 6.6. Is every localizable prime ideal P of H generated by its intersection with $Z(H)$?

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