ANZIAM J. 57(2016), 207–221
doi:10.1017/S1446181115000449

CPDO WITH FINITE TERMINATION: MAXIMAL RETURN UNDER CASH-IN AND CASH-OUT CONDITIONS

X. YANG[1], J. LIANG[✉1] and Y. WU[1]

(Received 21 October, 2014; accepted 23 May, 2015; first published online 10 February 2016)

Abstract

The maximal return and optimal leverage of a constant proportion debt obligation with finite termination and two boundaries are analysed by numerically solving Hamilton–Jacobi–Bellman equations. We discuss the probabilities of the asset value reaching the upper or lower bound under the optimal control and the optimal control problem with a time-varying boundary. Furthermore, we also analyse the relationship between the optimal return, the optimal policy and different parameters.

2010 *Mathematics subject classification*: primary 91G80; secondary 49L20, 65N06.

Keywords and phrases: CPDO, HJB equation, cash-in, cash-out, optimal leverage.

1. Introduction

The credit derivative market has been developing dramatically since the 1990s. Meanwhile, some credit derivatives have been improperly used in practice, and this finally led to the worldwide financial crisis which began in 2008. The financial crisis has influenced global economy profoundly and forced people to be more cautious in using credit derivatives. The issue of controlling leverage and using credit derivatives has attracted more interest in academia and industry.

As one of these leveraged derivatives, constant proportion debt obligation (CPDO) aims at paying high coupons and returning the principal to the investors by putting the capital into a bank and leveraging a nominal credit exposure to indices [1, 3]. The leverage needs to be adapted dynamically to generate high coupon payments (usually, 100–200 basis points above London interbank offered rate (LIBOR)) for investors [3]. Cash-out and cash-in terms are included in a CPDO contract in order to avoid substantial losses and reduce the risky exposure of the portfolio [4, 7]. The cash-out term is a minimal return guarantee to the CPDO investors, while the cash-in term sets the maximal payoff to them.

[1]Department of Mathematics, Tongji University, Shanghai 200092, PR China;
e-mail: changyansummer@163.com, liang_jin@tongji.edu.cn, wuyuan0720@hotmail.com.

The higher coupon payments of CPDO contracts compared to other products in the same rating class (typically AAA or Aaa) made them popular among investors. However, this led to more controversy among market participants about the methods used by major rating agencies [1, 3, 7]. Consequently, many researches have been carried out on this issue. Cont and Jessen [3] used a top-down approach to analyse the risks of a CPDO contract, pointing out that risk analyses based on rating or defaulting probabilities alone are not sensible. Gordy and Willemann [7] used the CPDO case as an example of model risk in rating derivatives with complex structures, and claimed that the rating of CPDOs according to the traditional rating models was too high. Jobst et al. in their dominion bond rating service (DBRS) commentary [10] used a Gaussian copula to assess the risks of CPDOs. They pointed out that the model risks, high path dependency, high credit spread volatility and the complex structure of CPDOs made it challenging to rate them. Torresetti et al. [18] used a generalized Poisson loss model instead of a Gaussian copula to study the possibility of a cluster of defaults occurring to the pool of underlying names.

There are other researches focusing on the pricing of CPDOs. Dorn [4] derived a dynamic closed-form pricing formula for a CPDO, which was useful in showing some mechanisms and measuring the risks. Varloot et al. [19] used a simple closed-form formula of a CPDO and calculated its basic risk measures in their investment research on The United Bank of Switzerland (UBS). In addition, they offered advice on leverage adjustment and credit rating. Çekiç and Uğur [2] applied Laplace transformation to obtain a closed-form CPDO pricing formula.

Meanwhile, the dynamically adjusting leverage in a CPDO contract makes it a good model of optimal control. Baydar et al. [1] derived a Hamilton–Jacobi–Bellman (HJB) equation to find the optimal leverage of a CPDO and compared it with that used in industry. Instead of modelling the cash-in and cash-out boundaries directly, they chose a special utility function to avoid them and obtained a closed-form solution by a duality method. Later, Wu [21] and Yang et al. [22] analysed these contracts in the optimal control framework as well, and cash-in and cash-out conditions were included explicitly and described in detail. Wu [21] studied a perpetual CPDO contract with a cash-out term, and discussed the cases of minimizing cash-out probability and maximizing total returns. Yang et al. [22] modelled the conditional redemption in the default probability minimization problem and found the optimal upper bound for the control policy.

In this paper, we still use the optimal control method and HJB equation to solve the problem. The HJB equations are widely used to solve control and optimization problems. Due to their nonlinearity, most HJB equations cannot be solved explicitly; only equations with special terminal conditions can be solved by the variable separation or duality method. However, it is usually difficult to apply these methods to equations with boundary conditions. In those cases, numerical methods are preferred.

The numerical methods to calculate HJB equations can be divided into two categories: the Markov-chain method [5, 14, 16] and the finite-difference method [6, 20]. The Markov-chain method is substantially a kind of explicit finite-difference method which is usually limited by its time step. That is, only minor time steps make sure the convergence of a difference scheme, which leads to heavy computation. Additionally, another difficulty needs to be overcome: both the optimal policy and the optimal value need to be solved at every time step, but the optimal value and the policy depend on each other. Forsyth and Labahn [6] proposed an implicit method to solve HJB and HJBI (Hamilton–Jacobi–Bellman–Isaacs) equations, in which Newton-type iteration is applied at every time step to determine the optimal policy and value. Wang and Forsyth [20] discussed the possibility of using central difference in this implicit scheme.

In this paper, we establish a stochastic optimal control model to find the optimal leverage and maximal payoff of a CPDO. Both the cash-in and cash-out boundaries are taken into consideration, and the requirements of a reasonable cash-in boundary are discussed. An HJB equation with two boundaries is deduced to solve the problem. Furthermore, we discuss the problem of a time-dependent cash-in boundary and solve the associated HJB equation. The properties of the optimal policy and payoff are shown and analysed in both models.

The main contributions of this article are the following: firstly, it applies stochastic optimal control theories to study CPDO contracts; secondly, both cash-out and cash-in terms are included in a CPDO with finite expiration leading to HJB equations with two boundary conditions. This kind of equation is difficult to solve explicitly. Thirdly, the equations with both fixed and time-dependent boundaries are solved numerically, and the properties of the solutions are discussed.

The rest of this paper is organized as follows. In Section 2, the model of maximizing the total payoff with fixed cash-in and cash-out boundaries is built, and numerical results of the optimal value and the optimal policy are given. In Section 3, the payoff-maximization model is improved to a regime-changing problem, numerical results are given and the relationships between optimal policy, optimal payoff and parameters are analysed. The conclusion is presented in Section 4.

2. Maximization of total payoff with fixed boundary

2.1. The model In a CPDO contract, an investor provides some principal to a CPDO manager (special purpose vehicle (SPV)) by buying a CPDO, and this capital is the initial investment [3]. The CPDO manager then builds a portfolio by putting the capital into a bank account and keeping a position in credit indices (for example, iTraxx or CDX) with the bank account as a nominal to obtain high returns [1]. The manager adjusts the leverage of the credit default swap (CDS) dynamically to pay coupons of LIBOR plus a constant spread to the investor, and return the principal at termination [1]. The performance of a CPDO can be characterized by three states: *cash-in*, *cash-out* and *failure to return the principal* at the termination. If the asset value is high enough to cover the present value of all future coupon payments and principal redemption at expiration, then the SPV reduces the exposure to credit indices (the risky exposure) to zero, and puts all the asset in the bank account to receive a risk-free return. This case is called *cash-in*. When the value of the capital falls below a determined lower boundary (smaller than initial capital), the manager stops the contract and returns all what is left to the investor. This case is called *cash-out*, which is similar to default [4, 7, 10].

Consider a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, on which a Brownian motion W_t is defined adapted to $\mathscr{F}_t$. Consider a CPDO contract with finite expiration time T and initial capital $X_0 = 1$ (dollar). If we use X_t to represent the capital process, then the capital in the bank account satisfies the equation

$$dX_t = rX_t\,dt, \tag{2.1}$$

where $r > 0$ is the constant risk-free interest rate. The CPDO manager uses the asset X_t as nominal principal to leverage credit indices. Let S_t be the risky process of a credit index and a_t be the nominal risk exposure to the credit index at time t; then the asset value satisfies

$$dX_t = rX_t + a_t\,dS_t. \tag{2.2}$$

Notice that the asset invested in the bank account acts as collateral to sell the CDS contracts, and a_t is the control process in this model.

Assume that S_t satisfies

$$dS_t = \mu\,dt + \sigma\,dW_t, \tag{2.3}$$

where μ stands for average index spread and σdW_t represents the mark-to-market gains or losses due to the changes in the default probability of the underlying index portfolio. Here, μ and σ are both positive constants. This model is the same as in the work by Baydar et al. [1], and it also appeared in the work of Varloot et al. [19]. If the CPDO manager is investing in a single CDS, default will cause a jump in the asset process. However, when the CPDO manager leverages a portfolio of many CDS contracts, the defaults can be modelled by a standard Brownian motion. From this point of view, this process is similar to a reserve process of an insurance company (see, for example, the articles by Hojgaard and Taksar [9] and Taksar and Markussen [17]).

By inserting (2.3) into (2.2), we obtain the process of the asset value

$$dX_t = (rX_t + a_t\mu)\,dt + a_t\sigma\,dW_t.$$

In this model, we assume that the coupon is paid continuously to the investor for simplicity. The coupon rate is the constant risk-free interest rate r as in equation (2.1) plus a spread δ, with δ being a positive constant. Since the initial principal is 1, the coupon paid during the time interval $(t, t + dt)$ is $(r + \delta)\,dt$. Then the asset value X_t satisfies the stochastic differential equation (SDE)

$$dX_t = (rX_t + a_t\mu - r - \delta)\,dt + a_t\sigma\,dW_t, \tag{2.4}$$

where all parameters are considered to be constants. Note that in further works, models with stochastic coefficients can be taken into consideration as well, but the simple model in this paper still reveals some fundamental characteristics of this kind of derivative.

Note that the admissible policy a_t in (2.4) is nonnegative and the $\mathscr{F}_t$-adapted process is such that the above SDE (2.4) with an initial value $X_0 = 1$ has a unique $\mathscr{F}_t$-adapted solution X_t which is right continuous with left-hand-side limitations. The set of all admissible policies is denoted by $\mathcal{A}$.

In a CPDO contract discussed in this paper, both the cash-out and cash-in terms are included. With a principal of 1 in this model, the cash-out boundary is set as a constant $K \in (0, 1)$ and the cash-out time is defined as

$$\tau_1 = \inf\{t > 0 \mid X_t \le K\},$$

which is a stopping time. Meanwhile, the cash-in boundary is set to be a constant $\bar{K} > K$. Similarly, the cash-in time is defined as

$$\tau_2 = \inf\{t > 0 \mid X_t \ge \bar{K}\}.$$

In this section, we assume that the CPDO manager returns the total asset $\bar{K}$ to the buyer immediately when the cash-in event happens. If neither the cash-in nor the cash-out case happens before termination, the CPDO manager returns the total value of the portfolio to the investor at time T.

It can be deduced from above that the payoff to the CPDO buyer consists of four parts:

(1) the coupon $r + \delta$ paid continuously during the period when the CPDO is still valid;
(2) the capital returned to the investor when the cash-out case happens, namely, K;
(3) the value of capital $\bar{K}$ returned to the investor when the cash-in case happens;
(4) the value X_T received by the investor if the CPDO contract remains valid until expiration.

We assume that the investors use utility functions to measure their satisfaction caused by those incomes. The independent variable of a utility function is usually the profit or loss of an investment, and the dependent variable is the satisfaction from the profit or aversion to the loss. A utility function $U(x)$ is usually increasing, since investors are more content with higher profits. Meanwhile, a utility function is usually concave down ($U''(x) \le 0$), meaning that the marginal effect of the income decreases as the total amount of income increases. More information on this type of function can be found in the book by Karatzas and Shreve [13].

In this article, we define the utility functions $U_i(x)$ as follows:

$$U_i(x) = \frac{1}{\gamma_i}(1 - e^{-\gamma_i x}), \quad i = 1, 2.$$

Other types of utility functions can also be applied in this model, as long as they are meaningful during the value interval of x and satisfy the basic properties of the utility functions.

Different investors may show different attitudes towards the coupon income and the terminal payoff. For example, some may be more pleased to receive a steady

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

income in the duration, while others may be more interested in the return at terminal, or when cash-in or cash-out happens. In this paper, different utility functions are used to measure the coupon income and the terminal income, that is, $U_1(x)$ and $U_2(x)$ are used to measure the utility from coupon and other incomes, respectively. Then, at time $t \in (0, T)$, the total expected utility of the investor should be

$$J(x,t;a) = \mathbb{E}\Big[\int_t^T e^{-\beta(s-t)} U_1(r+\delta)\mathbf{1}_{\{s<\tau_1\wedge\tau_2\}}\,ds + \mathbf{1}_{\{\tau_1<\tau_2\wedge T\}} e^{-\beta(\tau_1-t)} U_2(K) + \mathbf{1}_{\{\tau_2<\tau_1\wedge T\}} e^{-\beta(\tau_2-t)} U_2(\bar{K}) + \mathbf{1}_{\{T<\tau_1\wedge\tau_2\}} e^{-\beta(T-t)} U_2(X_T) | X_t = x\Big],$$

where β is the discount rate of utility. The usage of discounted utility can also be found in the works by Karatzas et al. [12] and Jonsson and Sircar [11] (also see the articles by Guiso and Paiella [8] and Neilson and Winter [15] for more information on the parameter calibration of utility functions). Here, $\mathbf{1}_A$ is the indicator function of a set A, defined as

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in A^C. \end{cases}$$

The CPDO manager adjusts the control policy dynamically in order to maximize the profits for the investor. Then the optimal value function is defined as

$$P(x,t) = \sup_{a \in \mathcal{A}} J(x,t;a). \tag{2.5}$$

In this model, the cash-out boundary K and cash-in boundary $\bar{K}$ should be chosen such that

$$U_2(K) < \frac{1}{\beta} U_1(r+\delta), \quad U_2(\bar{K}) = \frac{1}{\beta} U_1(r+\delta).$$

The reason for these assumptions is that if a CPDO contract has an infinite time horizon, and the contract remains valid until infinity, then the profit of the investor should be

$$\int_0^\infty e^{-\beta t} U_1(r+\delta)\,dt = \frac{1}{\beta} U_1(r+\delta).$$

It is reasonable to take this as the upper bound of an investor's payoff. Then the cash-in boundary $\bar{K}$ should be chosen such that

$$U_2(\bar{K}) \le \frac{1}{\beta} U_1(r+\delta),$$

otherwise stopping adjustment on the boundary may not be the optimal choice. Here, we assume that

$$U_2(\bar{K}) = \frac{1}{\beta} U_1(r+\delta)$$

and the cash-out boundary K, which is set to prevent the investor from losing everything in a bad investment, is set to satisfy

$$U_2(K) < \frac{1}{\beta} U_1(r+\delta).$$

From the above assumptions,

$$\bar{K} = -\frac{1}{\gamma_2}\ln\Big(1 - \frac{\gamma_2}{\beta}U_1(r+\delta)\Big), \tag{2.6}$$

which is used in the following sections.

2.2. Value function, HJB equation and numerical results The optimal return function $P(x,t)$ in (2.5) with $\bar{K}$ as in (2.6) satisfies the HJB equation, if it is continuous [5],

$$\begin{cases} P_t + \sup\limits_{a\in\mathcal{A}}\{(rx + a\mu - r - \delta)P_x + \frac{1}{2}a^2\sigma^2 P_{xx}\} + U_1(r+\delta) - \beta P = 0, \\ \qquad\qquad (x,t)\in\Omega = (K,\bar{K})\times(0,T), \\ P(K,t) = U_2(K), \quad P(\bar{K},t) = U_2(\bar{K}), \quad t\in[0,T], \\ P(x,T) = U_2(x), \quad x\in[K,\bar{K}], \end{cases}$$

where P_t, P_x, P_{xx} represent $\partial P/\partial t$, $\partial P/\partial x$, $\partial^2 P/\partial x^2$, respectively.

From Fleming and Soner's book [5, pages 196–197], $P(x,t)$ is a concave function of x: taking $x_1, x_2 \in [K,\bar{K}]$, $\lambda\in(0,1)$ yields

$$\lambda P(x_1,t) + (1-\lambda)P(x_2,t) \geq P(\lambda x_1 + (1-\lambda)x_2, t),$$

which is a common result in an optimal control problem if the value function is of a special type.

Assuming that $P_x > 0$, the optimal policy is

$$a^* = -\mu P_x/(\sigma^2 P_{xx}),$$

which is obtained as soon as the optimal value is solved.

As far as we know, the above nonlinear equation cannot be solved explicitly, yet some numerical results can be obtained. Here, we use the implicit central and forward (backward) difference and iteration method introduced by Wang and Forsyth [20] and Forsyth and Labahn [6] to numerically calculate the optimal value and the optimal leverage. The parameters are taken as follows:

$$\begin{gathered} K = 0.1\,(\text{dollar}), \quad T = 3\,(\text{year}), \quad \beta = 0.08/\text{year}, \quad \mu = 0.05/\text{year}, \\ \sigma = 0.1, \quad r = 0.03/\text{year}, \quad \delta = 0.015/\text{year}, \quad \gamma_1 = 2, \quad \gamma_2 = 1.5. \end{gathered}$$

Note that in this model, all the values of the parameters are given, but, in further works, if needed, the parameters related to the CPDO contract and asset value can be calibrated from real data. But the parameters related to the utility functions are difficult to calibrate because of the relatively abstract concept of *utility* and the different types of utility functions to choose from.

A transformation of time is made to make the terminal condition at time T to be the initial condition at time 0. Take the time step to be Δt. In this paper, the initial value of the control policy at every time node is chosen to be the optimal policy of the former time node. However, the initial policy at time Δt is chosen to be identically equal to 1,

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

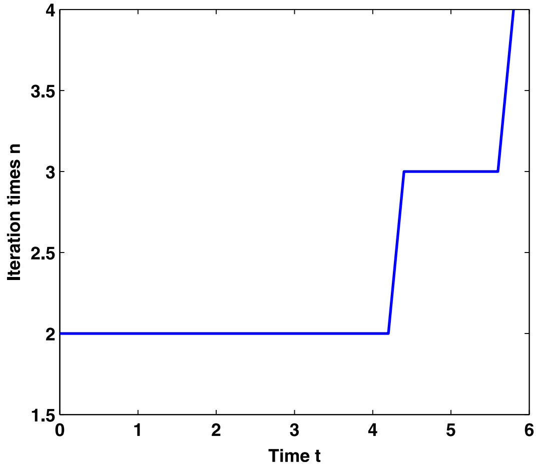

FIGURE 1. Iteration times at every time step.

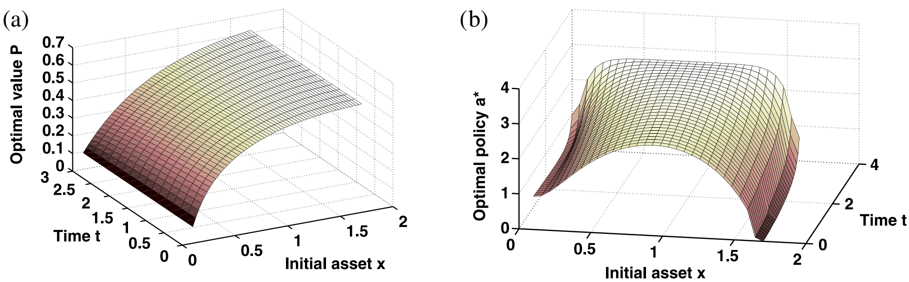

FIGURE 2. Optimal policy and corresponding payoff (colour available online). (a) Relationship between optimal payoff P, initial asset value x and time t. (b) Relationship between optimal policy a^*, initial asset value x and time t.

since the optimal control policy at time 0 does not exist. After choosing the initial policy, a relevant value vector is calculated implicitly, and the value vector is used to calculate the (iterated) value of the control. The value of the control is used to evaluate the iterated value again. This is the iteration method to obtain the optimal leverage and return. If the infinite norm of the difference in value vector from two iterations is sufficiently small, then the iteration of this time node is complete, and the optimal value and optimal policy at this node are obtained.

We have used the following values for iteration:

$$\Delta t = 0.2, \quad \Delta x = \tfrac{1}{160}(\bar{K} - K), \quad \epsilon = 10^{-5},$$

where ϵ stands for the tolerance of error in the iteration. The iteration times at every time node are shown in Figure 1.

The optimal value and the optimal leverage are shown in Figure 2.

The figures indicate that the optimal redemption $P(x,t)$ is an increasing, concave-down function of the initial asset value x, which agrees with the assumptions made

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

on P. The optimal leverage a^* is bounded, which is initially increasing and then decreasing with respect to x. More properties of the optimal payoff and strategy will be shown and discussed in the next section.

3. Maximization of total payoff with time-dependent boundary

3.1. Model establishment In this section, we replace the cash-in condition in the last section with a more realistic one.

In Section 2, it is assumed that the cash-in boundary is constant, independent of time and the asset is returned to the CPDO buyer immediately after the cash-in event. However, in a common CPDO contract, the CPDO manager should continue paying coupons after cash-out and return the asset left to the investor at terminal T. In other words, the cash-in boundary should be decided such that the money in the bank account is enough to cover all the coupon payments and the terminal payoffs in the future.

Denote the new cash-in boundary as $\bar{K}(t)$; the new cash-in time is defined as

$$\tau_2 = \inf\{t \geq 0 \mid X_t \geq \bar{K}(t)\}.$$

The process of the investment is separated into two parts.

(1) *From time $t = 0$ to the cash-in time τ_2:* The CPDO manager faces an optimal investment problem; he has to choose an optimal strategy in order to generate the maximal return to the CPDO buyer.

(2) *From the cash-in time τ_2 to the terminal time T:* The CPDO manager faces a determined investment problem; since the risky exposure is zero, he only needs to receive interest incomes and pay coupons to the CPDO buyer continuously until termination and return the capital to the buyer at T.

The asset processes and the strategies of the CPDO manager are different between those two processes.

Taking $P_0(t)$ to be the value function of the period $(\tau_2, T]$, the value function of the whole investment problem becomes

$$\begin{aligned} P(x,t) = \sup_{a\in\mathcal{A}} \mathbb{E}\bigg[\int_t^T e^{-\beta(s-t)} U_1(r+\delta)\mathbf{1}_{\{s<\tau_1\wedge\tau_2\}}\,ds + \mathbf{1}_{\{\tau_1<\tau_2\wedge T\}} e^{-\beta(\tau_1-t)} U_2(K) \\ + \mathbf{1}_{\{\tau_2<\tau_1\wedge T\}} e^{-\beta(\tau_2-t)} P_0(\tau_2) + \mathbf{1}_{\{T<\tau_1\wedge\tau_2\}} e^{-\beta(T-t)} U_2(X_T) | X_t = x\bigg]. \quad (3.1) \end{aligned}$$

Under this assumption, the asset value X_t (deterministic process) after the cash-in time follows the ordinary differential equation

$$\begin{cases} dX(t)/dt = rX(t) - r - \delta, & t \in [\tau_2, T], \\ X(T) = \bar{K}. \end{cases}$$

The above problem admits a unique solution

$$X(t) = \Big(\bar{K} - \frac{r+\delta}{r}\Big)e^{-r(T-t)} + \frac{r+\delta}{r} = \frac{r+\delta}{r}[1 - e^{-r(T-t)}] + \bar{K}e^{-r(T-t)} > 0,$$

from which one can deduce that

$$X(t) = \int_t^T e^{-r(s-t)} U_1(r+\delta)\, ds + e^{-r(T-t)} U_2(\bar{K}). \tag{3.2}$$

Since $X(t)$ defined in (3.2) equals the present value of all the future coupon payments and the capital payoff at termination, it is the new (time-dependent) cash-in boundary $\bar{K}(t)$ in this problem. The boundary $\bar{K}(t)$ is an extension of the constant cash-in boundary $\bar{K}$ in Section 2.1.

The payoff function after the cash-in time should be

$$\begin{aligned} P_0(t) &= \int_t^T e^{-\beta(s-t)} U_1(r+\delta) ds + e^{-\beta(T-t)} U_2(\bar{K}) \\ &= \frac{1}{\beta} U_1(r+\delta) + \Big[U_2(\bar{K}) - \frac{1}{\beta} U_1(r+\delta) \Big] e^{-\beta(T-t)}, \quad t \in [\tau_2, T]. \end{aligned}$$

If $\bar{K}$ is defined as (2.6), then $U_2(\bar{K}) = (1/\beta)U_1(r+\delta)$ and $P_0(t) = (1/\beta)U_1(r+\delta)$. As in Section 2, the new value function defined in (3.1), if continuous, is a viscosity solution to the following HJB equation:

$$\begin{cases} \dfrac{\partial P}{\partial t} + \sup\limits_{a \in \mathcal{A}} \Big\{ (rx + a\mu - r - \delta) \dfrac{\partial P}{\partial x} + \dfrac{1}{2} a^2 \sigma^2 \dfrac{\partial^2 P}{\partial x^2} \Big\} \\ \qquad + U_1(r+\delta) - \beta P = 0, \quad (x,t) \in (K, \bar{K}(t)) \times (0,T), \\ P(K,t) = U_2(K), \quad P(\bar{K}(t), t) = \dfrac{1}{\beta} U_1(r+\delta), \quad t \in [0,T], \\ P(x,T) = U_2(x), \quad x \in [K, \bar{K}]. \end{cases}$$

By using the same numerical method in Section 2, the equation can be solved.

3.2. Numerical results The parameters here are the same as in Section 2.2. The cash-in boundary changing with time t is shown in Figure 3.

The optimal value and the optimal policy are shown in Figure 4(a) and (b), respectively.

The relationships between the optimal payoff P and parameters are shown in Figure 5. Notice that the cash-in boundary (upper bound of asset value) increases and the solution domain of the optimal payoff expands as r or δ increases.

The above figures imply that a higher value of μ, r or δ leads to a higher optimal payoff. Since r and δ are components of the coupon returned to the investor, a higher value of r or δ means a higher coupon payoff to the investor. Consequently, the optimal value will be higher. A higher value of μ means a better performance of the CDS, which facilitates the CPDO manager to pay coupons while keeping a capital with a relatively high return.

Contrarily, a higher σ leads to a lower P, because the volatility in the CDS value forces the CPDO manager to choose a more conservative way of investment to prevent the portfolio from default. Accordingly, the optimal value for the investor will be lower.

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

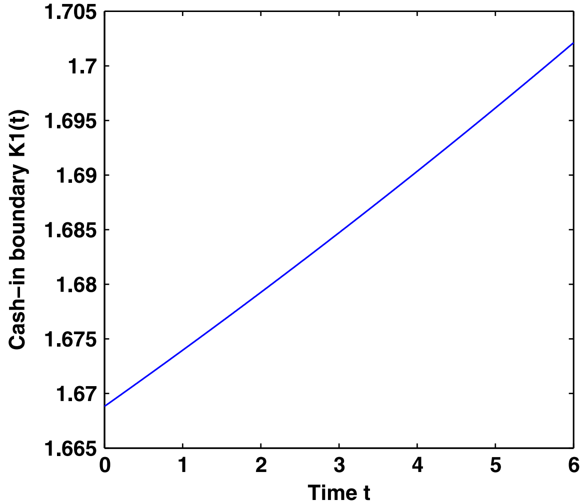

FIGURE 3. Cash-in boundary.

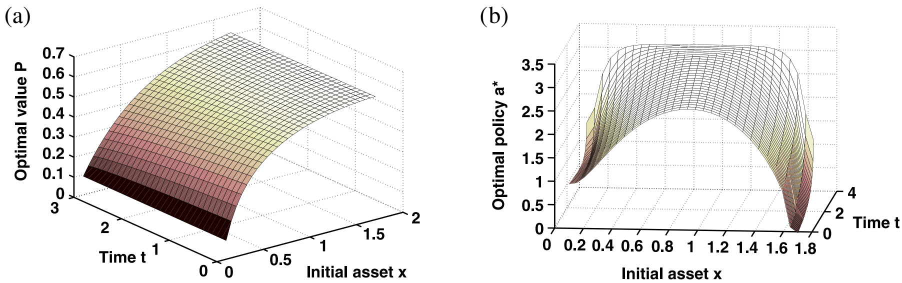

FIGURE 4. Optimal policy and optimal payoff under time-varying cash-in boundary (colour available online). (a) Relationship between P, x and t. (b) Relationship between a^*, x and t.

The relationships between the optimal policy a^* and the parameters are shown in Figure 6.

From Figure 6, it follows that similar to the performance of the optimal value P, the optimal leverage increases with r and δ and decreases with σ. Higher r or δ means higher coupon to be paid to the CPDO buyer, which forces the CPDO manager to use a more radical leverage. The higher uncertainty of the market represented by higher volatility of the CDS price forces the CPDO manager to be conservative and use a lower leverage. As is shown in Figure 5(b), the optimal return becomes lower.

However, the relationship between a^* and μ is influenced by the initial asset value x. When x is relatively small, the optimal leverage a^* decreases with μ. When x is bigger, a^* increases with μ. For the increasing part, the reason may be that higher return of the CDS contract (with the same uncertainty) enables the CPDO manager to choose a more risky leverage in order to obtain higher returns. However, when the present

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

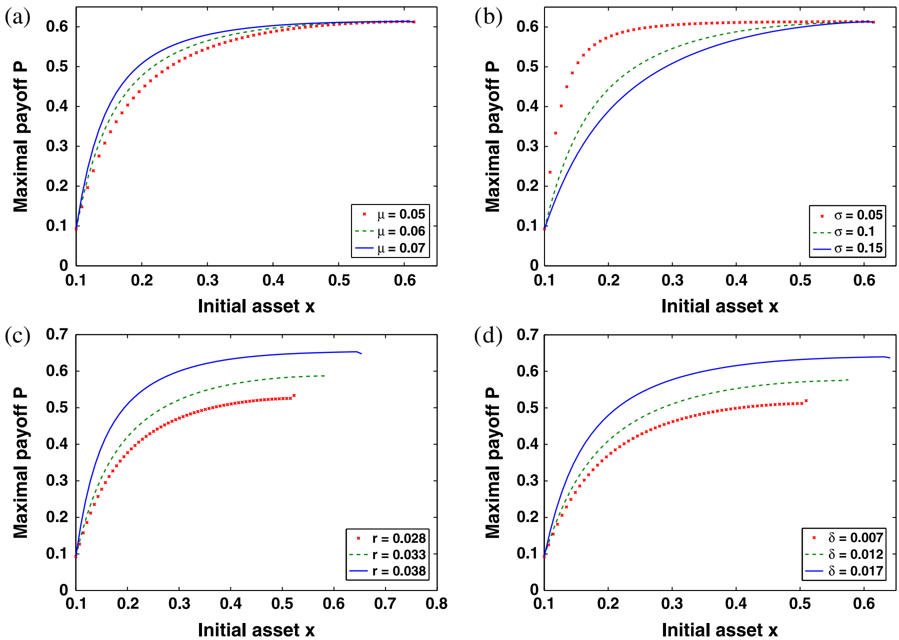

FIGURE 5. Relationships between optimal payoff P and the parameters. (a) P and μ. (b) P and σ. (c) P and r. (d) P and δ (colour available online).

value of the portfolio is very small, which may not be able to cover the coupon part ($rX_t - r - \delta < 0$ in (2.4)), the CPDO manager has to ensure that the asset value has an increasing trend ($rX_t + a\mu - r - \delta \geq 0$) in order to avoid the cash-out case. Then the CPDO manager has to choose a higher a^* with a lower μ.

The properties of optimal payoff and leverage with fixed cash-in boundary are similar.

The cash-out and cash-in probabilities under the above optimal control are shown in Figure 7.

4. Conclusion

In this paper, the optimal leverage of a CPDO contract for maximizing the expected return of the CPDO buyer is analysed. Inclusion of the cash-in and cash-out terms in our model, which are important in a CPDO contract, improves some of the previous work on this topic. These terms add boundary conditions to the derived HJB equation, making it difficult to obtain a closed-form solution. We solve the problem numerically. Furthermore, another problem with a time-dependent cash-in boundary is discussed, analysed and solved numerically.

Our results indicate that the optimal return function $P(x, t)$ is an increasing, concave-down function of the initial asset value x, while the optimal policy a^* is first increasing and then decreasing with respect to x.

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

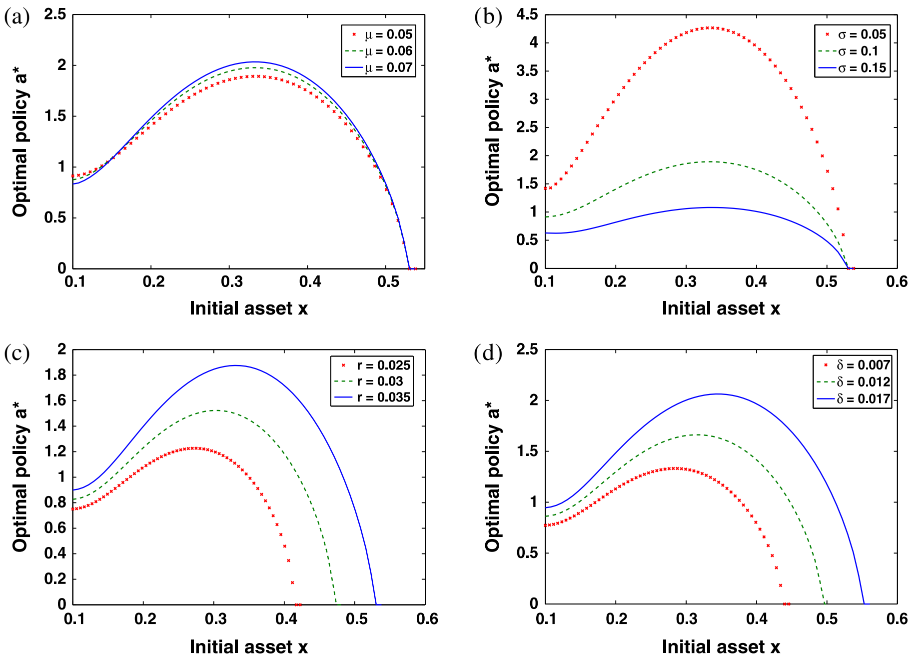

Figure 6. Relationships between optimal policy a^* and the parameters. (a) a^* and μ. (b) a^* and σ. (c) a^* and r. (d) a^* and δ (colour available online).

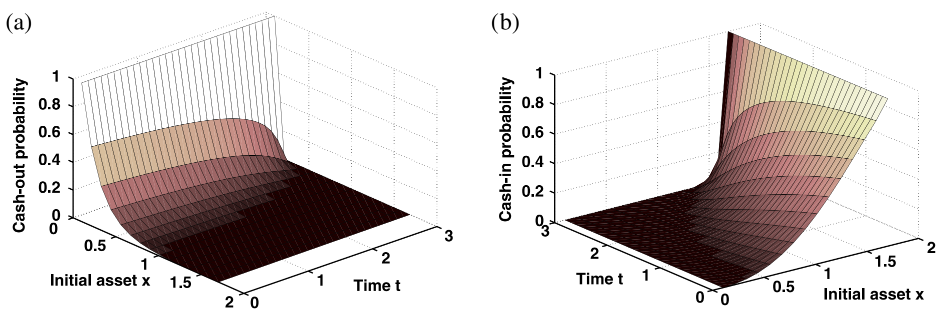

Figure 7. (a) Cash-out and (b) cash-in probabilities under optimal control (colour available online).

The optimal payoff and the optimal leverage will be affected by parameters. We observe that $P(x,t)$ increases with the return rate μ of the CDS contracts, the risk-free interest rate r and the exceeding part of the CPDO coupon rate over the risk-free interest rate, δ. This implies that a better performance of the CDS or a higher coupon rate will result in a higher return to the investors. Also, note that $P(x,t)$ decreases with a higher volatility σ in CDS price, meaning that higher uncertainty in the financial market will have negative effects on the payoff of the investors. The performance

https://doi.org/10.1017/S1446181115000449 Published online by Cambridge University Press

of the optimal policy a^* is similar to that of the optimal return, while high volatility influences the policy more evidently.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (No. 11271287).

References

[1] E. Baydar, G. Di-Graziano and R. Korn, "Theoretical solution versus industry standard: optimal leverage function for CPDOs", *Bl. DGVFM* **30** (2009) 15–29; doi:10.1007/s11857-009-0081-7.

[2] A. İ. Çekiç and Ö. Uğur, "Pricing formulae for constant proportion debt obligation notes: the Laplace transform technique", *J. Comput. Appl. Math.* **259** (2014) 362–370; doi:10.1016/j.cam.2013.06.006.

[3] R. Cont and C. Jessen, "Constant proportion debt obligations (CPDOs) modeling and risk analysis", *Quant. Finance* **12** (2012) 1199–1218; http://ssrn.com/abstract=1372414.

[4] J. Dorn, "Modeling of CPDOs—identifying optimal and implied leverage", *J. Bank. Finance* **34** (2010) 1371–1382; doi:10.1016/j.jbankfin.2009.12.005.

[5] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, 2nd edn (Springer, New York, 2003).

[6] P. A. Forsyth and G. Labahn, "Numerical methods for controlled Hamilton–Jacobi–Bellman PDEs in finance", 2007; https://cs.uwaterloo.ca/~glabahn/Papers/borrow-lend.pdf.

[7] M. B. Gordy and S. Willemann, "Constant proportion debt obligations: a post-mortem analysis of rating models", *Manag. Sci.* **58** (2012) 476–492; http://www.federalreserve.gov/pubs/feds/2010/201005/201005pap.pdf.

[8] L. Guiso and M. Paiella, "Risk aversion, wealth, and background risk", *J. Eur. Econ. Assoc.* **6** (2008) 1109–1150; doi:10.1162/JEEA.2008.6.6.1109.

[9] B. Hojgaard and M. Taksar, "Optimal proportional reinsurance policies for diffusion models with transaction costs", *Insurance Math. Econom.* **22** (1998) 41–51; doi:10.1016/S0167-6687(98)00007-9.

[10] N. Jobst, Y. Xuan, S. Zarya, N. Sandstrom and K. Gilkes, "CPDOs laid bare: structure, risk and rating sensitivity", 2007; http://www.math.ust.hk/~maykwok/courses/Adv_Num_Met/dbrs_cpdo.pdf.

[11] M. Jonsson and R. Sircar, "Optimal investment problems and volatility homogenization approximations", *NATO Sci. Ser. II Math.* **75** (2002) 314–328; http://www.princeton.edu/%7Esircar/Public/ARTICLES/montreal.pdf.

[12] I. Karatzas, J. P. Lehoczky and S. E. Shreve, "Optimal portfolio and consumption decisions for a "small investor" on a finite horizon", *SIAM J. Control Optim.* **25** (1987) 1557–1586; doi:10.1137/0325086.

[13] I. Karatzas and S. E. Shreve, *Methods of mathematical finance* (Springer, New York, 1998).

[14] X. Li and Q. Song, "Markov chain approximation methods on generalized HJB equation", *Proceedings of the 46th IEEE Conf. on Decision and Control*, (New Orleans, USA, 2007), 4069–4074; http://www.researchgate.net/publication/4312482_Markov_chain_approximation_methods_on_generalized_HJB_equation.

[15] W. S. Neilson and H. Winter, "A verification of the expected utility calibration theorem", *Econ. Lett.* **74** (2002) 347–351; http://web.utk.edu/~wneilson/EcLett-Calibration.pdf.

[16] Q. Song, "Convergence of Markov chain approximation on generalized HJB equation and its applications", *Automatica* **44** (2008) 761–766; doi:10.1016/j.automatica.2007.07.014.

[17] M. I. Taksar and C. Markussen, "Optimal dynamic reinsurance policies for large insurance portfolios", *Finance Stoch.* **7** (2003) 97–121; doi:10.1007/s007800200073.

[18] R. Torresetti, A. Pallavicini, B. B. V. Argentaria and B. Leonardo, "Stressing rating criteria allowing for default clustering: the CPDO case", 2007; http://mpra.ub.uni-muenchen.de/17104/.

[19] E. Varloot, G. Charpin and E. Charalampidou, "CPDO insights on rating actions and manager impact using a new formula", 2007;
http://www.defaultrisk.com/_pdf6J4/CPDO_UBS_CPDO_Insights_n_New_Formula.pdf.

[20] J. Wang and P. A. Forsyth, "Maximal use of central differencing for Hamilton–Jacobi–Bellman PDEs in finance", *SIAM J. Numer. Anal.* **46** (2008) 1580–1601; doi:10.1137/060675186.

[21] Y. Wu, "Valuation of credit derivatives and optimization research", Masters Dissertation, Tongji University, Shanghai, 2012.

[22] X. Yang, J. Liang and Y. Wu, "Optimal control of perpetual CPDO: minimal cash-out probability and maximal conditional return", *Optimal Control Appl. Methods* **35** (2014) 383–394; doi:10.1002/oca.2075.