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REPRESENTATIONS OF WEYL GROUPS OF TYPE B INDUCED FROM CENTRALISERS OF INVOLUTIONS

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Let G be a Weyl group of type B, and T a set of representatives of the conjugacy classes of self-inverse elements of G. For each t in T, we construct a (complex) linear character π_t of the centraliser of t in G, such that the sum of the characters of G induced from the π_t contains each irreducible complex character of G with multiplicity precisely 1. For Weyl groups of type A (that is, for the symmetric groups), a similar result was published recently by Inglis, Richardson and Saxl.

1.

An *involution model* for the irreducible complex characters of a finite group G is a set of linear characters λ_t of certain subgroups, such that the sum of the induced characters λ_t^G involves each irreducible character of G with multiplicity precisely 1. The relevant subgroups are the centralisers $\mathbf{C}_G(t)$ with t ranging through a complete set of representatives of the conjugacy classes of the self-inverse elements of G. In a January 1980 lecture which remains unpublished, Richardson had proved that all classical Weyl groups except those of type D_{2n} do have involution models. For type A, a much simpler proof was given eventually in Inglis, Richardson, Saxl [**2**]. In a paper [**1**] still to appear, Baddeley proves that if a finite group has an involution model then so does its wreath product with any symmetric group; this provides a new proof for the full result of Richardson.

This paper presents yet another proof for the type B case. This will be based on the theorem of Frobenius and Schur (see for example Isaacs [**3**]) which asserts that if all irreducible characters of G come from real representations and if $g \in G$ then the number of square roots of g in G is the value at g of the sum of the irreducible characters of G. Baddeley notes that (by another part of the Frobenius-Schur theorem) an involution model cannot exist unless all irreducible characters come from real representations: our argument makes use of the well-known fact that Weyl groups of type B satisfy this condition. (For that, an explicit and convenient reference is harder to find: we can only name Mayer [**4**].)

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2

For the rest of this paper let G be $W(B_n)$, the Weyl group of type B_n, which is the group of 'signed permutations' of n letters. It will be convenient to think of G as the semidirect product of the symmetric group $S_{\mathbf{n}}$ on $\mathbf{n} = \{1, \dots, n\}$ and the additive group $P_{\mathbf{n}}$ of the Boolean algebra of all subsets of $\mathbf{n}$. (The addition in question being symmetric difference of sets, it will be written as $\triangle$ instead of $+$.) Accordingly the elements of G will be written as $(\alpha, \mathbf{a}), (\beta, \mathbf{b}), \dots$ with $\alpha, \beta, \dots$ permutations on $\mathbf{n}$ and $\mathbf{a}, \mathbf{b}, \dots$ subsets of $\mathbf{n}$. The product of $(\alpha, \mathbf{a})$ and $(\beta, \mathbf{b})$ in G is given by

$$(\alpha, \mathbf{a})(\beta, \mathbf{b}) = (\alpha\beta, \beta(\mathbf{a}) \triangle \mathbf{b}).$$

By the support of a permutation α we mean the set of points which are actually moved (that is, not fixed) by α; we shall write it as $\operatorname{supp}\alpha$. Given $(\alpha, \mathbf{a})$ in G, a cycle γ of α is said to be a positive cycle of $(\alpha, \mathbf{a})$ if $|(\operatorname{supp}\gamma) \cap \mathbf{a}|$ is even, and a negative cycle of $(\alpha, \mathbf{a})$ otherwise. The signed cycle type of $(\alpha, \mathbf{a})$ is the function which counts the number of signed cycles of $(\alpha, \mathbf{a})$ of any given length and sign. Two elements of G are conjugate if and only if they have the same signed cycle type [4].

A straightforward calculation shows that if $(\beta, \mathbf{b}) \in \mathbf{C}_G((\alpha, \mathbf{a}))$ then $\operatorname{supp}\alpha$ and $\mathbf{a} \setminus \operatorname{supp}\alpha$ are invariant under β:

$$\beta(\operatorname{supp}\alpha) = \operatorname{supp}\alpha \text{ and } \beta(\mathbf{a} \setminus \operatorname{supp}\alpha) = \mathbf{a} \setminus \operatorname{supp}\alpha. \tag{1}$$

In particular, by restricting β one obtains a permutation $\beta \downarrow \operatorname{supp}\alpha$ of $\operatorname{supp}\alpha$, and the composite map

$$(\beta, \mathbf{b}) \mapsto \beta \mapsto \beta \downarrow \operatorname{supp}\alpha \mapsto \operatorname{sign}\beta \downarrow \operatorname{supp}\alpha$$

may be viewed as a linear character of $\mathbf{C}_G((\alpha, \mathbf{a}))$. It is also easy to see that

$$(\beta, \mathbf{b}) \mapsto (-1)^{|\mathbf{b} \cap \mathbf{a} \setminus \operatorname{supp}\alpha|}$$

is a linear character of $\mathbf{C}_G((\alpha, \mathbf{a}))$. (Use that if $(\beta, \mathbf{b}), (\beta', \mathbf{b}') \in \mathbf{C}_G((\alpha, \mathbf{a}))$, then by the Boolean distributive law

$$\begin{aligned}(\beta(\mathbf{b}') \triangle \mathbf{b}) \cap \mathbf{a} \setminus \operatorname{supp}\alpha &= (\beta(\mathbf{b}') \cap \mathbf{a} \setminus \operatorname{supp}\alpha) \triangle (\mathbf{b} \cap \mathbf{a} \setminus \operatorname{supp}\alpha) \\ &= \beta(\mathbf{b}' \cap \mathbf{a} \setminus \operatorname{supp}\alpha) \triangle (\mathbf{b} \cap \mathbf{a} \setminus \operatorname{supp}\alpha) \quad \text{by (1)}\end{aligned}$$

and that the parity of the cardinality of a symmetric difference is the parity of the sum of the cardinalities of the components.)

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The product of these two linear characters is what we want: define the linear character $\pi_{(\alpha,\mathrm{a})}$ of $\mathrm{C}_G((\alpha,\mathrm{a}))$ by

$$\pi_{(\alpha,\mathrm{a})}((\beta,\mathrm{b})) = (-1)^{|\mathrm{b}\cap\mathrm{a}\setminus\mathrm{supp}\,\alpha|}\mathrm{sign}\,\beta\downarrow\mathrm{supp}\,\alpha.$$

Our claim is that the $\pi_{(\alpha,\mathrm{a})}$ with $(\alpha,\mathrm{a}) = t\in T$ form an involution model for G.

In view of the Frobenius-Schur theorem, it will be sufficient to prove instead that

$$f((\alpha,\mathrm{a})) = \sum_{t\in T}\pi_t^G((\alpha,\mathrm{a})) \tag{2}$$

holds for all (α,a) in G. The proof will occupy the rest of the paper.

3.

STEP 1. By definition, the induced character π_t^G is given by

$$\pi_t^G((\alpha,\mathrm{a})) = \sum_{x\in G}|\mathrm{C}_G(t)|^{-1}\pi_t^0\big(x^{-1}(\alpha,\mathrm{a})x\big)$$

$$\text{where } \pi_t^0(g) = \begin{cases}\pi_t(g) & \text{if } g\in\mathrm{C}_G(t)\\ 0 & \text{otherwise.}\end{cases}$$

It is straightforward from the definition of π_t that $\pi_t^0\big(x^{-1}(\alpha,\mathrm{a})x\big) = \pi^0_{xtx^{-1}}((\alpha,\mathrm{a}))$. Using that G is the disjoint union of its cosets $\{x : xtx^{-1}\}$ modulo $\mathrm{C}_G(t)$ and that S is the disjoint union of the conjugacy classes t^G of the t in T, one can argue that

$$\begin{aligned}\sum_{t\in T}\pi_t^G((\alpha,\mathrm{a})) &= \sum_{t\in T}\sum_{x\in G}|\mathrm{C}_G(t)|^{-1}\pi_t^0\big(x^{-1}(\alpha,\mathrm{a})x\big)\\ &= \sum_{t\in T}\sum_{x\in G}|\mathrm{C}_G(t)|^{-1}\pi^0_{xtx^{-1}}((\alpha,\mathrm{a}))\\ &= \sum_{t\in T}\sum_{s\in t^G}\sum_{x:xtx^{-1}=s}|\mathrm{C}_G(s)|^{-1}\pi_s^0((\alpha,\mathrm{a}))\\ &= \sum_{t\in T}\sum_{s\in t^G}\pi_s^0((\alpha,\mathrm{a}))\\ &= \sum_{s\in S}\pi_s^0((\alpha,\mathrm{a}))\\ &= \sum_{s\in S\cap C_G((\alpha,\mathrm{a}))}\pi_s((\alpha,\mathrm{a})).\end{aligned} \tag{3}$$

Call this last sum $R_{\alpha,\mathrm{a}}$: what we have to prove is that $R_{\alpha,\mathrm{a}} = f((\alpha,\mathrm{a}))$.

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STEP 2. Given any decomposition of $\mathbf{n}$ as disjoint union

$$\mathbf{n} = \mathbf{d}_1 \dot{\cup} \ldots \dot{\cup} \mathbf{d}_m \tag{4}$$

of nonempty subsets, consider the subgroup of G consisting of all $(\alpha, \mathbf{a})$ in G such that $\alpha(\mathbf{d}_i) = \mathbf{d}_i$ for all i. This is the direct product of the subgroups G_i defined by

$$G_i = \{(\alpha, \mathbf{a}) \in G : \operatorname{supp} \alpha \subseteq \mathbf{d}_i,\ \mathbf{a} \subseteq \mathbf{d}_i\},$$

and of course restriction to $\mathbf{d}_i$ gives an isomorphism $G_i \cong W(B_{|\mathbf{d}_i|})$. To each g in G, there is a unique decomposition (4) such that

(i) $g \in G_1 \times \ldots \times G_m$: say, $g = \prod g_i$ with $g_i \in G_i$;
(ii) for each i, the cycles of g_i in $\mathbf{d}_i$ all have the same length and sign;
(iii) if $i \neq j$, the cycles of g_i in $\mathbf{d}_i$ differ from the cycles of g_j in $\mathbf{d}_j$ either in length or in sign (or in both).

It is easy to check case by case that the cycles of the square of a signed cyclic permutation all have the same length and the same sign; hence

$$f_G(g) = \prod_{i=i}^{m} f_{G_i}(g_i). \tag{5}$$

With $g = (\alpha, \mathbf{a})$ the set $\mathbf{a} \setminus \operatorname{supp} \alpha$ considered in (1) is either empty or one of the $\mathbf{d}_i$ here (the union of the negative orbits of length one); as we saw there, it is setwise invariant under $\mathbf{C}_G(g)$. Similiarly each $\mathbf{d}_i$ is setwise invariant; so

$$\mathbf{C}_G(g) = \mathbf{C}_{G_1}(g_1) \times \ldots \times \mathbf{C}_{G_m}(g_m).$$

It follows that $S \cap \mathbf{C}_G(g) = \{\prod s_i : s_i \in S(G_i) \cap \mathbf{C}_{G_i}(g_i)\}$. Of course if g_i is written as $(\alpha_i, \mathbf{a}_i)$ then $\mathbf{a}_i \cup \operatorname{supp} \alpha_i \subseteq \mathbf{d}_i$; and if $s = \prod s_i$ with $s = (\beta, \mathbf{b})$ and $s_i = (\beta_i, \mathbf{b}_i)$ then $\operatorname{supp} \beta_i = (\operatorname{supp} \beta) \cap \mathbf{d}_i$ and $\mathbf{b}_i \setminus \operatorname{supp} \beta_i = (\mathbf{b} \setminus \operatorname{supp} \beta) \cap \mathbf{d}_i$: thus it is is easy to see that $\pi_s(g_i) = \pi_{s_i}(g_i)$ and so

$$\pi_s(g) = \pi_s\left(\prod g_i\right) = \prod \pi_s(g_i) = \prod \pi_{s_i}(g_i).$$

Consequently,

$$\sum_s \pi_s(g) = \sum_{s_1} \cdots \sum_{s_m} \prod_{i=1}^{m} \pi_{s_i}(g_i) = \prod_{i=1}^{m} \sum_{s_i} \pi_{s_i}(g_i) \tag{6}$$

where the ranges of s and s_i are $S \cap \mathbf{C}_G(g)$ and $S_i \cap \mathbf{C}_{G_i}(g_i)$, respectively. It remains to note that the restriction of π_{s_i} to $\mathbf{C}_{G_i}(g_i)$ is the same as the linear character of this subgroup defined with reference to s_i when G_i is veiwed as a Weyl group of rank $|\mathbf{d}_i|$ in its own right. In veiw of (3), (5) and (6), this means that it suffices to prove (2) for the $(\alpha, \mathbf{a})$ whose cycles are all the same length and sign.

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STEP 3. Since both sides of (2) are characters and hence constant on conjugacy classes it is sufficient to prove (2) for representatives of conjugacy classes. Combining this with Step 2 it then follows that we only have to verify (2) for the $(\alpha, \mathbf{a})$ in G with

$$\begin{aligned} \alpha &= (1, \dots, r)(r+1, \dots, 2r) \dots ((k-1)r+1, \dots, kr) \\ \mathbf{a} &= \emptyset \quad \text{or} \quad \{1, r+1, \dots, (k-1)r+1\} \end{aligned} \tag{7}$$

where $kr = n$.

STEP 4. If $(\alpha, \mathbf{a})$ in G is of form (7) and is of odd order then r is odd and $\mathbf{a} = \emptyset$, thus $\pi_s((\alpha, \mathbf{a})) = 1$ for all s in $S \cap \mathrm{C}_G((\alpha, \mathbf{a}))$. Hence $R_{\alpha,\mathbf{a}} = |S \cap \mathrm{C}_G((\alpha, \mathbf{a}))|$. But if g is any element of odd order, $2q-1$ say, in any group, there is a bijection (namely $h \mapsto g^q h$) from the set of square roots of g to the set of self-inverse elements which commute with g. This verifies (2) for this case.

STEP 5. Let $X((\alpha, \mathbf{a}))$ be the set of those $(\beta, \mathbf{b})$ in $S \cap C_G((\alpha, \mathbf{a}))$ which act fixed point free on the set of orbits of α. The next point to establish is that if $(\alpha, \mathbf{a})$ in G is of form (7) and is of even order, then

$$R_{\alpha,\mathbf{a}} = |X((\alpha, \mathbf{a}))|. \tag{8}$$

First consider when k, the number of orbits of α, equals 1. Calculating $R_{\alpha,\mathbf{a}}$ will be done distinguishing three cases; in each case, $(\beta, \mathbf{b})$ ranges over $S \cap \mathrm{C}_G((\alpha, \mathbf{a}))$.

(i) Let n be odd and $\mathbf{a} = \{1\}$. As β is self-inverse, $|\operatorname{supp} \beta|$ is even, so $\operatorname{supp} \beta$ is a proper subset of $\mathbf{n}$. As $(\alpha, \mathbf{a})$ and $(\beta, \mathbf{b})$ commute, $\operatorname{supp} \beta$ is invariant under α: since α is transitive on $\mathbf{n}$, this is impossible unless $\operatorname{supp} \beta = \emptyset$, so $\beta = 1$. The equation

$$\beta(\{1\}) \,\triangle\, \mathbf{b} = \alpha(\mathbf{b}) \,\triangle\, \{1\}$$

then has two solutions, namely $\mathbf{b} = \emptyset$ and $\mathbf{b} = \mathbf{n}$. Thus

$$R_{\alpha,\mathbf{a}} = (-1)^{|\{1\} \cap \emptyset|} + (-1)^{|\{1\} \cap \mathbf{n}|} = 0$$

(ii) Let n be even and $\mathbf{a} = \emptyset$. The self-inverse elements of $S_\mathbf{n}$ which commute with $(1, \dots, n)$ are 1 and $(1, n/2+1)(2, n/2+2) \dots (n/2, n)$. The equation

$$\beta(\emptyset) \,\triangle\, \mathbf{b} = \alpha(\mathbf{b}) \,\triangle\, \emptyset$$

has the same two solutions for both values of β, namely $\mathbf{b} = \emptyset$ and $\mathbf{b} = \mathbf{n}$. Thus

$$R_{\alpha,\mathbf{a}} = 2(\operatorname{sign} \alpha \downarrow \emptyset + \operatorname{sign} \alpha \downarrow \mathbf{n}) = 0$$

since n is even.

(iii) Let n be even and $\mathbf{a} = \{1\}$. As in case (ii) we have that β is either 1 or $(1, n/2+1)\dots(n/2, n)$. If $\beta = 1$ then the equation

$$\beta(\{1\}) \triangle \mathbf{b} = \alpha(\mathbf{b}) \triangle \{1\}$$

has two solutions, namely $\mathbf{b} = \emptyset$ and $\mathbf{b} = \mathbf{n}$. However if $\beta = (1, n/2+1)\dots(n/2, n)$ then that equation has no solutions. Thus

$$R_{\alpha,\mathbf{a}} = (-1)^{|\{1\}\cap\emptyset|} + (-1)^{|\{1\}\cap\mathbf{n}|} = 0.$$

Now consider the general case with $k \geqslant 1$.

Let $\mathbf{d}_1$ be an orbit of α; set $\mathbf{d}_2 = \mathbf{n} \setminus \mathbf{d}_1$, and

$$G_i = \{ (\gamma, \mathbf{c}) \in G : \operatorname{supp}\gamma \subseteq \mathbf{d}_i,\ \mathbf{c} \subseteq \mathbf{d}_i \},$$

Then $(\alpha, \mathbf{a})$ is in $G_1 \times G_2$ with, say, $(\alpha, \mathbf{a}) = g_1 g_2$ where $g_i \in G_i$. Now

$$S \cap \mathbf{C}_G((\alpha, \mathbf{a})) \cap (G_1 \times G_2) = \{ s_1 s_2 : s_i \in S(G_i) \cap \mathbf{C}_{G_i}(g_i) \}.$$

So by (6)

$$\sum_s \pi_s((\alpha, \mathbf{a})) = \prod_{i=1}^{2} \sum_{s_i} \pi_{s_i}(g_i)$$

where s and s_i range over $S \cap \mathbf{C}_G((\alpha, \mathbf{a})) \cap (G_1 \times G_2)$ and $S(G_i) \cap \mathbf{C}_{G_i}(g_i)$, respectively. But by the argument we used in the case $k = 1$,

$$\sum_{s_1} \pi_{s_1}(g_1) = 0.$$

Repeating this for each orbit of α, we get that

$$R_{\alpha,\mathbf{a}} = \sum_{s \in X((\alpha, \mathbf{a}))} \pi_s((\alpha, \mathbf{a})). \tag{9}$$

The claim then follows by considering the following two cases.

(i) Let k be odd. Then $R_{\alpha,\mathbf{a}} = 0$ by (9), because no element of $S \cap \mathbf{C}_G((\alpha, \mathbf{a}))$ can act fixed point free on the set of orbits of α.

(ii) Let k be even and let s be an element of $X((\alpha, \mathbf{a}))$. Then $\operatorname{supp}\alpha = \mathbf{n}$ and $\mathbf{a} \setminus \operatorname{supp}\alpha = \emptyset$. So

$$\begin{aligned} \pi_s((\alpha, \mathbf{a})) &= \operatorname{sign}\alpha \\ &= 1 \end{aligned}$$

since k is even. Thus, by (9), the claim (8) follows.

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STEP 6. Let $(\alpha, \mathbf{a})$ in G have even order and be of form (7). The final step is to show that the right hand side of (8) equals $f((\alpha, \mathbf{a}))$.

If α has an odd number of orbits then $(\alpha, \mathbf{a})$ has no square roots and $X((\alpha, \mathbf{a}))$ is the empty set, so we are done.

Let α have an even number of orbits. There are two cases to consider.

(i) Let $\mathbf{a} = \emptyset$. The square roots of $(\alpha, \mathbf{a})$ are the $(\gamma, \mathbf{c})$ with $\gamma^2 = \alpha$ and $\mathbf{c}$ a union of orbits of γ. Given a square root γ of α in $S_{\mathbf{n}}$, we define a permutation β on $\mathbf{n}$ as follows. Each element x of $\mathbf{n}$ can be written unambiguously in the form $\gamma^i(j)$ where $0 \leqslant i < 2r$ and j is the smallest element of the orbit of γ containing x. Using this we set

$$\beta(x) = \beta(\gamma^i(j)) = \begin{cases} \gamma^{i+1}(j) & \text{if } i \text{ is even,} \\ \gamma^{i-1}(j) & \text{if } i \text{ is odd.} \end{cases}$$

It is easy to see that β is a self-inverse element of $\mathbf{C}_{S_{\mathbf{n}}}(\alpha)$ and is fixed point free on the set of orbits of α. Moreover, if $(\gamma, \mathbf{c})$ is a square root of $(\alpha, \mathbf{a})$ in G, then $(\beta, \mathbf{c})$ is in $X((\alpha, \mathbf{a}))$, and $(\gamma, \mathbf{c}) \mapsto (\beta, \mathbf{c})$ is a bijection between the relevant sets.

(ii) Let $\mathbf{a} = \{1, r+1, \ldots, (k-1)r+1\}$. If now γ is a square root of α in $S_{\mathbf{n}}$, then each orbit $\mathbf{d}$ of γ is the union of two orbits of α, so in particular $\mathbf{d} \cap \mathbf{a}$ is a doubleton. It is not hard to see that an element $(\gamma, \mathbf{c})$ of G is a square root of $(\alpha, \mathbf{a})$ if and only if $\gamma^2 = \alpha$ and for each orbit $\mathbf{d}$ of γ the following conditions hold: the intersection $\mathbf{d} \cap \mathbf{a} \cap \gamma^{-1}(\mathbf{a})$ is a singleton, and $\mathbf{d} \cap \mathbf{c}$ is either this singleton or its complement in $\mathbf{d}$. When $\mathbf{d} \cap \mathbf{a} \cap \gamma^{-1}(\mathbf{a})$ is a singleton, call its one element $y_{\mathbf{d}}$.

Given a square root γ of α in $S_{\mathbf{n}}$, we define a permutation β on $\mathbf{n}$ as in (i). Given a square root $(\gamma, \mathbf{c})$ of $(\alpha, \mathbf{a})$ in G, we define a subset $\mathbf{b}$ of $\mathbf{n}$ as follows. For each orbit $\mathbf{d}$ of γ, with $y_{\mathbf{d}}$ defined as above, and $z = \gamma(y_{\mathbf{d}})$,

$$\mathbf{b} \cap \mathbf{u} = \begin{cases} \{y_{\mathbf{d}}, y_{\mathbf{d}}+2, y_{\mathbf{d}}+4, \ldots, z, z+2, z+4, \ldots\} & \text{if } \mathbf{c} \cap \mathbf{u} = \{y_{\mathbf{d}}\}, \\ \{y_{\mathbf{d}}+1, y_{\mathbf{d}}+3, y_{\mathbf{d}}+5, \ldots, z+1, z+3, z+5, \ldots\} & \text{otherwise.} \end{cases}$$

Now $(\beta, \mathbf{b})$ is in $X((\alpha, \mathbf{a}))$, and $(\gamma, \mathbf{c}) \mapsto (\beta, \mathbf{b})$ is a bijection between the relevant sets.

This completes the proof. $\square$

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