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DEL PEZZO SURFACES IN WEIGHTED PROJECTIVE SPACES

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Abstract We study singular del Pezzo surfaces that are quasi-smooth and well-formed weighted hypersurfaces. We give an algorithm to classify all of them.

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1. Introduction

Fano varieties are the building blocks of rationally connected varieties. They have been studied for a long time and have often been used to produce counterexamples to long-standing conjectures. Classically, Fano varieties were assumed to be smooth. However, during the last decades, progress in the area has provided tools and posed problems dealing with mildly singular Fano varieties. The classification of singular Fano varieties seems to be hopeless in higher dimensions without bounding the singularities. Nevertheless, we know many partial classification-type results about two-dimensional Fano varieties, also known as del Pezzo surfaces, thanks to the combined efforts of many people (see [**1**, **3**, **17**, **22**, **23**, **25**, **26**, **34**]).

The classification problem for Fano manifolds is closely related to the problem of the existence of Kähler–Einstein metrics on them (see [**27**]). It has been conjectured by Yau, Tian and Donaldson that a Fano manifold admits a Kähler–Einstein metric if and only if it is K-polystable. One direction of this conjecture, the K-polystability of Kähler–Einstein Fano manifolds, follows from the works of Tian, Donaldson, Stoppa and Berman (see [**4**, **14**, **28**, **31**]). The other direction has been recently proved by Chen, Donaldson and Sun in [**10**–**12**] and independently by Tian in [**32**]. Unfortunately, this result is not easy to apply since K-polystability is usually very hard to check.

The problem of the existence of a Kähler–Einstein metric on smooth del Pezzo surfaces has been explicitly solved by Tian in [**30**]. For del Pezzo surfaces with quotient singularities, we do not have such an explicit solution (for orbifold metrics), since del Pezzo

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surfaces with quotient singularities have not been classified. So, it seems natural to consider this problem imposing some additional restrictions on the class of singular del Pezzo surfaces. In this paper, we will consider singular del Pezzo surfaces that are quasi-smooth and well-formed hypersurfaces in weighted projective spaces (see [**18**, Definition 6.9]).

Let S_d be a hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d, where $1 \leqslant a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$ are some natural numbers. Then S_d is given by

$$\phi(x, y, z, t) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t])$$

where $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, and ϕ is a quasi-homogeneous polynomial of degree d with respect to these weights. The equation

$$\phi(x, y, z, t) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x, y, z, t])$$

defines a three-dimensional hypersurface quasi-homogeneous singularity (V, O), where $O = (0, 0, 0, 0)$. Recall that S_d is called *quasi-smooth* if the singularity (V, O) is isolated. Recall that S_d is called *well-formed* if

$$\gcd(a_1, a_2, a_3) = \gcd(a_0, a_2, a_3) = \gcd(a_0, a_1, a_3) = \gcd(a_0, a_1, a_2) = 1$$

and each positive integer $\gcd(a_0, a_1)$, $\gcd(a_0, a_2)$, $\gcd(a_0, a_3)$, $\gcd(a_1, a_2)$, $\gcd(a_1, a_3)$, $\gcd(a_2, a_3)$ divides d. If the hypersurface S_d is quasi-smooth and well-formed, then it follows from [**24**, Theorem 7.9], [**24**, Proposition 8.13], [**24**, Remark 8.14.1], [**24**, Theorem 11.1] and the adjunction formula that the following conditions are equivalent:

- The inequality $d < a_0 + a_1 + a_2 + a_3$ holds.
- The singularity (V, O) is a rational singularity.
- The singularity (V, O) is a Kawamata log terminal singularity.
- The hypersurface S_d is a del Pezzo surface with quotient singularities.

Starting from now, we set $d < \sum_{i=0}^{n} a_i$ and set the hypersurface S_d as quasi-smooth and well-formed. We define $I = a_0 + a_1 + a_2 + a_3 - d$. This is usually called the index of the hypersurface $S_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$. For every positive integer I, we have infinitely many possibilities for the sextuple $(a_0, a_1, a_2, a_3, d, I)$ such that there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$. This is not surprising since we know there are infinitely many families of del Pezzo surfaces with quotient singularities.

Problem 1.1. Describe all sextuples $(a_0, a_1, a_2, a_3, d, I)$ such that there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$.

This problem was posed by Dmitri Orlov a long time ago to test his conjecture about the existence of a full exceptional collection on del Pezzo surfaces with quotient singularities. Later his conjecture was proved by Kawamata, Elagin, Ishii and Ueda (see [**15**, **19**, **21**]).

The first step in solving Problem 1.1 was done by Johnson and Kollár, who proved the following.

Theorem 1.2 ([20, Theorem 8]). *Suppose that $I = 1$. Then*

- *either* $(a_0, a_1, a_2, a_3, d) = (2, 2m+1, 2m+1, 4m+1, 8m+4)$ *for some* $m \in \mathbb{N}$,
- *or the quintuple* (a_0, a_1, a_2, a_3, d) *lies in the sporadic set*

$$\left\{\begin{array}{l} (1,1,1,1,3),(1,1,1,2,4),(1,1,2,3,6),(1,2,3,5,10),(1,3,5,7,15),(1,3,5,8,16),\\ (2,3,5,9,18),(3,3,5,5,15),(3,5,7,11,25),(3,5,7,14,28),(3,5,11,18,36),\\ (5,14,17,21,56),(5,19,27,31,81),(5,19,27,50,100),(7,11,27,37,81),\\ (7,11,27,44,88),(9,15,17,20,60),(9,15,23,23,69),(11,29,39,49,127),\\ (11,49,69,128,256),(13,23,35,57,127),(13,35,81,128,256) \end{array}\right\}.$$

Moreover, for each listed quintuple (a_0, a_1, a_2, a_3, d)*, there exists a quasi-smooth well-formed hypersurface in* $\mathbb{P}(a_0, a_1, a_2, a_3)$ *of degree* $d = a_0 + a_1 + a_2 + a_3 - 1$.

The second step in solving Problem 1.1 was done by Cheltsov and Shramov, who solved Problem 1.1 for $I = 2$ (see [9, Corollary 1.13]).

For Cheltsov, Johnson, Kollár and Shramov, the main motivation to prove Theorem 1.2 was the Calabi problem for del Pezzo surfaces with quotient singularities and, in particular, the Calabi problem for quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$. Regarding the latter, Gauntlett, Martelli, Sparks and Yau proved the following.

Theorem 1.3 ([16]). *The surface S_d does not admit an orbifold Kähler–Einstein metric if $I > 3a_0$.*

Thus, the Calabi problem for quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ has a negative solution if $I > 3a_0$. Conversely, Araujo, Boyer, Demailly, Galicki, Johnson, Kollár and Nakamaye proved that the Calabi problem for quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d = a_0 + a_1 + a_2 + a_3 - 1$ almost always has a positive solution.

Theorem 1.4 ([2, 5, 13, 20]). *Suppose that $I = 1$. Then S_d admits an orbifold Kähler–Einstein metric except possibly the case when* $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ *and the polynomial* $\phi(x_0, x_1, x_2, x_3)$ *does not contain the monomial* $x_1x_2x_3$.

The proof of Theorem 1.4 implicitly uses the α-invariant introduced by Tian for smooth Fano varieties in [29]. For S_d, its algebraic counterpart can be defined as

$$\alpha(S_d) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (S_d, \lambda D) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_{S_d}\end{array}\right\},$$

and one can easily extend this definition to any Fano variety with at most Kawamata log terminal singularities. Tian, Demailly and Kollár showed that the α-invariant plays an important role in the existence of an orbifold Kähler–Einstein metric on Fano varieties with quotient singularities (see [6, 13, 29], [8, Theorem A.3]). In particular, we have the following.

Theorem 1.5 ([13, 29], [8, Theorem A.3]). *If $\alpha(S_d) > 2/3$, then S_d admits an orbifold Kähler–Einstein metric.*

Araujo, Boyer, Demailly, Galicki, Johnson, Kollár and Nakamaye proved that $\alpha(S_d) > 2/3$ if $I = 1$ except exactly one case when $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the polynomial $\phi(x_0, x_1, x_2, x_3)$ does not contain the monomial $x_1x_2x_3$ (in this case $\alpha(S_d) = 7/10 < 2/3$ by [7]). A similar approach was used by Boyer, Cheltsov, Galicki, Nakamaye, Park and Shramov for $I \geqslant 2$ (see [5, 7, 9]).

It seems unlikely that Problem 1.1 has a *nice* solution for all I at once. However, the results by Cheltsov, Johnson, Kollár and Shramov indicate that it is possible to solve Problem 1.1 for any fixed I. The main purpose of the present paper is to prove this and to give an algorithm that solves Problem 1.1 for any fixed I, that is, that finds the set of quintuples (a_0, a_1, a_2, a_3, d) such that there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$.

We hope that our classification can be used to produce a vast number of examples of non-Kähler–Einstein del Pezzo surfaces with quotient singularities using different kinds of existing obstructions. For example, recently Spotti proved the following.

Theorem 1.6 ([27]). *Let S be a del Pezzo surface with at most quotient singularities, and let N be the biggest natural number such that S_d has a quotient singularity $\mathbb{C}^2/G$ with $N = |G|$, where G is a finite subgroup in $\mathrm{GL}_2(\mathbb{C})$ that does not contain quasi-reflections. Then S does not admit an orbifold Kähler–Einstein metric if $K_S^2 N \geqslant 12$.*

Using our classification, we immediately obtain a huge number of examples of quasi-smooth well-formed hypersurfaces in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ that do not admit an orbifold Kähler–Einstein metric by Theorem 1.6 such that the obstruction found by Gauntlett, Martelli, Sparks and Yau, i.e. Theorem 1.3, is not applicable. The following tuples (a_0, a_1, a_2, a_3, d) come from § 4.

- The tuple $(1, 3, 4, 8, 12)$ from the series $(1, 3, 3a + 1, 3b + 2, 3a + 3b + 3)$ with $I = 4$ satisfies both Theorem 1.3 and Spotti's inequality.
- The tuple $(1, 3, 7, 8, 15)$ from the series $(1, 3, 3a + 1, 3b + 2, 3a + 3b + 3)$ with $I = 4$ satisfies Theorem 1.3 but not Spotti's inequality.
- The tuple $(2, 2, 3, 7, 10)$ from the series $(2, 2, 2a + 1, 2b + 1, 2a + 2b + 2)$ with $I = 4$ does not satisfy Theorem 1.3 but satisfies Spotti's inequality.
- The tuple $(2, 2, 3, 3, 6)$ from the series $(2, 2, 2a + 1, 2b + 1, 2a + 2b + 2)$ with $I = 4$ satisfies neither Theorem 1.3 nor Spotti's inequality.

These examples show that the previous inequality by Gauntlett, Martelli, Sparks and Yau (Theorem 1.3) is independent of the new inequality discovered by Spotti (Theorem 1.6). Thus, Spotti's inequality is a new and powerful obstruction to the existence of orbifold Kähler–Einstein metrics on del Pezzo surfaces with quotient singularities.

Now, we state the main result of the paper.

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Theorem 1.7. *The tuples (a_0, a_1, a_2, a_3, d) such that there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$ are precisely those that either belong to Tables A1 and A2 (Appendix A), or are ordered, satisfy conditions (i)–(iv) from Theorem 2.2 and have one of the following forms where $x, y \in \mathbb{Z}$:*

- *Class 1 tuples $(a_0, a_1, b_2 + xm, b_3 + ym, b_2 + b_3 + (x + y)m)$, where*
  — a_0, a_1 *are positive integers such that* $I = a_0 + a_1$,

  — $m = \operatorname{lcm}(a_0, a_1)$,

  — b_2, b_3 *are positive integers.*
- *Class 2 tuples $(a_0, a_1, a_2, b_3 + xm, a_1 + b_3 + xm)$, where*
  — a_0, a_1, a_2 *are positive integers such that* $I = a_0 + a_2$ *and* $I > a_0 + a_1$,

  — $m = \operatorname{lcm}(a_0, a_1, a_2)$,

  — b_3 *is a positive integer.*
- *Class 3 tuples $(a_0, a_1, a_2, b_3 + xm, a_0 + b_3 + xm)$, where*
  — a_0, a_1, a_2 *are positive integers such that* $I = a_1 + a_2$ *and* $I > a_0 + a_2$,

  — $m = \operatorname{lcm}(a_0, a_1, a_2)$,

  — b_3 *is a positive integer.*
- *Class 4 tuples $(a_0, a_1, b_2 + xm, (a_1/2) + b_2 + xm, a_1 + 2b_2 + 2xm)$, where*
  — a_0, a_1 *are positive integers such that* $I = a_0 + (a_1/2)$,

  — $m = \operatorname{lcm}(a_0, (a_1/2))$,

  — b_2 *is a positive integer.*
- *Class 5 tuples $(a_0, a_1, b_2 + xm, (a_0/2) + b_2 + xm, a_0 + 2b_2 + 2xm)$, where*
  — a_0, a_1 *are positive integers such that* $I = (a_0/2) + a_1$ *and* $I > a_0 + (a_1/2)$,

  — $m = \operatorname{lcm}((a_0/2), a_1)$,

  — b_2 *is a positive integer.*
- *Class 6 tuples $(a_0, a_1, b_2 + xm, b_3 + xm, d + 2xm) = (I - k, I + k, a + xm, a + k + xm, 2a + I + k + 2xm)$, where*
  — I *is the index,*

  — k *is a positive integer such that* $I - k$ *is positive,*

  — $m = \operatorname{lcm}(a_0, a_1, k) = \operatorname{lcm}(I - k, I + k, k)$,

  — a *is a positive integer.*

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Quasi-smoothness and well-formedness conditions are given in Theorems 2.1 and 2.2. A tuple (a_0, a_1, a_2, a_3, d) is ordered if $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant d$. The fact that we need to consider only conditions (i) to (iv) of Theorem 2.2 is proved in Proposition 2.9.

The second main result is the following.

Theorem 1.8. *If there exists a quasi-smooth well-formed hypersurface for a tuple from a class n series given in Theorem 1.7, then there also exists a quasi-smooth well-formed hypersurface for all the other ordered tuples in the series, that is, all the other ordered tuples with the same a_i and b_i but different x or y.*

This result is proved in Theorem 2.10.

For any fixed I, there are only finitely many series in Theorem 1.7. Let us check this for class 1. A series in class 1 is given by ordered tuples $(a_0, a_1, b_2 + xm, b_3 + ym, b_2 + b_3 + (x + y)m)$ where $x, y \in \mathbb{Z}$ and a_0, a_1, b_2, b_3, m are fixed positive integers. Since $a_0 + a_1 = I$, there are only finitely many choices for a_0 and a_1. The number $m = \operatorname{lcm}(a_0, a_1)$ is uniquely determined by a_0 and a_1. Since $x, y \in \mathbb{Z}$, we are interested in only b_2 and b_3 modulo m. A series is uniquely determined by a_0, a_1, b_2, b_3 and m, so there are only finitely many series in class 1. Similarly for classes 2 to 6.

According to Theorem 1.8, checking one tuple from every such series determines whether every tuple in the series is such that there exists a quasi-smooth well-formed hypersurface. So, the algorithm to classify the hypersurfaces for an index I reduces to checking conditions (i)–(iv) of Theorem 2.2 for finitely many tuples.

In § 2, we prove Theorems 1.7 and 1.8, leaving computations to § 3. The main tool we use is Theorem 2.2, which lists the quasi-smoothness and well-formedness conditions. The complete lists of the classified hypersurfaces for indices I from 1 to 6 are given in § 4. Appendix B contains the computer code of the algorithm.

Theorems 1.7 and 1.8 provide the algorithm to solve Orlov's problem, Problem 1.1, for any fixed I, as well as giving the general form of the answer for any I. The surprisingly rigid form describes the hypersurfaces for all indices at once, without needing to calculate them explicitly.

2. Technical result

The following theorem describes the tuples (a_0, a_1, a_2, a_3, d) such that there exists a quasi-smooth well-formed hypersurface. Conditions (iv), (v) and (vi) are taken from [**20**, Conditions 2.1, 2.2 and 2.3].

Theorem 2.1. *There exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$ iff all of the following conditions hold:*

(i–ii) *The hypersurface is well-formed.*

(iii) *The hypersurface is not degenerate, that is, none of the weights a_i equals the degree of the hypersurface.*

(iv) *For every i, there exists j (j may equal i) such that there exists a monomial $x_i^{m_i} x_j$ of degree d, where $m_i \geqslant 1$.*

(v) *For every $i < j$ such that* $\gcd(a_i, a_j) > 1$, *there is a monomial* $x_i^{b_i} x_j^{b_j}$ *of degree* d, *where* $b_i, b_j \geqslant 0$ *and* $b_i + b_j \geqslant 2$.

(vi) *For every $i < j$, either there is a monomial $x_i^{b_i} x_j^{b_j}$ of degree d or there exist $k < l$ such that the indices i, j, k, l are pairwise different and there are monomials $x_i^{c_i} x_j^{c_j} x_k$ and $x_i^{d_i} x_j^{d_j} x_l$ of degree d, where $b_i, b_j, c_i, c_j, d_i, d_j \geqslant 0$ and $b_i + b_j \geqslant 2$ and $c_i + c_j \geqslant 1$ and $d_i + d_j \geqslant 1$.*

We will use the following form of the theorem, adding results from [9, Theorem 2.3] and [9, Definitions 1.10 and 2.2].

Theorem 2.2. *There exists a quasi-smooth well-formed hypersurface in* $\mathbb{P}(a_0, a_1, a_2, a_3)$ *of degree* $d < a_0 + a_1 + a_2 + a_3$ *and index* $I = a_0 + a_1 + a_2 + a_3 - d$ *iff all of the following conditions hold:*

(i) *For every $i < j$,* $\gcd(a_i, a_j) \mid d$.

(ii) *For every $i < j < k$,* $\gcd(a_i, a_j, a_k) = 1$.

(iii) $d > a_3$.

(iv) *For every i, there exists j (j may equal i) such that $a_i \mid d - a_j$.*

(v) *For every $i < j$ such that* $\gcd(a_i, a_j) > 1$, *one of the following holds:*
- $a_i \mid d$,
- $a_j \mid d$,
- $a_i \mid d - a_j$,
- $a_j \mid d - a_i$,
- *there exists $b_j \geqslant 2$ such that $a_i \mid d - a_j b_j$ and $d - a_j b_j \geqslant 0$.*

(vi) *For every $i < j$, (at least) one of the following holds:*
- *one of the following holds:*
  * $a_i \mid d$,
  * $a_j \mid d$,
  * $a_i \mid d - a_j$,
  * $a_j \mid d - a_i$,
  * *there exists $b_j \geqslant 2$ such that $a_i \mid d - a_j b_j$ and $d - a_j b_j \geqslant 0$.*
- *for pairwise different indices i, j, k, l satisfying $k < l$, both of the following hold:*
  * *one of the following holds:*
    · $a_i \mid d - a_k$,
    · $a_j \mid d - a_k$,

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· *there exists* $c_j \geqslant 1$ *such that* $a_i \mid d - a_k - a_j c_j$ and $d - a_k - a_j c_j \geqslant 0$.

∗ *one of the following holds:*

· $a_i \mid d - a_l$,

· $a_j \mid d - a_l$,

· *there exists* $d_j \geqslant 1$ *such that* $a_i \mid d - a_l - a_j d_j$ and $d - a_l - a_j d_j \geqslant 0$.

and one of the following conditions holds:

(type I) $I = a_i + a_j$ *for some* $i \neq j$,

(type II) $I = a_i + (a_j/2)$ *for some* $i \neq j$,

(type III) $(a_0, a_1, a_2, a_3, d) = (I - k, I + k, a, a + k, 2a + I + k)$, *where* $1 \leqslant k < I$ *and* $a \geqslant I + k$,

(type IV) (a_0, a_1, a_2, a_3, d) *belongs to one of the infinite series listed in Table A1 or to the sporadic set given in Table A2 (Appendix A).*

Throughout this paper, condition (x) refers to condition (x) of Theorem 2.2. Type X refers to type X of Theorem 2.2. Tuples of type IV are simply read from the tables, so the rest of the paper is concerned with types I–III.

Type III of Theorem 2.2 was changed from $0 \leqslant k < I$ in [9, Theorem 2.2] to $1 \leqslant k < I$, as the only tuple from the case $k = 0$ already exists in the tables for type IV.

The aim of this paper is to find all the tuples (a_0, a_1, a_2, a_3, d) such that there exists a quasi-smooth well-formed hypersurface, given a fixed index I. At first, it is easier to deal with a superset of such tuples, namely, when only conditions (i)–(iv) are satisfied. Proposition 2.9 shows that for tuples of type I–III, this weaker set of conditions suffices; that is, for every such tuple, there exists a quasi-smooth well-formed hypersurface.

Below, we consider a certain subset of the tuples of types I–III. If a tuple belongs to this subset, we assign a unique *class number* and an infinite series to it.

Definition 2.3. Given an index I, an ordered tuple (a_0, a_1, a_2, a_3, d) is assigned a unique *class number* if it satisfies one of the following:

- Class 1: $I = a_0 + a_1$.
- Class 2: $I = a_0 + a_2$ and $I > a_0 + a_1$.
- Class 3: $I = a_1 + a_2$ and $I > a_0 + a_2$.
- Class 4: $I = a_0 + (a_1/2)$ and $a_3 = (a_1/2) + a_2$.
- Class 5: $I = (a_0/2) + a_1$ and $I > a_0 + (a_1/2)$ and $a_3 = (a_0/2) + a_2$.
- Class 6: The tuple is of type III, that is, it satisfies $(a_0, a_1, a_2, a_3, d) = (I - k, I + k, a, a + k, 2a + I + k)$, where $1 \leqslant k < I$ and $a \geqslant I + k$.

Definition 2.4. Infinite series for tuples of classes 1–6.

- For class 1 tuples, the corresponding series is $(a_0, a_1, a_2 + xm, a_3 + ym, a_2 + a_3 + (x + y)m)$, where $m = \operatorname{lcm}(a_0, a_1)$ and $x, y \in \mathbb{Z}$.
- For class 2 tuples, the corresponding series is $(a_0, a_1, a_2, a_3 + xm, a_1 + a_3 + xm)$, where $m = \operatorname{lcm}(a_0, a_1, a_2)$ and $x \in \mathbb{Z}$.
- For class 3 tuples, the corresponding series is $(a_0, a_1, a_2, a_3 + xm, a_0 + a_3 + xm)$, where $m = \operatorname{lcm}(a_0, a_1, a_2)$ and $x \in \mathbb{Z}$.
- For class 4 tuples, the corresponding series is $(a_0, a_1, a_2 + xm, (a_1/2) + a_2 + xm, a_1 + 2a_2 + 2xm)$, where $m = \operatorname{lcm}(a_0, (a_1/2))$ and $x \in \mathbb{Z}$.
- For class 5 tuples, the corresponding series is $(a_0, a_1, a_2 + xm, (a_0/2) + a_2 + xm, a_0 + 2a_2 + 2xm)$, where $m = \operatorname{lcm}((a_0/2), a_1)$ and $x \in \mathbb{Z}$.
- For class 6 tuples, the corresponding series is $(a_0, a_1, a_2 + xm, a_3 + xm, d + 2xm) = (I - k, I + k, a + xm, a + k + xm, 2a + I + k + 2xm)$, where $m = \operatorname{lcm}(a_0, a_1, k) = \operatorname{lcm}(I - k, I + k, k)$ and $x \in \mathbb{Z}$.

The parameters x and y are bounded below such that the tuple is ordered.

By an infinite series $(a_0, a_1, a_2 + xm, a_3 + ym, a_2 + a_3 + (x + y)m)$ for a class 1 tuple, we mean the set $\{(a_0, a_1, a_2 + xm, a_3 + ym, a_2 + a_3 + (x + y)m) \mid x, y \in \mathbb{Z}\}$ where all the tuples (b_0, b_1, b_2, b_3, e) in the set are ordered, that is, $b_0 \leqslant b_1 \leqslant b_2 \leqslant b_3 \leqslant e$. Analogously for classes 2–6.

Below, we prove Proposition 2.8, which implies that every tuple of type I–III for which there exists a quasi-smooth well-formed hypersurface is of some class 1–6. In fact, we show more: every tuple of type I–III satisfying conditions (i)–(iv) is of some class 1–6. First, we state three lemmas, the proofs of which are in § 3.

Lemma 2.5. *Let (a_0, a_1, a_2, a_3, d) be an ordered tuple that satisfies conditions (i)–(iv) and $I = a_t + (a_u/2)$, where one of t and u is equal to 2 and the other is less than 2. Then, either $I = a_i + a_j$ for some $i < j \leqslant 2$ or $I = a_0 + (a_1/2)$ or $I = (a_0/2) + a_1$.*

Lemma 2.6. *Let (a_0, a_1, a_2, a_3, d) be an ordered tuple that satisfies conditions (i)–(iv) and $I = a_t + a_3$ or $I = (a_t/2) + a_3$ or $I = a_t + (a_3/2)$, where $t \in \{0, 1, 2\}$. Then, either $I = a_i + a_j$ for some $i < j \leqslant 2$ or $I = a_0 + (a_1/2)$ or $I = (a_0/2) + a_1$.*

Lemma 2.7. *Let (a_0, a_1, a_2, a_3, d) be an ordered tuple that satisfies conditions (i)–(iv) and $I = a_t + (a_u/2)$, where $t, u \in \{0, 1\}$ and $t \neq u$. Then $a_3 = (a_u/2) + a_2$.*

Proposition 2.8. *An ordered tuple of type I, II or III satisfying conditions (i)–(iv) is of some class 1–6.*

Proof. Follows directly from the three lemmas above. □

Using Proposition 2.8, we can show the following.

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Proposition 2.9. *Every tuple (a_0, a_1, a_2, a_3, d) of type I, II or III satisfying conditions (i)–(iv) also satisfies conditions (v) and (vi); that is, there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$.*

Using the last two propositions, we can prove the main theorem.

Theorem 2.10. *If an ordered tuple (b_0, b_1, b_2, b_3, e) is of type I, II or III and satisfies conditions (i)–(iv), then for every ordered tuple (a_0, a_1, a_2, a_3, d) in its corresponding infinite series given in Definition 2.4, there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$.*

The proofs of Proposition 2.9 and Theorem 2.10 are in § 3.

Proposition 2.8 implies that all the tuples for which there exists a quasi-smooth well-formed hypersurface lie in the infinite series given in Definition 2.4 or belong to Tables A1 or A2. Theorem 2.10 says that we need to check only one tuple for conditions (i)–(iv) to see whether all the tuples in the series are such that there exists a quasi-smooth well-formed hypersurface. Because there are only finitely many infinite series for any index I (by the argument given below Theorem 1.8), this provides an effective algorithm to classify the hypersurfaces for a fixed I.

3. Proofs

To prove Lemmas 2.5 and 2.6, we use condition (iv) of Theorem 2.2, choosing $i = 3$ to express the weight a_3 in terms of a_0, a_1 and a_2. Then, we try to express the index $I = a_0 + a_1 + a_2 + a_3 - d$ in terms of the weights as in the statement of the lemma. If unsuccessful, we use condition (iv) again, choosing $i = 2$ to express a_2 in terms of a_0 and a_1. The weights a_0, a_1, a_2, a_3 are ordered integers with $1 \leqslant a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$.

Lemma 2.5. *Let (a_0, a_1, a_2, a_3, d) be an ordered tuple that satisfies conditions (i)–(iv) and $I = a_t + (a_u/2)$, where one of t and u is equal to 2 and the other is less than 2. Then, either $I = a_i + a_j$ for some $i < j \leqslant 2$ or $I = a_0 + (a_1/2)$ or $I = (a_0/2) + a_1$.*

Proof. Let us define the index v with

$$\{t, u, v\} = \{0, 1, 2\}$$
$$v \in \{0, 1\}.$$

We find

$$d = a_v + \frac{a_u}{2} + a_3.$$

Using condition (iv) with $i = 3$, we find $a_3 \mid d - a_j$, giving $a_3 \mid a_v + (a_u/2) - a_j$, where $j \in \{t, u, v, 3\}$. Noting that $-a_3 < a_v + (a_u/2) - a_j < 2a_3$, we can define x such that

$$x \in \{0, a_3\}$$
$$x = a_v + \frac{a_u}{2} - a_j.$$

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The cases $(j,x) \in \{(v,0),(v,a_3),(u,a_3),(3,a_3)\}$ easily give contradictions. If $(j,x) = (u,0)$, then $(a_u/2) = a_v$ and $I = a_t + a_v$. If $(j,x) = (t,0)$, then $a_t = a_v + (a_u/2)$ and $I = a_u + a_v$. It is left to consider $(j,x) \in \{(3,0),(t,a_3)\}$. For both of these, we can define y such that

$$\begin{aligned} y &\in \{0, a_t\} \\ a_3 &= \frac{a_u}{2} + a_v - y \\ d &= a_u + 2a_v - y \\ I &= a_t + \frac{a_u}{2}. \end{aligned}$$

Using condition (iv) with $i = 2$, we find $a_2 \mid a_u + 2a_v - y - a_k$ where $k \in \{t,u,v,3\}$. It is easy to see that $a_u + 2a_v - y - a_k \in \{a_2, 2a_2\}$. We consider these cases separately.

First, we consider $a_1 = a_2$. If $\{t,u\} = \{0,2\}$, then $I = a_0 + (a_1/2)$ or $I = (a_0/2) + a_1$. This leaves the case $\{t,u\} = \{1,2\}$. We find

$$\begin{aligned} y &\in \{0, a_1\} \\ a_3 &= \frac{a_1}{2} + a_0 - y \\ d &= a_1 + 2a_0 - y \\ I &= \frac{3}{2}a_1. \end{aligned}$$

Since $I = (3/2)a_1$, we find $2 \mid a_1$. Using condition (i) and $a_1 = a_2$, we find $a_1 \mid 2a_0$. Since $(a_1/2) \mid a_0, a_1, a_2$, condition (ii) implies $(a_0,a_1,a_2) = (1,2,2)$. We find $a_3 = 2$, and condition (ii) gives us a contradiction.

Next, we consider $a_u + 2a_v - y - a_k = 2a_2$ and $a_1 < a_2$. By definition $v \in \{0,1\}$. Since $v = 0$ gives a contradiction, we find $v = 1$. Similarly, $k \geqslant 1$ gives a contradiction. We find $(k,t,v,u) = (0,0,1,2)$, giving

$$\begin{aligned} y &\in \{0, a_0\} \\ a_2 &= 2a_1 - a_0 - y \\ a_3 &= -\frac{a_0}{2} + 2a_1 - \frac{3}{2}y \\ d &= -a_0 + 4a_1 - 2y \\ I &= \frac{a_0}{2} + a_1 - \frac{y}{2}. \end{aligned}$$

If $y = 0$, then $I = (a_0/2) + a_1$. This leaves the case $y = a_0$. We find $a_3 = 2a_1 - 2a_0 = a_2$ and $d = 4a_1 - 3a_0$. Condition (i) implies $a_2 \mid a_0$, giving $a_1 = a_2$, a contradiction.

Finally, we consider $a_u + 2a_v - y - a_k = a_2$ and $a_1 < a_2$. We find $2a_3 + y = a_2 + a_k$, giving $y = 0$ and $a_k = a_2 = a_3$. Therefore

$$a_3 = a_2 = \frac{a_u}{2} + a_v$$
$$d = a_u + 2a_v$$
$$I = a_t + \frac{a_u}{2}.$$

If $t = 2$, then $I = a_u + a_v$. This leaves the case $u = 2$. We find $a_2 = 2a_v$. Since $a_v \mid a_v, a_2, a_3$, condition (ii) implies $(a_0, a_1, a_2, a_3) = (1, 1, 2, 2)$. We find $I = a_0 + a_1$. □

Lemma 2.6. *Let (a_0, a_1, a_2, a_3, d) be an ordered tuple that satisfies conditions (i)–(iv) and $I = a_t + a_3$ or $I = (a_t/2) + a_3$ or $I = a_t + (a_3/2)$, where $t \in \{0, 1, 2\}$. Then, either $I = a_i + a_j$ for some $i < j \leqslant 2$ or $I = a_0 + (a_1/2)$ or $I = (a_0/2) + a_1$.*

Proof. We define u, v with

$$\{t, u, v\} = \{0, 1, 2\}$$
$$u < v.$$

First, let us consider $I = a_t + a_3$. We find

$$d = a_u + a_v.$$

Condition (iv) gives $a_3 \mid d - a_j$ where $j \in \{0, 1, 2, 3\}$. Condition (iii) gives $d > a_3$, and we find

$$d - a_j = a_3.$$

This implies $I = a_t + a_u + a_v - a_j$. If $j \in \{0, 1, 2\}$, we find $I = a_p + a_q$ where $p < q \leqslant 2$, as required. This leaves the case $j = 3$, giving $a_u + a_v = 2a_3$. We find $a_u = a_v = a_3$, and condition (ii) implies $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$. So, $I = a_0 + a_1$.

Next, let us consider $I = (a_t/2) + a_3$. We find

$$d = a_u + a_v + \frac{a_t}{2}.$$

Condition (iv) gives us $a_3 \mid a_u + a_v + (a_t/2) - a_j$, where $j \in \{t, u, v, 3\}$. We find

$$a_3 = a_u + a_v + \frac{a_t}{2} - a_j.$$

This implies $I = a_t + a_u + a_v - a_j$. So, it suffices to consider $j = 3$, giving

$$d = 2a_3.$$

Using condition (iv), we find $a_2 \mid a_u + a_v + (a_t/2) - a_k$, where $k \in \{t, u, v, 3\}$, giving $a_2 = a_u + a_v + (a_t/2) - a_k$. Since $a_2 \leqslant a_3$, we find $a_2 = a_3$, giving $d = a_2 + a_3$. From the definition of the index, we find $I = a_0 + a_1$.

Finally, we consider $I = a_t + (a_3/2)$. We find

$$d = a_u + a_v + \frac{a_3}{2}.$$

Condition (iv) gives $a_3 \mid a_u + a_v + (a_3/2) - a_j$, where $j \in \{t, u, v, 3\}$. So, we can define x such that

$$x \in \left\{\frac{a_3}{2}, \frac{3a_3}{2}\right\}$$
$$x = a_u + a_v - a_j.$$

If $(j, x) \in \{(t, (a_3/2)), (u, (a_3/2)), (v, (a_3/2))\}$, then $I = a_p + a_q$, where $p < q \leqslant 2$, as required. The cases $(j, x) \in \{(u, (3a_3/2)), (v, (3a_3/2)), (3, (3a_3/2))\}$ give contradictions. This leaves the cases $(j, x) \in \{(3, (a_3/2)), (t, (3a_3/2))\}$. For both of these, we can define y such that

$$y \in \{0, a_t\}$$
$$a_3 = \frac{2a_u + 2a_v - 2y}{3}$$
$$d = \frac{4a_u + 4a_v - y}{3}.$$

Condition (iv) implies $a_2 \mid (4a_u + 4a_v - y - 3a_k/3)$, where $k \in \{0, 1, 2, 3\}$. We can define z such that

$$z \in \{a_2, 2a_2\}$$
$$z = \frac{4a_u + 4a_v - y - 3a_k}{3}.$$

We consider the values of z separately.

- If $z = a_2$, then $d = 2a_3 + y = a_k + a_2$. We find $y = 0$ and $a_2 = a_3$, giving $d = a_2 + a_3$. From the definition of the index, we find $I = a_0 + a_1$.
- If $z = 2a_2$, then $a_2 \geqslant a_u, a_v, a_t$ implies $k = 0$ and $k \neq u$. By definition $u < v$, so $(k, t, u, v) = (0, 0, 1, 2)$. We find

$$y \in \{0, a_0\}$$
$$a_2 = \frac{-3a_0 + 4a_1 - y}{2}$$
$$a_3 = -a_0 + 2a_1 - y$$
$$d = -2a_0 + 4a_1 - y$$
$$I = \frac{a_0}{2} + a_1 - \frac{y}{2}.$$

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If $y = 0$, then $I = (a_0/2) + a_1$. If $y = a_0$, we find

$$a_2 = a_3 = 2a_1 - 2a_0$$
$$d = 4a_1 - 3a_0.$$

Condition (i) implies $a_3 \mid a_0$, and condition (ii) implies $a_3 = 1$, which is a contradiction since $2 \mid a_3$. □

Lemma 2.7. *Let (a_0, a_1, a_2, a_3, d) be an ordered tuple that satisfies conditions (i)–(iv) and $I = a_t + (a_u/2)$, where $t, u \in \{0, 1\}$ and $t \neq u$. Then $a_3 = (a_u/2) + a_2$.*

Proof. We have

$$d = \frac{a_u}{2} + a_2 + a_3.$$

Using condition (iv), we find $a_3 \mid (a_u/2) + a_2 - a_j$, where $j \in \{0, 1, 2, 3\}$, giving $(a_u/2) + a_2 - a_j \in \{0, a_3\}$. If $(a_u/2) + a_2 - a_j = 0$, we find $a_3 = (a_u/2) + a_2$, as required. This leaves the case $(a_u/2) + a_2 - a_j = a_3$, giving $u = 1$ and $j = 0$. We have

$$a_3 = -a_0 + \frac{a_1}{2} + a_2$$
$$d = -a_0 + a_1 + 2a_2.$$

Using condition (iv), we find $a_2 \mid -a_0 + a_1 - a_k$, where $k \in \{0, 1, 2, 3\}$, giving $-a_0 + a_1 - a_k \in \{-a_2, 0\}$. Since $a_3 \geqslant a_k$, the case $-a_0 + a_1 - a_k = -a_2$ gives a contradiction. Therefore, we find $a_1 = a_0 + a_k$, giving

$$a_1 = 2a_0$$
$$a_3 = a_2$$
$$d = a_0 + 2a_2.$$

Condition (i) implies $a_2 \mid a_0$. Condition (ii) implies $a_2 = 1$, which is a contradiction since $2 \mid a_1$. □

Proposition 2.9. *Every tuple (a_0, a_1, a_2, a_3, d) of type I, II or III satisfying conditions (i)–(iv) also satisfies conditions (v) and (vi); that is, there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d < a_0 + a_1 + a_2 + a_3$ and index $I = a_0 + a_1 + a_2 + a_3 - d$.*

Proof. By Proposition 2.8, it suffices to consider tuples of classes 1–6.

Classes 1–3: $I = a_t + a_u$, where $t < u \leqslant 2$. We can define v such that

$$\{t, u, v\} = \{0, 1, 2\}$$
$$d = a_v + a_3.$$

If $(i, j) = (v, 3)$, then conditions (v) and (vi) are satisfied, since $a_3 \mid d - a_v$.

If $(i, j) = (t, u)$, then condition (iv) implies $a_u \mid d - a_k$ where $k \in \{t, u, v, 3\}$. If $k \in \{t, u\}$, conditions (v) and (vi) hold. If $k \in \{v, 3\}$, then $a_u \mid a_v$ or $a_u \mid a_3$. Condition (i) implies $a_u \mid a_v, a_3$ and (ii) implies $a_u = 1$. Therefore $a_u \mid d$, so (v) and (vi) hold.

If $(i, j) \neq (t, u), (v, 3)$, then either $\gcd(a_i, a_j) \mid a_v$ or $\gcd(a_i, a_j) \mid a_3$. Condition (i) implies $\gcd(a_i, a_j) \mid a_v, a_3$, and (ii) implies $\gcd(a_i, a_j) = 1$, so (v) is satisfied.

It is left to consider condition (vi) for pairs $(i, j) \neq (t, u), (v, 3)$. Note that the order of i, j is not important in (vi). Similarly, the order of k, l is not important in the second part of (vi). So, it suffices to consider

$$i \in \{t, u\}$$
$$j \in \{v, 3\}.$$

Using condition (iv), we find

$$a_i \mid d - a_k$$

where $k \in \{0, 1, 2, 3\}$. If $k \in \{i, j\}$, then (vi) is satisfied. If $k \in \{v, 3\}$, then $a_i \mid a_v$ or $a_i \mid a_3$, and as before we find $a_i \mid d$, so (vi) is satisfied. This leaves the case

$$k \in \{t, u\}$$
$$k \neq i.$$

We define l with

$$l \in \{v, 3\}$$
$$l \neq j.$$

The indices i, j, k, l are pairwise different, and $a_i \mid d - a_k$ and $a_j \mid d - a_l$. So, by the second part of (vi), condition (vi) holds.

Classes 4–5: $I = a_t + (a_u/2)$, where $t, u \in \{0, 1\}$ with $t \neq u$. By Lemma 2.7, we have

$$a_3 = \frac{a_u}{2} + a_2$$
$$d = 2a_3 = a_u + 2a_2.$$

If $j = 3$, then $a_j \mid d$, so conditions (v) and (vi) are satisfied.

If $(i, j) = (u, 2)$, then $a_2 \mid d - a_u$, so (v) and (vi) hold.

If $(i, j) = (t, 2)$, then (iv) implies $a_t \mid d - a_p$ where $p \in \{t, u, 2, 3\}$. If $p \in \{t, 2, 3\}$, then (v) and (vi) are satisfied. If $p = u$, then $a_t \mid 2a_2$. Using conditions (i) and (ii), we find $a_t \in \{1, 2\}$, giving $a_t \mid d$. So, (v) and (vi) are satisfied.

If $(i, j) = (t, u)$, ignoring the order of i and j, then (iv) implies $a_u \mid d - a_q$, where $q \in \{t, u, 2, 3\}$. If $q \in \{t, u, 3\}$, then (v) and (vi) are satisfied. If $q = 2$, then $a_u \mid a_2$, giving $a_u \mid d$, so (v) and (vi) are satisfied.

Class 6: $(a_0, a_1, a_2, a_3, d) = (I - k, I + k, a, a + k, 2a + I + k)$, where $1 \leqslant k < I$ and $a \geqslant I + k$. We have

$$d = a_0 + 2a_3 = a_1 + 2a_2.$$

If $(i, j) \in \{(0, 3), (1, 2)\}$, then (v) and (vi) are satisfied.

If $(i,j)=(2,3)$, we find $a_3 \mid d-a_0$ and $a_2 \mid d-a_1$. By denoting $(i,j,k,l)=(2,3,0,1)$ in the second part of (vi), condition (vi) holds. For condition (v), note that (i) implies $\gcd(a_2,a_3) \mid a_0,a_1$; therefore $\gcd(a_2,a_3)=1$, and (v) is satisfied.

If $(i,j) \in \{(0,1),(0,2),(1,3)\}$, then (iv) implies

$$a_0 \mid d-a_p$$

where $p \in \{0,1,2,3\}$. We show that for any p, we have either $a_0 \mid d$ or $a_0 \mid d-a_2$.

- If $p \in \{0,3\}$, then $a_0 \mid d$.
- If $p=1$, then (i) and (ii) imply $a_0 \in \{1,2\}$, giving $a_0 \mid d$.
- If $p=2$, then $a_0 \mid d-a_2$.

Similarly, we can show that either $a_1 \mid d$ or $a_1 \mid d-a_3$. Therefore, conditions (v) and (vi) are satisfied for $(i,j) \in \{(0,2),(1,3)\}$. Also, conditions (v) and (vi) are satisfied for $(i,j)=(0,1)$ if either $a_0 \mid d$ or $a_1 \mid d$.

It is left to consider $(i,j)=(0,1)$ with

$$a_0 \mid d-a_2$$
$$a_1 \mid d-a_3.$$

Choosing $(i,j,k,l)=(0,1,2,3)$, we see by the second part of condition (vi) that (vi) is satisfied. Next, we check (v). From the above, we see that

$$\gcd(a_0,a_1) \mid d-a_2$$
$$\gcd(a_0,a_1) \mid d-a_3.$$

Therefore, $\gcd(a_0,a_1) \mid a_2,a_3$, and condition (ii) gives $\gcd(a_0,a_1)=1$. So, (v) is satisfied. □

Theorem 2.10. *If an ordered tuple (b_0,b_1,b_2,b_3,e) is of type I, II or III and satisfies conditions (i)–(iv), then for every ordered tuple (a_0,a_1,a_2,a_3,d) in its corresponding infinite series given in Definition 2.4, there exists a quasi-smooth well-formed hypersurface in $\mathbb{P}(a_0,a_1,a_2,a_3)$ of degree $d < a_0+a_1+a_2+a_3$ and index $I=a_0+a_1+a_2+a_3-d$.*

Proof. By Proposition 2.9, it is sufficient to show the tuple (a_0,a_1,a_2,a_3,d) satisfies conditions (i)–(iv). Note that (iii) clearly holds for all classes 1–6. By Proposition 2.8, it suffices to consider tuples of classes 1–6.

Class 1: $I=a_0+a_1$, giving

$$d=a_2+a_3.$$

From Definition 2.4, there exist x and y such that

$$m=\operatorname{lcm}(a_0,a_1)$$
$$(a_0,a_1,a_2,a_3,d)=(b_0,b_1,b_2+xm,b_3+ym,e+(x+y)m)$$

and the tuple is ordered, that is, $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant d$.

First, we check condition (i). Since $i < j$, we find $i \in \{0, 1, 2\}$.

- If $i \in \{0, 1\}$, then $a_i = b_i$ and $a_i \mid m$, giving $\gcd(a_i, a_j) = \gcd(b_i, b_j)$. By assumption, the tuple (b_0, b_1, b_2, b_3, e) satisfies conditions (i)–(iv), so $\gcd(b_i, b_j) \mid e$, giving $\gcd(a_i, a_j) \mid d$.
- For $(i, j) = (2, 3)$, we note $d = a_2 + a_3$.

Next, we check condition (ii). Since $i < j < k$, we find $i \in \{0, 1\}$, giving $a_i = b_i$ and $a_i \mid m$. Therefore, $\gcd(a_i, a_j, a_k) = \gcd(b_i, b_j, b_k)$, and by assumption $\gcd(b_i, b_j, b_k) = 1$.

Finally, we check condition (iv).

- If $i \in \{0, 1\}$, then $a_i = b_i$ and $a_i \mid m$. By assumption, there exists j such that $b_i \mid e - b_j$, giving $a_i \mid d - a_j$.
- If $i \in \{2, 3\}$, we note $d = a_2 + a_3$.

Classes 2–3: $I = a_t + a_2$, where $t \in \{0, 1\}$. We can define u such that

$$\{t, u\} = \{0, 1\}$$
$$d = a_u + a_3.$$

From Definition 2.4, there exist x and y such that

$$m = \operatorname{lcm}(a_0, a_1, a_2)$$
$$(a_0, a_1, a_2, a_3, d) = (b_0, b_1, b_2, b_3 + xm, e + xm).$$

First, we check condition (i). Since $i < j$, we find $i \in \{0, 1, 2\}$, giving $a_i = b_i$ and $a_i \mid m$. We have $\gcd(a_i, a_j) = \gcd(b_i, b_j)$. By assumption $\gcd(b_i, b_j) \mid e$, giving $\gcd(a_i, a_j) \mid d$.

For condition (ii), we similarly find $\gcd(a_i, a_j, a_k) = \gcd(b_i, b_j, b_k) = 1$.

Finally, we consider condition (iv). It holds for $i = 3$, since $d = a_u + a_3$. If $i \neq 3$, then by assumption there exists j such that $b_i \mid e - b_j$. Since $a_i = b_i$ and $a_i \mid m$, we find $a_i \mid d - a_j$.

Classes 4–5: $I = a_t + a_u/2$, where t and u are such that

$$\{t, u\} = \{0, 1\}.$$

From Definition 2.4, there exist x and y such that

$$a_3 = \frac{a_u}{2} + a_2$$
$$d = 2a_3 = a_u + 2a_2$$
$$m = \operatorname{lcm}\left(a_t, \frac{a_u}{2}\right)$$
$$(a_0, a_1, a_2, a_3, d) = (b_0, b_1, b_2 + xm, b_3 + xm, e + 2xm).$$

First, we check condition (i). We have $i < j$.

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- If $j = 3$, we note $d = 2a_3$.
- If $(i, j) = (u, 2)$, we note $d = a_u + 2a_2$.
- If $i = t$ or $j = t$, then $a_t = b_t$ and $a_t \mid m$, giving $\gcd(a_i, a_j) = \gcd(b_i, b_j)$. By assumption $\gcd(b_i, b_j) \mid e$, giving $\gcd(a_i, a_j) \mid d$.

Next, we check condition (ii).

- If $i = t$ or $j = t$, we find $a_t = b_t$ and $a_t \mid m$, giving $\gcd(a_i, a_j, a_k) = \gcd(b_i, b_j, b_k) = 1$.
- If $(i, j, k) = (u, 2, 3)$, then using $a_3 = (a_u/2) + a_2$, we find $\gcd(a_u, a_2, a_3) = \gcd((a_u/2), a_2)$. We have $a_u = b_u$ and $(a_u/2) \mid m$, therefore $\gcd((a_u/2), a_2) = \gcd((b_u/2), b_2) = \gcd(b_u, b_2, b_3)$. By assumption $\gcd(b_u, b_2, b_3) = 1$, giving $\gcd(a_u, a_2, a_3) = 1$.

Finally, we check condition (iv).

- If $i = 3$, we note $d = 2a_3$.
- If $i = 2$, we note $d = a_u + 2a_2$.
- If $i = t$, then $a_i = b_i$ and $a_i \mid m$. By assumption there exists j such that $b_i \mid e - b_j$, therefore $a_i \mid d - a_j$.
- If $i = u$, then

$$\begin{aligned} a_u &= b_u \\ a_u &\mid 2m. \end{aligned}$$

  From Definition 2.4

$$e = 2b_3 = b_u + 2b_2.$$

  By assumption there exists j such that $b_u \mid e - b_j$. We show there exists k such that $a_u \mid e - a_k$.

  - If $j \in \{0, 1\}$, then $a_j = b_j$, giving $a_u \mid e - a_j$.
  - If $j \in \{2, 3\}$, then either $b_u \mid b_2$ or $b_u \mid b_3$, giving $b_u \mid e$. This implies $a_u \mid e - a_u$.

  Now, since $a_u \mid 2m$, we find $a_u \mid d - a_k$. So, condition (iv) is satisfied.

Class 6: there exist a and k such that

$$\begin{aligned} 1 &\leqslant k < I \\ a &\geqslant I + k \\ (a_0, a_1, a_2, a_3, d) &= (I - k, I + k, a, a + k, 2a + I + k). \end{aligned}$$

We have

$$d = a_0 + 2a_3 = a_1 + 2a_2.$$

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From Definition 2.4, we have

$$m = \mathrm{lcm}(a_0, a_1, k)$$
$$(a_0, a_1, a_2, a_3, d) = (b_0, b_1, b_2 + xm, b_3 + xm, e + 2xm).$$

First, we check condition (i). Since $i < j$, we have $i \in \{0, 1, 2\}$.

- If $i \in \{0, 1\}$, then $a_i = b_i$ and $a_i \mid m$, giving $\gcd(a_i, a_j) = \gcd(b_i, b_j)$. By assumption $\gcd(b_i, b_j) \mid e$, giving $\gcd(a_i, a_j) \mid d$.
- If $(i, j) = (2, 3)$, then $\gcd(a_2, a_3) = \gcd(a_2, k)$. Since $k \mid m$, we find $\gcd(a_2, k) = \gcd(b_2, k) = \gcd(b_2, b_3)$. By assumption $\gcd(b_2, b_3) \mid e$, giving $\gcd(a_2, a_3) \mid d$.

Next, we check condition (ii). Since $i < j < k$, we have $i \in \{0, 1\}$, giving $a_i = b_i$ and $a_i \mid m$. We find $\gcd(a_i, a_j, a_k) = \gcd(b_i, b_j, b_k) = 1$.

Finally, we check condition (iv).

- If $i = 3$, we note $d = a_0 + 2a_3$.
- If $i = 2$, we note $d = a_1 + 2a_2$.
- If $i \in \{0, 1\}$, then $a_i = b_i$ and $a_i \mid m$. By assumption $b_i \mid e - b_j$, giving $a_i \mid d - a_j$. □

4. Small index cases

In this section, we give the complete lists of quasi-smooth well-formed hypersurfaces for indices $I = 1, 2, \ldots, 6$. The parameters x and y are non-negative integers with $x \leqslant y$. We first list the two-parameter series, then one-parameter series and lastly sporadic cases.

Tables 1 and 2 for indices 1 and 2, respectively, are already known from [**20**, Theorem 8] and [**9**, Corollary 1.13], respectively.

Tables 3–6 for indices 3–6, respectively, are new.

The author has computed the lists for all $I \leqslant 100$. The lists grow as the cube of the index and the computation time grows as the fifth power of I.

Differences from [**9**, Corollary 1.13]. There is a misprint in the list for $I = 2$ in [**9**], namely, the second occurrence of $(3, 4, 6, 7, 18)$ should instead be $(3, 4, 5, 7, 17)$. In the

Table 1. *Index 1.*

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(2, 2x+3, 2x+3, 4x+5)$	$8x+12$	$(1, 1, 1, 1)$	3	$(1, 1, 1, 2)$	4
$(1, 1, 2, 3)$	6	$(1, 2, 3, 5)$	10	$(1, 3, 5, 7)$	15
$(1, 3, 5, 8)$	16	$(2, 3, 5, 9)$	18	$(3, 3, 5, 5)$	15
$(3, 5, 7, 11)$	25	$(3, 5, 7, 14)$	28	$(3, 5, 11, 18)$	36
$(5, 14, 17, 21)$	56	$(5, 19, 27, 31)$	81	$(5, 19, 27, 50)$	100
$(7, 11, 27, 37)$	81	$(7, 11, 27, 44)$	88	$(9, 15, 17, 20)$	60
$(9, 15, 23, 23)$	69	$(11, 29, 39, 49)$	127	$(11, 49, 69, 128)$	256
$(13, 23, 35, 57)$	127	$(13, 35, 81, 128)$	256		

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list below (Table 2), the tuple $(3, 4, 5, 7, 17)$ is contained in the series $(3, 3x + 4, 3x + 5, 6x + 7, 12x + 17)$, which has been extended to include $x = 0$. The tuple $(1, 1, 2, 2, 4)$ is contained in the series $(1, 1, x + 1, y + 1, x + y + 2)$.

Table 2. *Index 2.*

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 1, x + 1, y + 1)$	$x + y + 2$	$(1, 2, x + 2, x + 3)$	$2x + 6$
$(1, 3, 3x + 3, 3x + 4)$	$6x + 9$	$(1, 3, 3x + 4, 3x + 5)$	$6x + 11$
$(3, 3x + 3, 3x + 4, 3x + 4)$	$9x + 12$	$(3, 3x + 4, 3x + 5, 3x + 5)$	$9x + 15$
$(3, 3x + 4, 3x + 5, 6x + 7)$	$12x + 17$	$(3, 3x + 4, 6x + 7, 9x + 9)$	$18x + 21$
$(3, 3x + 4, 6x + 7, 9x + 12)$	$18x + 24$	$(4, 2x + 5, 2x + 5, 4x + 8)$	$8x + 20$
$(4, 2x + 5, 4x + 10, 6x + 13)$	$12x + 30$	$(1, 3, 4, 6)$	12
$(1, 4, 5, 7)$	15	$(1, 4, 5, 8)$	16
$(1, 4, 6, 9)$	18	$(1, 5, 7, 11)$	22
$(1, 6, 9, 13)$	27	$(1, 6, 10, 15)$	30
$(1, 7, 12, 18)$	36	$(1, 8, 13, 20)$	40
$(1, 9, 15, 22)$	45	$(2, 3, 4, 5)$	12
$(2, 3, 4, 7)$	14	$(3, 4, 5, 10)$	20
$(3, 4, 6, 7)$	18	$(3, 4, 10, 15)$	30
$(5, 13, 19, 22)$	57	$(5, 13, 19, 35)$	70
$(6, 9, 10, 13)$	36	$(7, 8, 19, 25)$	57
$(7, 8, 19, 32)$	64	$(9, 12, 13, 16)$	48
$(9, 12, 19, 19)$	57	$(9, 19, 24, 31)$	81
$(10, 19, 35, 43)$	105	$(11, 21, 28, 47)$	105
$(11, 25, 32, 41)$	107	$(11, 25, 34, 43)$	111
$(11, 43, 61, 113)$	226	$(13, 18, 45, 61)$	135
$(13, 20, 29, 47)$	107	$(13, 20, 31, 49)$	111
$(13, 31, 71, 113)$	226	$(14, 17, 29, 41)$	99

Table 3. *Index 3.*

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 2, 2x + 3, 2y + 3)$	$2(x + y) + 6$	$(1, 1, 2, 2x + 3)$	$2x + 4$
$(1, 5, 10x + 5, 10x + 7)$	$20x + 15$	$(1, 5, 10x + 7, 10x + 9)$	$20x + 19$
$(1, 7, 9, 13)$	27	$(1, 7, 9, 14)$	28
$(1, 9, 13, 20)$	40	$(1, 13, 22, 33)$	66
$(1, 14, 23, 35)$	70	$(1, 15, 25, 37)$	75
$(5, 7, 11, 13)$	33	$(5, 7, 11, 20)$	40
$(11, 21, 29, 37)$	95	$(11, 37, 53, 98)$	196
$(13, 17, 27, 41)$	95	$(13, 27, 61, 98)$	196
$(15, 19, 43, 74)$	148		

Table 4. *Index 4.*

(a_0, a_1, a_2, a_3)	d
$(1, 3, 3x + 4, 3y + 5)$	$3(x + y) + 9$
$(1, 3, 3x + 5, 3y + 5)$	$3(x + y) + 10$
$(1, 3, 3x + 5, 3y + 7)$	$3(x + y) + 12$
$(2, 2, 2x + 3, 2y + 3)$	$2(x + y) + 6$
$(1, 1, 3, 3x + 5)$	$3x + 6$
$(1, 2, 2, 2x + 3)$	$2x + 4$
$(1, 2, 3, 6x + 4)$	$6x + 6$
$(1, 2, 3, 6x + 5)$	$6x + 7$
$(1, 2, 3, 6x + 7)$	$6x + 9$
$(1, 2, 3, 6x + 8)$	$6x + 10$
$(1, 7, 21x + 7, 21x + 10)$	$42x + 21$
$(1, 7, 21x + 10, 21x + 13)$	$42x + 27$
$(1, 7, 21x + 14, 21x + 17)$	$42x + 35$
$(1, 7, 21x + 17, 21x + 20)$	$42x + 41$
$(2, 3, 3x + 4, 3x + 5)$	$6x + 10$
$(2, 3, 3x + 5, 3x + 6)$	$6x + 12$
$(2, 4, 2x + 5, 2x + 7)$	$4x + 14$
$(2, 6, 6x + 9, 6x + 11)$	$12x + 24$
$(3, 5, 15x + 5, 15x + 6)$	$30x + 15$
$(3, 5, 15x + 10, 15x + 11)$	$30x + 25$
$(3, 5, 15x + 11, 15x + 12)$	$30x + 27$
$(3, 5, 15x + 16, 15x + 17)$	$30x + 37$
$(6, 6x + 9, 6x + 11, 6x + 11)$	$18x + 33$
$(6, 6x + 11, 12x + 20, 18x + 27)$	$36x + 60$
$(6, 6x + 11, 12x + 20, 18x + 33)$	$36x + 66$
$(1, 10, 13, 19)$	39
$(1, 10, 13, 20)$	40
$(1, 13, 19, 29)$	58
$(1, 14, 21, 31)$	63
$(1, 19, 32, 48)$	96
$(1, 20, 33, 50)$	100
$(1, 21, 35, 52)$	105
$(2, 7, 10, 15)$	30
$(2, 9, 12, 17)$	36
$(5, 6, 8, 9)$	24
$(5, 6, 8, 15)$	30
$(9, 11, 12, 17)$	45
$(10, 13, 25, 31)$	75
$(11, 17, 20, 27)$	71
$(11, 17, 24, 31)$	79
$(11, 31, 45, 83)$	166
$(13, 14, 19, 29)$	71
$(13, 14, 23, 33)$	79
$(13, 23, 51, 83)$	166

Table 5. *Index 5.*

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 4, 4x + 5, 4y + 7)$	$4(x + y) + 12$	$(1, 4, 4x + 7, 4y + 9)$	$4(x + y) + 16$
$(2, 3, 6x + 5, 6y + 7)$	$6(x + y) + 12$	$(2, 3, 6x + 7, 6y + 7)$	$6(x + y) + 14$
$(2, 3, 6x + 7, 6y + 11)$	$6(x + y) + 18$	$(1, 1, 4, 4x + 7)$	$4x + 8$
$(1, 2, 3, 6x + 5)$	$6x + 6$	$(1, 2, 3, 6x + 7)$	$6x + 8$
$(1, 3, 4, 12x + 5)$	$12x + 8$	$(1, 3, 4, 12x + 9)$	$12x + 12$
$(1, 3, 4, 12x + 13)$	$12x + 16$	$(1, 9, 36x + 9, 36x + 13)$	$72x + 27$
$(1, 9, 36x + 13, 36x + 17)$	$72x + 35$	$(1, 9, 36x + 27, 36x + 31)$	$72x + 63$
$(1, 9, 36x + 31, 36x + 35)$	$72x + 71$	$(3, 7, 42x + 7, 42x + 9)$	$84x + 21$
$(3, 7, 42x + 23, 42x + 25)$	$84x + 53$	$(3, 7, 42x + 35, 42x + 37)$	$84x + 77$
$(3, 7, 42x + 37, 42x + 39)$	$84x + 81$	$(1, 13, 17, 25)$	51
$(1, 13, 17, 26)$	52	$(1, 17, 25, 38)$	76
$(1, 25, 42, 63)$	126	$(1, 26, 43, 65)$	130
$(1, 27, 45, 67)$	135	$(6, 7, 9, 10)$	27
$(11, 13, 19, 25)$	63	$(11, 25, 37, 68)$	136
$(13, 19, 41, 68)$	136		

Table 6. *Index 6.*

(a_0, a_1, a_2, a_3)	d	(a_0, a_1, a_2, a_3)	d
$(1, 5, 5x + 6, 5y + 9)$	$5(x + y) + 15$	$(1, 5, 5x + 7, 5y + 8)$	$5(x + y) + 15$
$(1, 5, 5x + 7, 5y + 9)$	$5(x + y) + 16$	$(1, 5, 5x + 8, 5y + 8)$	$5(x + y) + 16$
$(1, 5, 5x + 8, 5y + 12)$	$5(x + y) + 20$	$(1, 5, 5x + 9, 5y + 11)$	$5(x + y) + 20$
$(1, 5, 5x + 9, 5y + 12)$	$5(x + y) + 21$	$(2, 4, 4x + 5, 4y + 5)$	$4(x + y) + 10$
$(2, 4, 4x + 5, 4y + 7)$	$4(x + y) + 12$	$(2, 4, 4x + 7, 4y + 7)$	$4(x + y) + 14$
$(2, 4, 4x + 7, 4y + 9)$	$4(x + y) + 16$	$(3, 3, 3x + 4, 3y + 5)$	$3(x + y) + 9$
$(3, 3, 3x + 5, 3y + 7)$	$3(x + y) + 12$	$(1, 1, 5, 5x + 9)$	$5x + 10$
$(1, 2, 4, 4x + 5)$	$4x + 6$	$(1, 2, 4, 4x + 7)$	$4x + 8$
$(1, 2, 5, 10x + 8)$	$10x + 10$	$(1, 2, 5, 10x + 9)$	$10x + 11$
$(1, 2, 5, 10x + 13)$	$10x + 15$	$(1, 2, 5, 10x + 14)$	$10x + 16$
$(1, 3, 3, 3x + 5)$	$3x + 6$	$(1, 3, 5, 15x + 7)$	$15x + 10$
$(1, 3, 5, 15x + 8)$	$15x + 11$	$(1, 3, 5, 15x + 12)$	$15x + 15$
$(1, 3, 5, 15x + 13)$	$15x + 16$	$(1, 3, 5, 15x + 17)$	$15x + 20$
$(1, 3, 5, 15x + 18)$	$15x + 21$	$(1, 4, 5, 20x + 6)$	$20x + 10$
$(1, 4, 5, 20x + 7)$	$20x + 11$	$(1, 4, 5, 20x + 11)$	$20x + 15$
$(1, 4, 5, 20x + 12)$	$20x + 16$	$(1, 4, 5, 20x + 16)$	$20x + 20$
$(1, 4, 5, 20x + 17)$	$20x + 21$	$(1, 4, 5, 20x + 21)$	$20x + 25$
$(1, 4, 5, 20x + 22)$	$20x + 26$	$(1, 11, 55x + 11, 55x + 16)$	$110x + 33$
$(1, 11, 55x + 16, 55x + 21)$	$110x + 43$	$(1, 11, 55x + 22, 55x + 27)$	$110x + 55$
$(1, 11, 55x + 27, 55x + 32)$	$110x + 65$	$(1, 11, 55x + 33, 55x + 38)$	$110x + 77$
$(1, 11, 55x + 38, 55x + 43)$	$110x + 87$	$(1, 11, 55x + 44, 55x + 49)$	$110x + 99$
$(1, 11, 55x + 49, 55x + 54)$	$110x + 109$	$(2, 3, 3, 6x + 4)$	$6x + 6$

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Table 6. *Continued*

$(2,3,3,6x+7)$	$6x+9$	$(2,3,4,12x+5)$	$12x+8$
$(2,3,4,12x+7)$	$12x+10$	$(2,3,4,12x+9)$	$12x+12$
$(2,3,4,12x+11)$	$12x+14$	$(2,3,4,12x+13)$	$12x+16$
$(2,3,4,12x+15)$	$12x+18$	$(2,5,5x+8,5x+9)$	$10x+18$
$(2,5,5x+9,5x+10)$	$10x+20$	$(2,8,4x+9,4x+13)$	$8x+26$
$(2,10,20x+15,20x+19)$	$40x+40$	$(2,10,20x+25,20x+29)$	$40x+60$
$(5,7,35x+8,35x+9)$	$70x+23$	$(5,7,35x+14,35x+15)$	$70x+35$
$(5,7,35x+28,35x+29)$	$70x+63$	$(5,7,35x+29,35x+30)$	$70x+65$
$(8,4x+9,4x+11,4x+13)$	$12x+35$	$(9,3x+11,3x+14,6x+19)$	$12x+47$
$(1,16,21,31)$	63	$(1,16,21,32)$	64
$(1,21,31,47)$	94	$(1,22,33,49)$	99
$(1,31,52,78)$	156	$(1,32,53,80)$	160
$(1,33,55,82)$	165	$(2,13,18,27)$	54
$(2,15,20,29)$	60	$(3,7,8,12)$	24
$(7,10,15,19)$	45	$(11,19,29,53)$	106
$(13,15,31,53)$	106		

Appendix A. Tables

Tables A1 and A2 are taken from [**9**, Appendix B]. They contain one-parameter infinite series and sporadic cases, respectively, of values of $(a_0, a_1, a_2, a_3, d, I)$. The last columns represent the cases in [**33**] from which the sextuples $(a_0, a_1, a_2, a_3, d, I)$ originate (note that sometimes a sextuple $(a_0, a_1, a_2, a_3, d, I)$ originates from several cases in [**33**]). The parameter n is any positive integer.

Differences from [**9**, Appendix B]: the tuple $(3,3,4,4,12)$ with $I=2$ has been removed from Table A2, because it already appears in the series $(3,3n,3n+1,3n+1,9n+3)$ with $I=2$ in Table A1.

Table A1. *Infinite series.*

(a_0,a_1,a_2,a_3)	d	I	Source
$(1,3n-2,4n-3,6n-5)$	$12n-9$	n	VII.2(3)
$(1,3n-2,4n-3,6n-4)$	$12n-8$	n	II.2(2)
$(1,4n-3,6n-5,9n-7)$	$18n-14$	n	VII.3(1)
$(1,6n-5,10n-8,15n-12)$	$30n-24$	n	III.1(4)
$(1,6n-4,10n-7,15n-10)$	$30n-20$	n	III.2(2)
$(1,6n-3,10n-5,15n-8)$	$30n-15$	n	III.2(4)
$(1,8n-2,12n-3,18n-5)$	$36n-9$	$2n$	IV.3(3)
$(2,6n-3,8n-4,12n-7)$	$24n-12$	$2n$	II.2(4)
$(2,6n+1,8n+2,12n+3)$	$24n+6$	$2n+2$	II.2(1)
$(3,6n+1,6n+2,9n+3)$	$18n+6$	$3n+3$	II.2(1)
$(7,28n-18,42n-27,63n-44)$	$126n-81$	$7n-1$	XI.3(14)
$(7,28n-17,42n-29,63n-40)$	$126n-80$	$7n+1$	X.3(1)

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Table A1. *Continued*

$(7, 28n-13, 42n-23, 63n-31)$	$126n-62$	$7n+2$	X.3(1)
$(7, 28n-10, 42n-15, 63n-26)$	$126n-45$	$7n+1$	XI.3(14)
$(7, 28n-9, 42n-17, 63n-22)$	$126n-44$	$7n+3$	X.3(1)
$(7, 28n-6, 42n-9, 63n-17)$	$126n-27$	$7n+2$	XI.3(14)
$(7, 28n-5, 42n-11, 63n-13)$	$126n-26$	$7n+4$	X.3(1)
$(7, 28n-2, 42n-3, 63n-8)$	$126n-9$	$7n+3$	XI.3(14)
$(7, 28n-1, 42n-5, 63n-4)$	$126n-8$	$7n+5$	X.3(1)
$(7, 28n+2, 42n+3, 63n+1)$	$126n+9$	$7n+4$	XI.3(14)
$(7, 28n+3, 42n+1, 63n+5)$	$126n+10$	$7n+6$	X.3(1)
$(7, 28n+6, 42n+9, 63n+10)$	$126n+27$	$7n+5$	XI.3(14)
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	1	II.3(4)
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	2	III.5(1)
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	2	II.5(1)
$(3, 3n+1, 3n+2, 6n+1)$	$12n+5$	2	XVIII.2(2)
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	2	VII.3(2)
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	2	II.2(2)
$(4, 2n+3, 2n+3, 4n+4)$	$8n+12$	2	V.3(4)
$(4, 2n+3, 4n+6, 6n+7)$	$12n+18$	2	XII.3(17)
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	4	III.5(1)
$(6, 6n+5, 12n+8, 18n+9)$	$36n+24$	4	VII.3(2)
$(6, 6n+5, 12n+8, 18n+15)$	$36n+30$	4	IV.3(1)
$(8, 4n+5, 4n+7, 4n+9)$	$12n+23$	6	XIX.2(2)
$(9, 3n+8, 3n+11, 6n+13)$	$12n+35$	6	XIX.2(2)

Table A2. *Sporadic cases.*

(a_0, a_1, a_2, a_3)	d	I	Source	(a_0, a_1, a_2, a_3)	d	I	Source
$(1, 3, 5, 8)$	16	1	VIII.3(5)	$(2, 3, 5, 9)$	18	1	II.2(3)
$(3, 3, 5, 5)$	15	1	I.19	$(3, 5, 7, 11)$	25	1	X.2(3)
$(3, 5, 7, 14)$	28	1	VII.4(4)	$(3, 5, 11, 18)$	36	1	VII.3(1)
$(5, 14, 17, 21)$	56	1	XI.3(8)	$(5, 19, 27, 31)$	81	1	X.3(3)
$(5, 19, 27, 50)$	100	1	VII.3(3)	$(7, 11, 27, 37)$	81	1	X.3(4)
$(7, 11, 27, 44)$	88	1	VII.3(5)	$(9, 15, 17, 20)$	60	1	VII.6(3)
$(9, 15, 23, 23)$	69	1	III.5(1)	$(11, 29, 39, 49)$	127	1	XIX.2(2)
$(11, 49, 69, 128)$	256	1	X.3(1)	$(13, 23, 35, 57)$	127	1	XIX.2(2)
$(13, 35, 81, 128)$	256	1	X.3(2)	$(1, 3, 4, 6)$	12	2	I.3
$(1, 4, 6, 9)$	18	2	IV.3(3)	$(1, 6, 10, 15)$	30	2	I.4
$(2, 3, 4, 7)$	14	2	IX.3(1)	$(3, 4, 5, 10)$	20	2	II.3(2)
$(3, 4, 6, 7)$	18	2	VII.3(10)	$(3, 4, 10, 15)$	30	2	II.2(3)
$(5, 13, 19, 22)$	57	2	X.3(3)	$(5, 13, 19, 35)$	70	2	VII.3(3)
$(6, 9, 10, 13)$	36	2	VII.3(8)	$(7, 8, 19, 25)$	57	2	X.3(4)
$(7, 8, 19, 32)$	64	2	VII.3(3)	$(9, 12, 13, 16)$	48	2	VII.6(2)
$(9, 12, 19, 19)$	57	2	III.5(1)	$(9, 19, 24, 31)$	81	2	XI.3(20)

Table A2. *Continued*

(10, 19, 35, 43)	105	2	XI.3(18)	(11, 21, 28, 47)	105	2	XI.3(16)
(11, 25, 32, 41)	107	2	XIX.3(1)	(11, 25, 34, 43)	111	2	XIX.2(2)
(11, 43, 61, 113)	226	2	X.3(1)	(13, 18, 45, 61)	135	2	XI.3(14)
(13, 20, 29, 47)	107	2	XIX.3(1)	(13, 20, 31, 49)	111	2	XIX.2(2)
(13, 31, 71, 113)	226	2	X.3(2)	(14, 17, 29, 41)	99	2	XIX.2(3)
(5, 7, 11, 13)	33	3	X.3(3)	(5, 7, 11, 20)	40	3	VII.3(3)
(11, 21, 29, 37)	95	3	XIX.2(2)	(11, 37, 53, 98)	196	3	X.3(1)
(13, 17, 27, 41)	95	3	XIX.2(2)	(13, 27, 61, 98)	196	3	X.3(2)
(15, 19, 43, 74)	148	3	X.3(1)	(5, 6, 8, 9)	24	4	VII.3(2)
(5, 6, 8, 15)	30	4	IV.3(1)	(9, 11, 12, 17)	45	4	XI.3(20)
(10, 13, 25, 31)	75	4	XI.3(14)	(11, 17, 20, 27)	71	4	XIX.3(1)
(11, 17, 24, 31)	79	4	XIX.2(2)	(11, 31, 45, 83)	166	4	X.3(1)
(13, 14, 19, 29)	71	4	XIX.3(1)	(13, 14, 23, 33)	79	4	XIX.2(2)
(13, 23, 51, 83)	166	4	X.3(2)	(6, 7, 9, 10)	27	5	XI.3(14)
(11, 13, 19, 25)	63	5	XIX.2(2)	(11, 25, 37, 68)	136	5	X.3(1)
(13, 19, 41, 68)	136	5	X.3(2)	(11, 19, 29, 53)	106	6	X.3(1)
(13, 15, 31, 53)	106	6	X.3(2)	(11, 13, 21, 38)	76	7	X.3(1)

Appendix B. Source code

The computer code below classifies the hypersurfaces of index I. For simplicity, the tuples from the tables are left out. The full program and source code are available from the author. It is written in the functional programming language Haskell.

```
-- Classifying quasi-smooth well-formed weighted hypersurfaces.
-- Erik Paemurru

data Tuple = Quint Int Int Int Int Int Int Int deriving( Eq,Ord,Show )
--                 (a0, a1, a2, a3, dd, mm, cc)
-- dd - degree
-- mm - series modulo-number (lcm of smaller weights)
-- cc - series class-number (from 1 to 6)

main = do
  putStr ("Enter index, for which to solve:\n" ++ "Index = ")
  strL <- getLine
  mapM_ putStrLn (map show (solve (read strL :: Int)))

-- The 'solve' function classifies the hypersurfaces. It selects the well-formed
-- quasi-smooth tuples from the list of all tuples. The input 'ii' is the index.
-- The result is a list of 7-tuples in the above form. Using the definition of the
-- infinite series, it is easy to write down the corresponding series. Tuples from
-- the tables must also be added, which has not been implemented here.
solve ii = map (filter conds) (makeTuples ii)

lcm3 a b c = lcm a (lcm b c)
gcd3 a b c = gcd a (gcd b c)

-- div   a b gives a/b rounded down
-- divUp a b gives a/b rounded up
divUp a b = -((-a) `div` b)

-- makeTuples - generate all tuples for given index, without checking conditions
makeTuples ii = [makeClass cc ii | cc <- [1..6]]
```

```
-- makeClass - generate all tuples for given index and class
makeClass cc ii
  | cc == 1  = concat [makeClassWei a0 (ii - a0)  0 (lcm  a0    (ii - a0)) cc ii |
                       a0 <- [1..(ii `div` 2)]]
  | cc == 2  = concat [makeClassWei a0 a1 (ii - a0) (lcm3 a0 a1 (ii - a0)) cc ii |
                       a0 <- [1..(ii `div` 2)], a1 <- [a0..(ii-a0-1)]]
  | cc == 3  = concat [makeClassWei a0 a1 (ii - a1) (lcm3 a0 a1 (ii - a1)) cc ii |
                       a1 <- [2..(ii `div` 2)], a0 <- [1..(a1-1)]]
  | cc == 4  = concat [makeClassWei (ii-k) (2*k) 0 (lcm (ii-k) k) cc ii |
                       k <- reverse [(max (ii `divUp` 3) 1)..(ii-1)]]
  | cc == 5  = concat [makeClassWei (2*k) (ii-k) 0 (lcm (ii-k) k) cc ii |
                       k <- [1..((ii `divUp` 3)-1)]]
  | cc == 6  = concat [makeClassWei (ii-k) (ii+k) k (lcm3 (ii-k) (ii+k) k) cc ii |
                       k <- reverse [1..(ii-1)]]

-- makeClassWei - create tuples, given smaller weights a0,a1,xx and the number mm
makeClassWei a0 a1 xx mm cc ii
-- for c==1, xx is not used
  | cc == 1  = [Quint a0 a1 b2 b3 (b2+b3) mm cc | b2 <- [a1..(a1+mm-1)],
               b3 <- [b2..(b2+mm-1)]]
-- for c==2, xx = a2
  | cc == 2  = [Quint a0 a1 xx b3 (a1+b3) mm cc | b3 <- [xx..(xx+mm-1)]]
-- for c==3, xx = a2
  | cc == 3  = [Quint a0 a1 xx b3 (a0+b3) mm cc | b3 <- [xx..(xx+mm-1)]]
-- for c==4, xx is not used
  | cc == 4  = [Quint a0 a1 b2 (b2 + (a1 `div` 2)) (2*(b2 + (a1 `div` 2))) mm cc |
               b2 <- [a1..(a1+mm-1)]]
-- for c==5, xx is not used
  | cc == 5  = [Quint a0 a1 b2 (b2 + (a0 `div` 2)) (2*(b2 + (a0 `div` 2))) mm cc |
               b2 <- [a1..(a1+mm-1)]]
-- for c==6, xx = k
  | cc == 6  = [Quint a0 a1 b2 (b2+xx) (a1 + 2*b2) mm cc | b2 <- [a1..(a1+mm-1)]]

-- conds - checks all the well-formedness, quasi-smoothness conditions for a tuple
conds tuple = and[cond j tuple | j <- [1..4]]

-- cond j - checks condition (j) for a given tuple
cond j (Quint a0 a1 a2 a3 dd _ _)
  | j == 1  = and[dd `mod` (gcd ai aj) == 0 | (ai,aj) <- pairs]
  | j == 2  = and[gcd3 ai aj ak == 1 | (ai,aj,ak) <- triples]
  | j == 3  = a3 < dd
  | j == 4  = and[or[(dd - aj) `mod` ai == 0 | aj <- weights] | ai <- weights]
  where
  weights = [a0,a1,a2,a3]
  pairs = [(a0,a1),(a0,a2),(a0,a3),(a1,a2),(a1,a3),(a2,a3)]
  triples = [(a0,a1,a2),(a0,a1,a3),(a0,a2,a3),(a1,a2,a3)]
```

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