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Fubini's Theorem for Ultraproducts of Noncommutative L_p-Spaces

Marius Junge

Abstract. Let $(\mathcal{M}_i)_{i\in I}$, $(\mathcal{N}_j)_{j\in J}$ be families of von Neumann algebras and $\mathcal{U}$, $\mathcal{U}'$ be ultrafilters in I, J, respectively. Let $1 \le p < \infty$ and $n \in \mathbb{N}$. Let $x_1,\dots,x_n$ in $\prod L_p(\mathcal{M}_i)$ and $y_1,\dots,y_n$ in $\prod L_p(\mathcal{N}_j)$ be bounded families. We show the following equality

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^{n} x_k(i) \otimes y_k(j) \Big\|_{L_p(\mathcal{M}_i \otimes \mathcal{N}_j)} = \lim_{j,\mathcal{U}'} \lim_{i,\mathcal{U}} \Big\| \sum_{k=1}^{n} x_k(i) \otimes y_k(j) \Big\|_{L_p(\mathcal{M}_i \otimes \mathcal{N}_j)}.$$

For $p = 1$ this Fubini type result is related to the local reflexivity of duals of C^*-algebras. This fails for $p = \infty$.

0 Introduction

Fubini's Theorem is a fundamental tool in measure theory, probability and analysis. Our aim is to prove a version of Fubini's Theorem in the context of noncommutative L_p-spaces associated with von Neumann algebras. More precisely, we will extend the Fubini Theorem to ultraproducts of noncommutative L_p-spaces. Although ultraproducts might appear less natural in the context of measure theory, they are a standard tool in the context of von Neumann algebras, we refer for example to the use of Dixmier traces in Connes' work [C3] on noncommutative geometry. Ultraproducts appear also rather naturally in the investigation of L_p-spaces associated to residually finite non-amenable groups, for example free groups. The aim of this paper and the forthcoming paper [J2] is to extend Pisier's [P7] theory of vector-valued noncommutative L_p-spaces to L_p-spaces associated to free groups and an even more general class of von Neumann algebras. Our proof of the Fubini theorem for ultraproducts requires operator algebraic methods and is related to the local reflexivity of duals of C^*-algebras, see [EJR]. In combination with the factorization theory of linear maps, the noncommutative Fubini theorem provides an important ingredient for the investigation of the local structure of noncommutative L_p-spaces associated to von Neumann algebras, see the forthcoming papers [J2, JNRX].

Let us illustrate these applications by considering the free group $\mathbb{F}_n$ in n generators. Using an idea of S. Wassermann [Wa], it can easily be shown that the noncommutative L_p-space $L_p(VN(\mathbb{F}_n))$ associated to the von Neumann algebra $VN(\mathbb{F}_n)$ obtained from the left regular representation embeds naturally in an ultraproduct of L_p-spaces

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of matrix algebras. Although $VN(\mathbb{F}_n)$ is not hyperfinite, *i.e.*, does not admit an increasing weakly dense family of matrix algebras, we can nevertheless use genuinely finite dimensional techniques to obtain information about $L_p(VN(\mathbb{F}_n))$. We will show in [JNRX, J2] that this approach is generic for a large class of von Neumann algebras. Indeed, the Fubini Theorem is a central tool in the proof of the following applications.

(i) [JNRX] An analysis of the local structure of noncommutative L_p-spaces.
(ii) [JR] For $1 < p < \infty$ the space $L_p(VN(\mathbb{F}_n))$ has a basis.
(iii) [J2] Let $1 \le p < \infty$ and $V\colon L_p(\mathcal{N}_1) \to L_p(\mathcal{N}_2)$, $W\colon L_p(\mathcal{M}_1) \to L_p(\mathcal{M}_2)$ be linear maps, then

$$\|V \otimes W\colon L_p(\mathcal{N}_1 \otimes \mathcal{M}_1) \to L_p(\mathcal{N}_2 \otimes \mathcal{M}_2)\| \le \|V\|_{cb}\|W\|_{cb},$$

where

$$\|V\|_{cb} = \|V \otimes id_{L_p(B(\ell_2))}\colon L_p(\mathcal{N}_1 \otimes B(\ell_2)) \to L_p(\mathcal{N}_2 \otimes B(\ell_2))\|.$$

(iv) [J2] Let $1 \le p \le q < \infty$ and $V\colon L_q(\mathcal{N}_1) \to L_p(\mathcal{M}_1)$, $W\colon L_q(\mathcal{N}_2) \to L_p(\mathcal{M}_2)$ be completely positive maps, then

$$\|V \otimes W\colon L_q(\mathcal{N}_1 \otimes \mathcal{N}_2) \to L_p(\mathcal{M}_1 \otimes \mathcal{M}_2)\| \le \|V\|\|W\|.$$

Here $\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}_1$ and $\mathcal{M}_2$ are assumed to be von Neumann algebras with the QWEP and $L_p(\mathcal{N}_1 \otimes \mathcal{M}_1)$ refers to Haagerup L_p-space associated to the von Neumann algebra $\mathcal{N}_1 \bar{\otimes} \mathcal{M}_1$. Let us recall that a C^*-algebra has the *weak expectation property of Lance (WEP)*, if $A \subset A^{**} \subset B(H)$ is given in its universal representation and there exists a contraction $P\colon B(H) \to A^{**}$ such that $P|_A = id_A$. A C*-algebra B is a *quotient of a C^*-algebra with WEP, in short B is QWEP*, if there exists a C^*-algebra A with WEP and a two sided ideal I in A such that $B \cong A/I$. It is open whether every C^*-algebra is QWEP. (See Kirchberg's work [Ki] for many important equivalent conjectures.) In the classical commutative theory the applications (iii) and (iv) are easy consequences of Fubini's theorem and basic facts about vector-valued L_p-spaces. Our proof of (iii) and (iv) requires an extension of Pisier's [P7] notion of noncommutative vector-valued L_p-spaces to QWEP algebras and will be given in [J2] based on the following main result of this paper.

Theorem 0.1 *Let $1 \le p < \infty$ and $(\mathcal{M}_i)_{i\in I}$ $(\mathcal{N}_j)_{j\in J}$ be families of von Neumann algebras. Let $\mathcal{U}$, $\mathcal{U}'$ be ultrafilters on I, J, respectively. Let $n \in \mathbb{N}$, $x_1,\dots,x_n$ in $\prod L_p(\mathcal{M}_i)$ and $y_1,\dots,y_n$ in $\prod L_p(\mathcal{N}_j)$, then*

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j)\Big\|_{L_p(\mathcal{M}_i \otimes \mathcal{N}_j)} = \lim_{j,\mathcal{U}'} \lim_{i,\mathcal{U}} \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j)\Big\|_{L_p(\mathcal{M}_i \otimes \mathcal{N}_j)}$$

A reader not familiar with Haagerup's L_p-spaces is advised to assume that all the von Neumann algebras $\mathcal{M}_i$ and $\mathcal{N}_j$ are semifinite. However, even for finite von

Neumann algebras our proof of Fubini's theorem relies on modular theory and an approximation result derived from Kaplansky's density Theorem. However, in the type III case, we first have to define and study the canonical tensor product map $I_p\colon L_p(\mathcal{M}) \otimes L_p(\mathcal{N}) \to L_p(\mathcal{M} \otimes \mathcal{N})$ (see Proposition 3.6) which is (implicitly) used in the formulation of Theorem 0.1. The proof of Theorem 0.1 is based on Raynaud's fundamental work on ultraproducts of noncommutative L_p-spaces. Following a suggestion of G. Elliott, we prove in the appendix that the conclusion fails for $p = \infty$.

The paper is organized as follows: after some preliminaries in section one, we develop a theory of vector-valued L_1 spaces for duals of injective von Neumann algebras in section 2. In this case tensor norms for operator spaces can be used to prove the Fubini theorem. We refer to [J2] for the extension of the vector-valued theory to the case $p > 1$ and further applications. Section 2 is devoted to prove Theorem 0.1 for $p = 1$ in the general case. In the third part, we prove the Fubini Theorem 0.1 for $p > 1$ and show that both expressions coincide with the norm of finite rank tensors in the L_p-tensor product of two (suitable chosen) noncommutative L_p-spaces.

1 Preliminaries

We use standard notation in operator algebras as in [Tk, KR]. In particular, $B(H)$ denotes the bounded operators on a Hilbert H. For $n \in \mathbb{N}$, we denote by $M_n = B(\ell_2^n)$ the space of $n \times n$ matrices. Given two C^*-algebras $A \subset B(H_1)$ and $B \subset B(H_2)$, the *minimal tensor product* $A \otimes_{\min} B$ is the closure of the algebraic tensor product $A \otimes B$ with respect to the norm induced by the inclusion $A \otimes_{\min} B \subset B(H_1 \otimes H_2)$. A *von Neumann algebra* is a σ-weakly closed unital *-subalgebra of the bounded operators on a Hilbert space. For von Neumann algebras $\mathcal{N} \subset B(H_1)$, $\mathcal{M} \subset B(H_2)$, the von Neumann algebra tensor product $\mathcal{N}\bar{\otimes}\mathcal{M}$ is the σ-weak closure of $\mathcal{N} \otimes_{\min} \mathcal{M} \subset B(H_1 \otimes H_2)$. We refer to [Tk, KR] for the relevant locally convex topologies on operator algebras. Every von Neumann algebra $\mathcal{N}$ has a unique predual $\mathcal{N}_*$ consisting of the *normal* functionals. Indeed, a functional $\phi \in \mathcal{N}^*$ is normal if and only if it is the restriction of the functional $T \mapsto \operatorname{tr}(AT)$ to $\mathcal{N}$ for some A in the trace class operators $S_1(H)$ on H. Let us recall that every functional in $\mathcal{N}^*$ has a decomposition in its normal and singular part. Moreover, the projection of $\mathcal{N}^*$ onto the normal functionals $\mathcal{N}_*$ is given by multiplication with a central projection in $\mathcal{N}^{**}$, see [Tk]. In the following we will often use the existence of a strictly, semifinite normal weight. Indeed, given a maximal family $(q_j, \phi_j)_{j\in J}$ of mutually disjoint projections (q_j) and normal faithful state ϕ_j on $q_j\mathcal{N}q_j$, we will have $\sum_j q_j = 1$. Then, $w(x) = \sum_j \phi_j(x)$ is defined on a σ-weakly dense subalgebra. Indeed, we can use as index sets the finite subsets $P_f(J)$ of J and for $J' \in P_f(J)$, we set $q_{J'} = \sum_{j\in J'} q_j$. Then $w_{J'} = \sum_{j\in J'} \phi_j$ is a positive functional on $\mathcal{N}$ such that $w = \lim_{J'} w_{J'}$. In the following, we will also use the obvious facts that $q_{J'}w = wq_{J'}$ and the modular group σ_t^w satisfies $\sigma_t^w(q_{J'}) = q_{J'}$. Given a family of Banach spaces $(X_i)_{i\in I}$, we denote by

$$\prod X_i = \prod\nolimits_{i\in I} X_i = \{(x_i) \mid \forall_{i\in I} x_i \in X_i, \|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty\},$$

$$\sum X_i = \sum\nolimits_{i\in I} X_i = \{(x_i) \mid \forall_{i\in I} x_i \in X_i, \|(x_i)\| = \sum_{i\in I} \|x_i\| < \infty\}$$

the corresponding *product and sum space*. Note that if (A_i) is a family of C^*-algebras, then $\prod A_i$ is again a C^*-algebra. Similarly, the sum $\sum_i (\mathcal{N}_i)_*$ of preduals of von Neumann algebras is the predual of $\prod \mathcal{N}_i$.

Let us also recall some basic notions in the theory of operator spaces as they can be found in [ER3]. An *operator space* X is a Banach space together with a specified isometric embedding $J\colon X \to B(H)$. This embedding induces matrix norms $\| \ \|_{M_n(X)}$ on $M_n \otimes X$ defined for $x = (x_{ij})$ by

$$\|x\|_{M_n(X)} = \|(J(x_{ij}))_{i,j=1}^n\|_{B(\ell_2^n(H))}.$$

These matrix norms satisfy Ruan's axioms

(R1) $\|(a \otimes id_X)x(b \otimes id_X)\|_{M_n(X)} \le \|a\|\|x\|_{M_n(X)}\|b\|$ for all $x \in M_n(X)$ and $a, b \in M_n$.
(R2) For $x \in M_n(X)$ and $y \in M_m(X)$ the norm of $x \oplus y = \left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]$ satisfies

$$\|x \oplus y\|_{M_{n+m}(X)} = \max\{\|x\|_{M_n(X)}, \|y\|_{M_m(X)}\}.$$

Conversely, every sequence $\alpha = (\| \ \|_n)_{n\in\mathbb{N}}$ of norms on $(M_n \otimes X)_{n\in\mathbb{N}}$ satisfying the axioms (R1) and (R2) can be obtained from a suitable embedding $J\colon X \to B(H_\alpha)$, see [R1]. Therefore, we will say that an *operator space structure* is either given by a sequence of norms α satisfying Ruan's axioms or a concrete embedding J_α. The morphisms in the category of operator spaces are the *completely bounded linear maps*, *i.e.* the linear maps $T\colon X_1 \to X_2$ such that

$$\|T\|_{cb} = \sup_n \|id_{M_n} \otimes T\colon M_n(X_1) \to M_n(X_2)\| < \infty.$$

We denote by $CB(X_1, X_2)$ the Banach (operator) space of completely bounded maps between X_1 and X_2. A linear map T with $\|T\|_{cb} \le 1$ is called a *complete contraction*. A linear map $T\colon X_1 \to X_2$ is called *completely isometric* if $id_{M_n} \otimes T$ is isometric for all $n \in \mathbb{N}$. Similarly, a map $T\colon X_1 \to X_2$ is called a *complete quotient map* if $id_{M_n} \otimes T$ maps the open unit ball in $M_n(X_1)$ onto the open unit ball of $M_n(X_2)$. An important feature in the theory of operator spaces is duality. More precisely, for every operator space X and a matrix $x^* = (x_{ij}^*)_{i,j=1}^n \subset X^*$ of functionals we may consider the linear map $T_{x^*}\colon X \to M_n$ defined by $T_{x^*}(x) = (x_{ij}^*(x))_{i,j=1}^n$. Then the sequence of norms on $M_n(X^*)$ given by

$$\|x^*\|_{M_n(X^*)} = \|T_{x^*}\colon X \to M_n\|_{cb}$$

satisfies Ruan's axioms (R1) and (R2) and defines an operator space structure on X^*. This is called the *standard dual* of X, see [BP, ER3]. Using this duality, X is completely isometrically embedded into its bidual X^{**} and for every completely bounded map $T\colon X_1 \to X_2$, we have

$$\|T\colon X_1 \to X_2\|_{cb} = \|T^*\colon X_2^* \to X_1^*\|_{cb}.$$

Clearly, the adjoint of a complete quotient map is a complete isometry. Using Wittstock's extension Theorem, see [Pa], for M_m, it is easily seen that the dual of a complete isometry is a complete quotient map.

Natural examples of operator spaces are C^*-algebras, von Neumann algebras and their duals, preduals, respectively. As for C^*-algebras the *minimal tensor product* $X_1 \otimes_{\min} X_2$ of operator spaces $X_1 \subset B(H_1)$ and $X_2 \subset B(H_2)$ is defined as the closure of $X_1 \otimes X_2$ with respect to the norm induced by the inclusion

$$X_1 \otimes_{\min} X_2 \subset B(H_1) \otimes_{\min} B(H_2) \subset B(H_1 \otimes H_2).$$

Note that every element $x \in X_1 \otimes X_2$ induces a linear map $T_x \colon X_1^* \to X_2$ such that $\|x\|_{X_1 \otimes_{\min} X_2} = \|T_x \colon X_1^* \to X_2\|_{cb}$. As in the category of Banach spaces there is also a largest tensor norm $\|I\|_\wedge$ on $X_1 \otimes X_2$ such that

$$(X_1 \widehat{\otimes} X_2)^* \cong CB(X_1, X_2^*) \cong CB(X_2, X_1^*).$$

There are several ways to define this *operator space projective tensor product*, see [ER3, BP]. For example, it is easily checked that for every operator space X there is a complete quotient map $q \colon S_1(H) \to X$. Let $q_1 \colon S_1(H_1) \to X_1$ and $q_2 \colon S_1(H_2) \to X_2$ be such complete quotient maps, respectively, then

$$q_1 \otimes q_2 \colon S_1(H_1 \otimes H_2) \to X_1 \widehat{\otimes} X_2$$

is a quotient map and defines the norm in $X_1 \widehat{\otimes} X_2$. The operator space projective tensor product is functorial in the sense that

$$\|T_1 \otimes T_2 \colon X_1 \widehat{\otimes} X_2 \to Y_1 \widehat{\otimes} Y_2\| \le \|T_1\|_{cb} \|T_2\|_{cb}$$

for all operator spaces X_1, X_2, Y_1, Y_2 and completely bounded maps $T_1 \colon X_1 \to Y_1$, $T_2 \colon X_2 \to Y_2$. Let us note that the operator space projective tensor product is symmetric, *i.e.,* the flip map constitutes a complete isometry between $X_1 \widehat{\otimes} X_2$ and $X_2 \widehat{\otimes} X_1$. Moreover, $S_1(H_1) \widehat{\otimes} S_1(H_2) = S_1(H_1 \otimes H_2)$, see [ER3] for more information.

A von Neumann algebra $\mathcal{N}$ is called *injective*, if for every completely isometric inclusion of operator spaces $X_1 \subset X_2$ and every completely bounded map $T \colon X_1 \to \mathcal{N}$, there exists an extension $\widehat{T} \colon X_2 \to \mathcal{N}$ such that $\widehat{T}|_{X_1} = T$ and $\|\widehat{T}\|_{cb} = \|T\|_{cb}$. We refer to Connes' and Haagerup's work [C1, Ha1] for the equivalence between injectivity and hyperfiniteness for von Neumann algebras. We refer to [Se, Ne, Ko, Te1, Fi] for general information on noncommutative L_p-spaces and their operator space structure (see also section 3 below).

2 QWEP Algebras and Tensor Norms

In this part, we will prove the Fubini Theorem for $p = 1$ and start the investigation of vector-valued noncommutative L_1-spaces for QWEP algebras. The extension to vector-valued noncommutative L_p-spaces will be discussed in a forthcoming paper [J2] and allows us to obtain the applications mentioned in the introduction. However, the case $p = 1$ is fundamental for this investigation and relies on the operator space projective tensor product introduced by Effros/Ruan [ER3]. Pisier [P7] extended the notion of noncommutative vector-valued L_p-spaces for $1 < p < \infty$ first in the discrete case, *i.e.*, the underlying von Neumann algebra is $B(\ell_2)$, and then by approximation to the hyperfinite case [P7]. In this paper, we will use ultraproducts to extend this concept to the QWEP case in the case $p = 1$ and provide the fundamental tools for the general case $p > 1$ in [J2]. Vector-valued L_1-spaces and the Fubini theorem for duals of injective von Neumann algebras are closely related and can be obtained by introducing suitable tensor norms. The general case in Fubini's theorem is based on similar ideas. We will first discuss some equivalent reformulations of QWEP for von Neumann algebras.

Lemma 2.1 *Let B be a C^*-algebra. B is QWEP if and only if B^{**} is isomorphic to a von Neumann subalgebra of $B(H)^{**}$ (for some H) which is the range of a normal (not necessarily faithful) conditional expectation. If B is in addition a von Neumann algebra, then B is QWEP if B is itself isomorphic to such a von Neumann subalgebra of $B(H)^{**}$.*

Proof Let $A \subset A^{**} \subset B(H)$ be a C^*-algebra with WEP. Consider $P\colon B(H) \to A^{**}$ such that $P|_A = id_A$. Let $j\colon A^* \to B(H)^*$ be the restriction $j = P^*|_{A^*}$ of the dual map to A^*. Then $E = j^*\colon B(H)^{**} \to B^{**}$ is normal. For every $a \in A$, we have $E(a) = a$. Since every element $a^{**} \in A^{**}$ can be approximated in the σ-weak topology by a bounded net of elements in A, the weak*-continuity then implies $E(a^{**}) = a^{**}$ and in particular $E(1) = 1$. Hence, E is a normal conditional expectation, see [Tk, Theorem III 3.4.], and the assertion holds for A. If $B = A/I$ is a quotient of A, then there exists a central projection z such that $B^{**} \cong zA^{**}$ and $B^* \cong zA^*$. Hence, $E_B(x) = zE(x)$ yields a normal conditional expectation from $B(H)^{**}$ onto B^{**}. Let $\mathcal{N}$ be a von Neumann algebra which is QWEP and let $E\colon B(H)^{**} \to \mathcal{N}^{**}$ be a normal conditional expectation onto $\mathcal{N}^{**}$. If $i\colon \mathcal{N}_* \to \mathcal{N}^*$ denotes the natural inclusion map, then $E_1 = i^*\colon \mathcal{N}^{**} \to \mathcal{N}$ is a normal conditional expectation and hence $E_1E\colon B(H)^{**} \to \mathcal{N}$ is a normal conditional expectation onto $\mathcal{N}$. For the converse, we assume that $B \subset B^{**} \subset B(H)^{**}$ and $E\colon B(H)^{**} \to B^{**}$ is a normal conditional expectation. We note that $B(H)$ is injective and hence has WEP. Therefore $B(H)^{**}$ is QWEP according to [Ki, Corollary 3.3.]. (Indeed, $B(H)^{**}$ is a quotient of $C^{st}_\infty(S; B(H))$ the strong* convergent bounded sequences in $B(H)$ over a suitable index set S.) Then B is relatively weakly injective (r.w.i.) in $B(H)^{**}$. According to [Ki, Corollary 3.3.], we deduce that B is QWEP. ■

Our first step in extending the theory of noncommutative vector-valued L_1-spaces will be to define a suitable norm on the tensor product of an operator space and the dual of an injective von Neumann algebra. To be more precise let $\mathcal{N}$ be an injective von Neumann algebra and F be a finite dimensional operator space, then we define the space $\mathcal{N}^*[F]$ by specifying a norm on the tensor product $\mathcal{N}^* \otimes F$ as follows

$$\mathcal{N}^*[F] = (F^* \otimes_{\min} \mathcal{N})^*.$$

In other words, given a tensor $x = \sum_{i=1}^m r_i^* \otimes f_i$, we consider the associated linear map $S_x\colon \mathcal{N} \to F$ defined by $S_x(r) = \sum_{i=1}^m r_i^*(r) f_i$ and have

$$\|x\|_{\mathcal{N}^*[F]} = \sup\{tr(S_x T) \mid \|T\|_{cb} \le 1\} = \iota(S_x).$$

Here $\iota(S)$ denotes the operator integral norm, see [EJR, ER3]. By the definition of the operator space dual F^* and using that F^* is finite dimensional, we obtain a natural isomorphism ([ER2, BP])

$$F^* \otimes_{\min} \mathcal{N} \cong cb(F, \mathcal{N}).$$

Indeed, a tensor $y = \sum_{i=1}^m f_i^* \otimes r_i$ corresponds to the linear map defined by

$$T_y(f) = \sum_{i=1}^m f_i^*(f) r_i \text{ and } \|y\|_{F^* \otimes_{\min} \mathcal{N}} = \|T_y\|_{cb}. \tag{2.1}$$

Lemma 2.2 *Let $\mathcal{N}$ be an injective von Neumann algebra, $F_1 \subset F_2$ finite dimensional operator spaces, $T\colon F_2 \to F_3$ a linear map between finite dimensional operator spaces and $q\colon G_1 \to G_2$ a complete quotient map between finite dimensional spaces, then*

(i) $\mathcal{N}^*[F_1]$ *is isometrically embedded into* $\mathcal{N}^*[F_2]$.
(ii) $id_{\mathcal{N}^*} \otimes T\colon \mathcal{N}^*[F_2] \to \mathcal{N}^*[F_3]$ *satisfies*

$$\|id_{\mathcal{N}^*} \otimes T\colon \mathcal{N}^*[F_2] \to \mathcal{N}^*[F_3]\| \le \|T\|_{cb}.$$

(iii) $id_{\mathcal{N}^*} \otimes q\colon \mathcal{N}^*[G_1] \to \mathcal{N}^*[G_2]$ *is a quotient map.*

Proof (i) Let $i\colon F_1 \to F_2$ be the natural, completely isometric inclusion map. Then $q = i^*\colon F_2^* \to F_1^*$ is a quotient map. The assertion follows by duality provided

$$q^* \otimes id_N\colon F_2^* \otimes_{\min} \mathcal{N} \to F_1^* \otimes_{\min} \mathcal{N}$$

is a quotient map. Indeed, let $y \in F_1^* \otimes_{\min} \mathcal{N}$ and let $T_y\colon F_1 \to \mathcal{N}$ be its associated linear map. By injectivity of $\mathcal{N}$, we can find an extension $T\colon F_2 \to \mathcal{N}$ such that

$$\|T\|_{cb} \le \|T_y\|_{cb} = \|y\|_{F_1^* \otimes_{\min} \mathcal{N}}.$$

Since F_2^* is finite dimensional there exists an element $\widehat{y} \in F_2^* \otimes_{\min} \mathcal{N}$ such that $T_{\widehat{y}} = T$ and $(q \otimes id_{\mathcal{N}})(\widehat{y}) = y$. Using (2.1), we deduce the assertion.

(ii) Let $T\colon F_2 \to F_3$ be a linear map, then

$$\|T^* \otimes id_{\mathcal{N}}\colon F_3^* \otimes_{\min} \mathcal{N} \to F_2^* \otimes_{\min} \mathcal{N}\| \le \|T^*\|_{cb} = \|T\|_{cb}.$$

By duality, we deduce

$$\|id_{\mathcal{N}^*} \otimes T\colon \mathcal{N}^*[F_2] \to \mathcal{N}^*[F_3]\| \le \|T^* \otimes id_{\mathcal{N}}\colon F_3^* \otimes_{\min} \mathcal{N} \to F_2^* \otimes_{\min} \mathcal{N}\| \le \|T\|_{cb}.$$

(iii) Since $q\colon G_1 \to G_2$ is a complete quotient map, we see that $q^*\colon G_2^* \to G_1^*$ is a completely isometric inclusion. Therefore

$$q^* \otimes id_N\colon G_2^* \otimes_{\min} \mathcal{N} \to G_1^* \otimes_{\min} \mathcal{N}$$

is an isometric embedding. The Hahn-Banach theorem implies the assertion. ■

So far, the space $\mathcal{N}^*[F]$ is only defined for a finite dimensional space. Let us recall that for a Banach space X, the space $L_1(\Omega,\mu;X)$ of Bochner integrable functions is the norm closure of simple functions and therefore

$$L_1(\Omega,\mu;X) = L_1(\Omega,\mu)\otimes_\pi X.$$

Here π refers to the Banach space projective tensor product and the norm of a finite tensor can be calculated using a finite dimensional subspace of X. We pursue a similar approach in the noncommutative case.

Definition and Remark 2.3 Let X be an operator space and $\mathcal{N}$ be an injective von Neumann algebra, then $\mathcal{N}^*[X]$ is defined to be the completion of $\bigcup_{F\subset X}\mathcal{N}^*[F]$ where the union is taken over all finite dimensional subspaces F of X. Due to Lemma 2.2, $\mathcal{N}^*[X]$ is well-defined.

(i) If $X_1\subset X_2$ is a subspace, then $\mathcal{N}^*[X_1]\subset\mathcal{N}^*[X_2]$ completely isometrically.
(ii) If $T\colon X_2\to X_3$ is completely bounded then

$$\|id_{\mathcal{N}^*}\otimes T\colon \mathcal{N}^*[X_2]\to\mathcal{N}^*[X_3]\|\le\|T\|_{cb}.$$

In the following, we want to identify $\mathcal{N}^*[A^*]$ for the dual of a C^*-algebra A. These results are related to local reflexivity of duals of C^*-algebras [EJR]. We will have to use two facts related to this problem. The first fact is a direct consequence of Kaplansky's density Theorem [Tk, Theorem II.4.8]. For the second fact, we refer to [ER2, BP].

Fact 2.4 Let A and B be two C^*-algebras. Then the unit ball of $A\otimes_{\min}B$ is dense in the unit ball of the von Neumann algebra tensor product $A^{**}\bar{\otimes}B^{**}$ with respect to the $\sigma(A^{**}\bar{\otimes}B^{**},A^*\otimes B^*)$-topology.

Fact 2.5 (Effros-Ruan) Let A and B be C^*-algebras. Then

$$(A^*\widehat{\otimes}B^*)^* = CB(A^*,B^{**}) = A^{**}\bar{\otimes}B^{**}$$

holds isometrically. Here for an element $z\in A^{**}\bar{\otimes}B^{**}$, the linear map $T_z\colon A^*\to B^{**}$ is defined by

$$T_z(a^*)(b^*) = \langle z, a^*\otimes b^*\rangle. \tag{2.2}$$

Lemma 2.6 *Let $\mathcal{N}$ be an injective von Neumann algebra, A a C^*-algebra and $F\subset A^*$ a finite dimensional subspace with quotient map $q\colon A^{**}\to F^*$, then*

$$q\otimes id_{\mathcal{N}}\colon A^{**}\otimes_{\min}\mathcal{N}\to F^*\otimes_{\min}\mathcal{N}$$

maps the open unit ball onto the open unit ball.

Proof Let $x \in F^* \otimes_{\min} \mathcal{N}$ be an element of norm $\|x\| < 1$ and $T_x \colon F \to \mathcal{N}$ be the corresponding linear map. Let $(f_i)_{i=1}^n \subset F$, $(f_i^*)_{i=1}^n \subset F^*$ be an Auerbach basis of F, *i.e.*, $\|f_i\| \le 1$, $\|f_j^*\| \le 1$ and $f_j^*(f_i) = \delta_{ij}$. Let $a_j \in A^{**}$, $j = 1, \dots, n$ be norm preserving extensions of the f_j^*'s. Using the injectivity of $\mathcal{N}$, there exists an extension $T \colon A^* \to \mathcal{N}$ of T_x with the same cb-norm. According to Fact 2.5 the map T corresponds to a norm one element z in $A^{**}\bar{\otimes}\mathcal{N}$ and according to Fact 2.4, we can find a net (z_λ) in the unit ball of $A \otimes_{\min} \mathcal{N}$ converging to z with respect to the $\sigma(A^{**}\bar{\otimes}\mathcal{N}^{**}, A^* \otimes \mathcal{N}^*)$-topology. Hence, the corresponding net (T_λ) of maps $T_\lambda \colon A^* \to \mathcal{N}$ converges in the point-weak topology to T. A convex combination of those maps converges in the point-norm topology to T. Therefore, given $\varepsilon = \frac{1-\|x\|}{2}$, we can find a finite rank map $T_1 \colon A^* \to \mathcal{N}$ such that $\|T_1\|_{cb} \le \|x\|$ and

$$\|T_1(f_i) - T(f_i)\| \le \frac{\varepsilon}{n}$$

for $i = 1, \dots, n$. Consider

$$T_2 = \sum_{i=1}^n a_i \otimes [T(f_i) - T_1(f_i)],$$

then $T_1 + T_2|_F = T_x$. Following Fact 2.5, we see that the finite rank map $T = T_1 + T_2$ corresponds to a tensor $z' \in A^{**} \otimes \mathcal{N}$ such that $T_{z'} = T$. In particular, $T|_F = T_x$ implies $(q \otimes id_{\mathcal{N}})(z') = z$ and

$$\|z'\|_{A^{**}\otimes_{\min}\mathcal{N}} = \|T_1 + T_2\|_{cb} \le \|x\| + \sum_{i=1}^n \|a_i\|\|T_1(f_i) - T(f_i)\| \le \|x\| + \varepsilon < 1. \blacksquare$$

Lemma 2.7 *Let $\mathcal{N}$ be an injective von Neumann algebra and A be a C^*-algebra, then $\mathcal{N}^*[A^*] = \mathcal{N}^* \widehat{\otimes} A^*$ holds isometrically.*

Proof Since both spaces $\mathcal{N}^* \widehat{\otimes} A^*$ and $\mathcal{N}^*[A^*]$ are defined as the closure of the finite rank tensors, we only have to prove that for finite rank tensors the norms coincide. Let $x = \sum_{j=1}^m a_j^* \otimes r_j^*$ and $F = \operatorname{span}\{a_1^*, \dots, a_n^*\} \subset A^*$ be given. According to Lemma 2.6 and using the fact that a Banach space is isometrically embedded in its bidual, we get

$$\begin{aligned}\|x\|_{(F^*\otimes_{\min}\mathcal{N})^*} &= \|x\|_{(A^{**}\otimes_{\min}\mathcal{N})^*} \le \|x\|_{(A^{**}\bar{\otimes}\mathcal{N}^{**})^*} \\ &= \|x\|_{(\mathcal{N}^*\widehat{\otimes}A^*)^{**}} = \|x\|_{\mathcal{N}^*\widehat{\otimes}A^*}.\end{aligned}$$

Conversely, we apply the Hahn-Banach theorem and find an element in the unit ball of $A^{**}\bar{\otimes}\mathcal{N}^{**}$ such that

$$\|x\|_{\mathcal{N}^*\widehat{\otimes}A^*} = |\langle x, z\rangle|\,.$$

According to Fact 2.4, we can find a net z_λ in the unit ball of $\mathcal{N} \otimes_{\min} A$ such that

$$|\langle x, z\rangle| = \lim_\lambda |\langle x, z_\lambda\rangle|\,.$$

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Let $q\colon A^{**} \to F^*$ be the quotient map, then $id_{\mathcal{N}} \otimes q(z_\lambda)$ is in the unit ball of $F^* \otimes_{\min} \mathcal{N}$ and hence

$$|\langle x, z_\lambda \rangle| = |\langle x, id_{\mathcal{N}} \otimes q(z_\lambda)\rangle| \leq \|x\|_{\mathcal{N}^*[F]} \|id_{\mathcal{N}} \otimes q(z_\lambda)\|_{\mathcal{N} \otimes_{\min} F^*} \leq \|x\|_{\mathcal{N}^*[F]}.$$

Passing to the limit, we deduce $\|x\|_{\mathcal{N}^* \widehat{\otimes} A^*} = |\langle x, z \rangle| \leq \|x\|_{\mathcal{N}^*[F]}$. The assertion is proved. ■

Corollary 2.8 *Let $\mathcal{N}$ and $\mathcal{M}$ be injective von Neumann algebras, then*

$$\mathcal{N}^*[\mathcal{M}^*] \cong \mathcal{M}^*[\mathcal{N}^*]$$

isometrically.

Proof By symmetry in the definition of the operator space projective tensor product, we get

$$\mathcal{N}^*[\mathcal{M}^*] = \mathcal{N}^* \widehat{\otimes} \mathcal{M}^* \cong \mathcal{M}^* \widehat{\otimes} \mathcal{N}^* = \mathcal{M}^*[\mathcal{N}^*]$$ ■

Definition and Remark 2.9 Let A be a C^*-algebra with QWEP and let $E\colon B(H)^{**} \to A^{**}$ be a normal conditional expectation with predual map $E_*\colon A^* \to B(H)^*$ (for the existence, we refer to Lemma 2.1), then for every operator space X we define

$$A^*[X] = cl(A^* \otimes X)$$

to be the closure of the finite rank tensors $A^* \otimes X$ with respect to the norm

$$\|x\|_{A^*[X]} = \|E_* \otimes id_X(x)\|_{B(H)^*[X]}.$$

If $X_1 \subset X_2$ subspace (in the sense of operator spaces), then the inclusion map

$$A^*[X_1] \subset A^*[X_2]$$

is isometric. For any completely bounded map $T\colon X_2 \to X_3$

$$\|id_{A^*} \otimes T\colon A^*[X_2] \to A^*[X_3]\| \leq \|T\|_{cb}.$$

Formally this definition depends on E. However, the results in this paper are independent of the particular choice of E. In applications, we will have a canonical choice of E by verifying that A is QWEP.

Corollary 2.10 *Let A be C^*-algebra with QWEP and B be an arbitrary C^*-algebra, then $A^*[B^*] = A^* \widehat{\otimes} B^*$ isometrically. If moreover, B is QWEP, then $A^*[B^*] = A^* \widehat{\otimes} B^* \cong B^* \widehat{\otimes} A^* = B^*[A^*]$ isometrically.*

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Proof Fix $E_*\colon A^* \to B(H)^*$ as it is used to define the norm in $A^*[X]$. By Lemma 2.7 applied to $B(H)$, we have $B(H)^* \widehat{\otimes} B^* = B(H)^*[B^*]$. Hence,

$$E_* \otimes id_{B^*}\colon A^* \widehat{\otimes} B^* \to B(H_1)^* \widehat{\otimes} B^* = B(H)^*[B^*]$$

is continuous. Therefore, we get

$$\|x\|_{A^*[B^*]} = \|E_* \otimes id_{B^*}(x)\|_{B(H)^* \widehat{\otimes} B^*} \leq \|x\|_{A^* \widehat{\otimes} B^*}.$$

Let $\pi\colon A^{**} \to B(H)^{**}$ be the normal embedding and $\pi_*\colon B(H)^* \to A^*$ its predual map. Then $\pi_* E_* = id_{A^*}$ and π_* is completely contractive. Hence

$$\|x\|_{A^* \widehat{\otimes} B^*} = \|(\pi_* E_* \otimes id_{B^*})(x)\|_{A^* \widehat{\otimes} B^*} \leq \|\pi_*\|_{cb} \|E_* \otimes id_{B^*}(x)\|_{B(H)^* \widehat{\otimes} B^*} \leq \|x\|_{A^*[B^*]}.$$

If B has QWEP, the assertion follows from the symmetry of the operator space projective tensor product. ■

We want to apply similar techniques to ultraproducts. Let us recall that the ultraproduct $\prod_{\mathcal{U}} X_i$ of a family of Banach spaces is defined as the quotient $\prod_{i\in I} X_i / \operatorname{Ker}_{\mathcal{U}}$ where

$$\operatorname{Ker}_{\mathcal{U}} = \{(x_i)_{i\in I} \mid \lim_{i,\mathcal{U}} \|x_i\|_{X_i} = 0\}.$$

Let (A_i) be a family of C^*-algebras. According to a result of Groh [Gr], we know that

$$\mathcal{N}_{\mathcal{U}} = \Big(\prod\nolimits_{\mathcal{U}} A_i^* \Big)^*$$

is a von Neumann algebra. Indeed,

$$\prod\nolimits_{\mathcal{U}} A_i^* \subset \Big(\prod\nolimits_{i\in I} A_i \Big)^*$$

is invariant under the multiplication from the left and the right by elements in $\prod_{i\in I} A_i$ and according to [Tk, Theorem 2.7.] there exists a central projection $z_{\mathcal{U}} \in (\prod_{i\in I} A_i)^{**}$ such that

$$\prod\nolimits_{\mathcal{U}} A_i^* = z_{\mathcal{U}} \Big(\prod\nolimits_{i\in I} A_i \Big)^*.$$

Therefore $\mathcal{N}_{\mathcal{U}} \cong z_{\mathcal{U}} (\prod_{i\in I} A_i)^{**}$ is a von Neumann algebra such that $(\mathcal{N}_{\mathcal{U}})_* = \prod_{\mathcal{U}} A_i^*$.

Proposition 2.11 *Let (A_i), (B_j) be two families of C^*-algebras and $\mathcal{U}$, $\mathcal{U}'$ be two ultrafilters on the index set I, J, respectively. Let $\sum_{k=1}^n a_k^* \otimes b_k^* \in \prod_{\mathcal{U}} A_i^* \otimes \prod_{\mathcal{U}'} B_j^*$ be a finite rank tensor. Then*

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^n a_k^*(i) \otimes b_k^*(j) \Big\|_{A_i^* \widehat{\otimes} B_j^*} = \Big\| \sum_{k=1}^n a_k^* \otimes b_k^* \Big\|_{(\prod_{\mathcal{U}} A_i^*) \widehat{\otimes} (\prod_{\mathcal{U}'} B_j^*)}$$
$$= \lim_{j,\mathcal{U}'} \lim_{i,\mathcal{U}} \Big\| \sum_{k=1}^n a_k^*(i) \otimes b_k^*(j) \Big\|_{A_i^* \widehat{\otimes} B_j^*}.$$

Proof For $i \in I$ and $j \in J$, we denote by $\mathcal{M}_i = A_i^{**}$ and $\mathcal{N}_j = B_j^{**}$, the corresponding von Neumann algebras. Let $P_i\colon \mathcal{M}_i^* \to A_i^* = (\mathcal{M}_i)_*$, $Q_j\colon \mathcal{N}_j^* \to B_j^* = (\mathcal{N}_j)_*$ be the projections onto the normal parts, respectively. Then $P = (P_i)\colon \prod_{\mathcal{U}} \mathcal{M}_i^* \to \prod_{\mathcal{U}} A_i^*$, $Q = (Q(j))\colon \prod_{\mathcal{U}'} \mathcal{N}_j^* \to \prod_{\mathcal{U}'} B_j^*$ are completely contractive projections. Let $d = \sum_{k=1}^n a_k^* \otimes b_k^*$. From $P \otimes Q(d) = d$, we deduce

$$\Big\| \sum_{k=1}^n a_k^* \otimes b_k^* \Big\|_{(\prod_{\mathcal{U}} \mathcal{M}_i^*)\widehat{\otimes}(\prod_{\mathcal{U}'} \mathcal{N}_j^*)} = \Big\| \sum_{k=1}^n a_k^* \otimes b_k^* \Big\|_{(\prod_{\mathcal{U}} A_i^*)\widehat{\otimes}(\prod_{\mathcal{U}'} B_j^*)}.$$

Let $z_{\mathcal{U}}, z_{\mathcal{U}'}$ be the central projections such that $\prod_{\mathcal{U}} \mathcal{M}_i^* = z_{\mathcal{U}} \left(\prod \mathcal{M}_i\right)^*$, $\prod_{\mathcal{U}'} \mathcal{N}_j^* = z_{\mathcal{U}'} \left(\prod \mathcal{N}_j\right)^*$, respectively. Define the functional f by

$$\langle f, m \otimes n\rangle = \sum_{k=1}^n \lim_{i,\mathcal{U}} \langle m(i), a_k^*(i)\rangle \lim_{j,\mathcal{U}'} \langle n(j), b_k^*(j)\rangle,$$

then $(z_{\mathcal{U}} \otimes z_{\mathcal{U}'}) f = f$ and

$$\|f\|_{(\prod_{i\in I} \mathcal{M}_i)^* \widehat{\otimes} (\prod_{j\in J} \mathcal{N}_j)^*} = \Big\| \sum_{k=1}^n a_k^* \otimes b_k^* \Big\|_{(\prod_{\mathcal{U}} \mathcal{M}_i^*)\widehat{\otimes}(\prod_{\mathcal{U}'} \mathcal{N}_j^*)}.$$

By symmetry it therefore suffices to show the following equality

$$\|f\|_{(\prod_{i\in I} \mathcal{M}_i)^* \widehat{\otimes} (\prod_{j\in J} \mathcal{N}_j)^*} = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^n a_k^*(i) \otimes b_k^*(j) \Big\|_{A_i^* \widehat{\otimes} B_j^*}. \tag{2.3}$$

First, we consider an arbitrary element $\tilde{d} = (\tilde{d}(i,j)) \in \prod_{\mathcal{U}} \prod_{\mathcal{U}'} A_i^* \widehat{\otimes} B_j^*$. Let $(\varepsilon(i,j))$ be a family of positive real numbers such that $\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \varepsilon(i,j) = 0$. For each coordinate (i,j) we apply Fact 2.5 and find a norm one element $x(i,j) \in \mathcal{M}_i \bar{\otimes} \mathcal{N}_j$ such that

$$\|x(i,j)\|_{\mathcal{M}_i \bar{\otimes} \mathcal{N}_j} \le (1 + \varepsilon(i,j)) \left| \langle \tilde{d}(i,j), x(i,j)\rangle \right|.$$

This shows that map

$$\Phi_1\colon \prod\nolimits_{\mathcal{U}} \prod\nolimits_{\mathcal{U}'} A_i^* \widehat{\otimes} B_j^* \to \Big(\prod_{(i,j)\in I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j \Big)^*$$

defined by

$$\Phi_1(\tilde{d})(x) = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \langle \tilde{d}(i,j), x(i,j)\rangle$$

is isometric. We specialize to $d = \sum_{k=1}^n a_k^* \otimes b_k^*$ and want to relate this to f. According to Fact 2.5 every norm one element $x \in \prod_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j$ defines a complete contraction $T_x\colon \sum_i (\mathcal{M}_i)_* \to \prod_j \mathcal{N}_j$ defined according to (2.2) as follows. Let $i \in I$ and $a^* \in A_i^* = (\mathcal{M}_i)_*$. Consider the family $s_i(a^*) \in \sum_i (\mathcal{M}_i)_*$ defined by

$$s_i(a^*)(i') = \begin{cases} a^* & \text{if } i' = i, \\ 0 & \text{else.} \end{cases}$$

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By definition of $\sum_i (\mathcal{M}_i)_*$ the span of such elements is norm dense. Then T_x satisfies for all $b^* \in \mathcal{N}_j^*$

$$\langle b_j^*, T_x(s_i(a^*))(j)\rangle = \langle a^* \otimes b^*, x(i,j)\rangle.$$

We observe that $T_x^* : (\prod_j \mathcal{N}_j)^* \to \prod_i \mathcal{M}_i$ is also a complete contraction such that for every $\psi \in (\prod \mathcal{N}_j)^*$, and for every $i \in I$ and $a^* \in A_i^*$, we have

$$\begin{aligned}\langle a^*, T_x^*(\psi)(i)\rangle &= \langle s_i(a^*), T_x^*(\psi)\rangle = \psi \circ T_x(s_i(a^*)) \\ &= \langle \psi, (T_x(s_i(a^*))(j))_{j\in J}\rangle.\end{aligned}$$

Let $\iota\colon \prod \mathcal{M}_i \to (\prod \mathcal{M}_i)^{**}$ be the canonical embedding. Using Fact 2.5 (and an obvious flip), we get an isometric isomorphism

$$\beta\colon CB\Big(\big(\prod \mathcal{N}_j\big)^*, \big(\prod \mathcal{M}_i\big)^{**}\Big) \to \big(\prod \mathcal{M}_i\big)^{**}\bar{\otimes}\big(\prod \mathcal{N}_j\big)^{**}$$

such that for every $\phi \in (\prod \mathcal{M}_i)^*$ and $\psi \in (\prod \mathcal{N}_j)^*$, we have

$$\langle \beta(\iota T_x^*), \phi \otimes \psi\rangle = \langle \phi, T_x^*(\psi)\rangle.$$

Recall $f = \sum_{k=1}^{n} a_k^* \otimes b_k^*$. Let us fix $k \in \{1,\dots,n\}$ and consider the functionals $\langle \phi_k, m\rangle = \lim_{i,\mathcal{U}} \langle m(i), a_k^*(i)\rangle$ and $\langle \psi_k, n\rangle = \lim_{j,\mathcal{U}'} \langle n(j), b_k^*(j)\rangle$. Then we deduce

$$\begin{aligned}\langle \beta(\iota T_x^*), \phi_k \otimes \psi_k\rangle &= \langle \phi_k, T_x^*(\psi_k)\rangle \\ &= \lim_{i,\mathcal{U}} \langle a_k^*(i), T_x^*(\psi_k)(i)\rangle \\ &= \lim_{i,\mathcal{U}} \langle \psi_k, [(a_k^*(i) \otimes id_{\mathcal{N}_j})(x(i,j))]_{j\in J}\rangle \\ &= \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \langle a_k^*(i) \otimes b_k^*(j), x(i,j)\rangle \\ &= \Phi_1(a_k^* \otimes b_k^*)(x).\end{aligned}$$

Let us define the contraction $\alpha\colon \prod_{I\times J} \mathcal{N}_j \bar{\otimes} \mathcal{M}_i \to (\prod \mathcal{N}_j)^{**} \bar{\otimes} (\prod \mathcal{M}_i)^{**}$ by $\alpha(x) = \beta(\iota T_x^*)$. By linearity, we obtain

$$\Phi_1(d)(x) = \langle \beta(\iota T_x^*), f\rangle = \langle \alpha(x), f\rangle$$

for all $x \in \prod_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j$. Then α extends the natural embedding $\pi\colon (\prod_i \mathcal{M}_i) \otimes_{\min} (\prod \mathcal{N}_j) \to (\prod \mathcal{M}_i)^{**} \bar{\otimes} (\prod_j \mathcal{N}_j)^{**}$. From Fact 2.4, we deduce

$$\begin{aligned}\|\Phi_1(d)\| &= \sup_{\|x\|_{\prod_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j}} |\langle \alpha(x), f\rangle| \le \|f\| \sup_{\|x\|_{\prod_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j}} \|\alpha(x)\| \\ &\le \|f\| = \sup_{\|x\|_{(\prod \mathcal{M}_i)\otimes_{\min}(\prod \mathcal{N}_j)} \le 1} |\langle f, \pi(x)\rangle| \\ &\le \sup_{\|x\|_{\prod_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j} \le 1} |\langle f, \alpha(x)\rangle| = \|\Phi_1(d)\|.\end{aligned}$$

Hence, we obtain equality (2.3). This completes the proof. ■

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Remark 2.12 In the special case where $\mathcal{M}_i = B(H_i)$ and $\mathcal{N}_j = B(K_j)$, we can define the injective algebras $\mathcal{M} = \prod_i B(H_i)$ and $\mathcal{N} = \prod_j B(K_j)$. Then it is easy to deduce from Lemma 2.6 that

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^{n} a_k^*(i) \otimes b_k^*(j) \Big\|_{B(H_i)^* \widehat{\otimes} B(K_j)^*} = \Big\| \sum_{k=1}^{n} a_k^* \otimes b_k^* \Big\|_{\mathcal{N}^*[\mathcal{M}^*]},$$

where a_k^* and b_k^* denote the functional $a_k^*((T_i)) = \lim_{i,\mathcal{U}} \langle a_k^*(i), T_i\rangle$ and $b_k^*((S_j)) = \lim_{j,\mathcal{U}'} \langle b_k^*(j), S_j\rangle$, respectively. Therefore, in this special case the assertion of Proposition 2.11 follows from Corollary 2.8.

In general double duals and ultraproducts are closely related. This will be used in [J2] and in our situation this reads as follows.

Lemma 2.13 *Let $\mathcal{N}$ be a von Neumann algebra, then there exist an index set I, an ultrafilter $\mathcal{U}$, a von Neumann algebra $\mathcal{N}_{\mathcal{U}} = (\prod_{\mathcal{U}} \mathcal{N}_*)^*$, a normal contractive map $E\colon \mathcal{N}_{\mathcal{U}} \to \mathcal{N}^{**}$ and a normal faithful representation $\pi\colon \mathcal{N}^{**} \to \mathcal{N}_{\mathcal{U}}$ such that $E\pi = id_{\mathcal{N}^{**}}$.*

Proof Let us consider the index set

$$I = \{(F,G) | 1 \in F \subset \mathcal{N}, \dim(F) < \infty, G \subset \mathcal{N}^*, \dim(G) < \infty\}.$$

Using the principle of local reflexivity, see [EJR], we can find for every pair $i = (F,G)$ a map $u_i\colon G^* \to \mathcal{N}_*$ such that

$$\|u_i\|_{cb} \le 1 + \frac{1}{\dim(F) + \dim(G)},$$

and for all $x \in F$ and $y^* \in G$

$$|\langle y^*, x\rangle - \langle u(y^*), x\rangle| \le (\dim(F) + \dim(G))^{-1}.$$

For $x \in \mathcal{N}$, $y^* \in \mathcal{N}^*$ and $m \in \mathbb{N}$, we consider $I_{x,y^*,m} = \{(F,G) \in I | x \in F, y^* \in G, \dim(F) + \dim(G) \ge m\}$. Then finite intersections $I_{x_1,y_1^*,m_1} \cap \cdots \cap I_{x_k,y_k^*,m_k}$ are not empty and hence there exists an ultrafilter $\mathcal{U}$ such that

$$\{I_{x,y^*,m} | x \in \mathcal{N}, y^* \in \mathcal{N}^*, m \in \mathbb{N}\} \subset \mathcal{U}.$$

We define a map $u\colon \mathcal{N}^* \to \prod_{\mathcal{U}} \mathcal{N}_*$ by

$$u(y^*) = (u_i(y^*))_{i \in I}$$

where $u_i(y^*) = 0$ if $i = (F,G)$ and $y^* \notin G$. Obviously, $\|u\|_{cb} \le 1$ and u is well-defined. Let us consider $w\colon \prod_{\mathcal{U}} \mathcal{N}_* \to \mathcal{N}^*$ defined for $x \in \mathcal{N}$ by

$$\langle w((y_i)), x\rangle = \lim_{i,\mathcal{U}} \langle y_i, x\rangle.$$

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Then w is a complete contraction and by the definition of I and the choice of $\mathcal{U}$, we deduce for all $y^* \in \mathcal{N}^*$ and $x \in \mathcal{N}$

$$\langle wu(y^*), x\rangle = \langle y^*, x\rangle.$$

Thus, we get $wu = id_{\mathcal{N}^*}$ and by duality $u^*w^* = id_{\mathcal{N}^{**}}$. Let $\mathcal{N}_{\mathcal{U}} = (\prod_{\mathcal{U}} \mathcal{N}_*)^*$ and $z_{\mathcal{U}}$ be the corresponding central projection in $(\prod \mathcal{N})^{**}$. We have a natural representation $\pi_0 \colon \mathcal{N} \to \prod_{i\in I} \mathcal{N}$ defined by $\pi_0(x) = 1 \otimes x$. Let and $\pi_{\mathcal{U}} \colon \mathcal{N} \to \mathcal{N}_{\mathcal{U}}$ be given by $\pi_{\mathcal{U}}(x) = z_{\mathcal{U}}\pi_0(x)$. Then for every $x \in \mathcal{N}$ and $(y_i) \in \prod_{\mathcal{U}} \mathcal{N}_*$, we have

$$\langle \pi_{\mathcal{U}}(x), (y_i)\rangle = \lim_{i,\mathcal{U}} \langle x, y_i\rangle = \langle w((y_i)), x\rangle.$$

Hence, $\pi^*_{\mathcal{U}}|_{\prod_{\mathcal{U}} \mathcal{N}_*} = w$ and therefore the normalization $(\pi_{\mathcal{U}})_{nor} \colon \mathcal{N}^{**} \to \mathcal{N}_{\mathcal{U}}$ satisfies $(\pi_{\mathcal{U}})_{nor} = w^*$. In particular, w^* is a normal C^*-homomorphism. The map $u^* \colon \mathcal{N}_{\mathcal{U}} \to \mathcal{N}^{**}$ is a complete contraction and

$$u^*(1) = u^*w^*(1) = 1.$$

According to Tomiyama's Theorem [Tk, Theorem III 3.4.], we deduce that $E = w^*u^*$ is a normal conditional expectation onto $w^*(\mathcal{N}^{**})$. ∎

Remark 2.14 (1) Let us consider the special case $\mathcal{N} = B(H)$. Then $\mathcal{N}_{\mathcal{U}} = \mathcal{N}_{\mathcal{U}}(H)$ is a quotient of the QWEP algebra $\ell_\infty(B(H))^{**}$ and therefore $\mathcal{N}_{\mathcal{U}}(H)$ is QWEP. Lemma 2.1 implies that *a C^*-algebra A is QWEP if and only if there exists a normal conditional expectation $E\colon \mathcal{N}_{\mathcal{U}}(H) \to A^{**}$* for a suitable $\mathcal{N}_{\mathcal{U}}(H)$.

(2) The preconjugate $E_* \colon \prod_{\mathcal{U}} S_1 \to (\prod B(H))^*$ defined in the proof of Lemma 2.13, provides us with a conditional expectation $E = (E_*)^* \colon (\prod B(H))^{**} \to \mathcal{N}_{\mathcal{U}}$ such that

$$(\mathcal{N}_{\mathcal{U}})_*[F] = \prod_{\mathcal{U}} S_1[F]$$

for every finite dimensional operator space F. This will be our preferred choice of E_* for $\mathcal{N}_{\mathcal{U}}(H)$ in the forthcoming paper [J2].

3 The Fubini Theorem for $p > 1$

In this section we will extend Fubini's Theorem for ultraproducts of noncommutative L_p-spaces to the range $p > 1$. We will use the concept of the Haagerup L_p-spaces as a fundamental tool. We refer to [Ha2, Te1, C2] for more details and properties of the abstract Haagerup L_p-spaces, whose definition we recall now. Let $\mathcal{N}$ be a von Neumann algebra with an n.s.f. (normal, semifinite, faithful weight) w on $\mathcal{N}$. Consider the crossed product $\mathcal{N} \rtimes_{\sigma^w} \mathbb{R}$ with respect to the modular automorphism group σ^w. Indeed, if $\mathcal{N}$ acts faithfully on a Hilbert space H, then the crossed product $\mathcal{N} \rtimes_{\sigma^w} \mathbb{R}$ is a von Neumann algebra acting on $L_2(\mathbb{R}, H)$ and generated by all the elements

$$\pi(x)(\xi(t)) = \sigma^w_{-t}(x)(\xi(t)) \text{ and } \lambda(s)\xi(t) = \xi(t-s)$$

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$x \in \mathcal{N}$ and $s \in \mathbb{R}$. For $s \in \mathbb{R}$ let $W(s)$ be the unitary defined by the phase shift

$$\big(W(s)\xi\big)(t) = e^{-ist}\xi(t).$$

Then $\mathcal{N} \rtimes_{\sigma^w} \mathbb{R}$ is semifinite and admits a unique trace τ such that the dual action

$$\theta_s(x) = W(s)xW(s)^*$$

satisfies $\tau(\theta_s(x)) = e^{-s}\tau(x)$, see [PT]. Moreover, for every $x \in \mathcal{N}$ and $s \in \mathbb{R}$ we have $\theta_s(\pi(x)) = \pi(x)$ and even

(3.1) $$\pi(\mathcal{N}) = \{x \in \mathcal{N} \rtimes_{\sigma^w} \mathbb{R} \mid \forall s \in \mathbb{R} : \theta_s(x) = x\}$$

Let us agree to identify $\mathcal{N}$ with $\pi(\mathcal{N})$ in the following. Haagerup's L_p space $L_p(\mathcal{N}, w)$ (or in short $L_p(\mathcal{N})$) is defined to be the space of (unbounded) τ-measurable operators affiliated to $\mathcal{N} \rtimes_{\sigma^w} \mathbb{R}$ such that for all $s \in \mathbb{R}$

$$\theta_s(x) = e^{-\frac{s}{p}}x.$$

Note that the intersection $L_p(\mathcal{N}) \cap L_q(\mathcal{N})$ is $\{0\}$ for different values $p \neq q$. There is a natural isomorphism between $\mathcal{N}_*$ and $L_1(\mathcal{N})$ such that for every normal functional $\phi \in \mathcal{N}_*$ there is a unique operator $a_\phi \in L_1(\mathcal{N})$ satisfying

$$\phi\Big(\int_{\mathbb{R}} \theta_s(x)\, ds\Big) = \tau(a_\phi x)$$

for all positive elements $x \in \mathcal{N} \rtimes_{\sigma^w} \mathbb{R}$. The trace functional $\mathrm{tr}\colon L_1(\mathcal{N}) \to \mathbb{C}$ (corresponding to the integral in the commutative case) is given by

$$\mathrm{tr}(a_\phi) = \phi(1).$$

$\mathcal{N}$ acts as a left and right module on $L_p(\mathcal{N})$ and more generally Hölder's inequality

(3.2) $$\|xy\|_r \leq \|x\|_p\|y\|_q$$

holds whenever $x \in L_p(\mathcal{N})$, $y \in L_q(\mathcal{N})$ and $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}$. Let $\frac{1}{p}+\frac{1}{p'} = 1$ and $x \in L_p(\mathcal{N})$, $y \in L_{p'}(\mathcal{N})$. Then we have the tracial property

$$\mathrm{tr}(xy) = \mathrm{tr}(yx).$$

The polar decomposition $x = u|x|$ of $x \in L_p(\mathcal{N})$ satisfies $u \in N$ and

$$\|x\|_p = \mathrm{tr}(|x|^p)^{\frac{1}{p}}.$$

In particular, for every $x \in L_{2p}(\mathcal{N})$

(3.3) $$\|x\|_{2p} = \|x^*x\|_p^{\frac{1}{2}}.$$

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For $0 < p \leq 1$, the space $L_p(\mathcal{N})$ is p-normed, *i.e.*,

$$\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p \tag{3.4}$$

holds for all $x, y \in L_p(\mathcal{N})$ (see [Ko2]). As for semifinite von Neumann algebras, there is a positive cone $L_p(\mathcal{N})_+$ in $L_p(\mathcal{N})$ consisting of elements in $L_p(\mathcal{N})$ which are positive as unbounded operators affiliated to $\mathcal{N} \rtimes_{\sigma^w} \mathbb{R}$. Following [Te1, Proposition 33, Theorem 32], we deduce for $0 \leq x \leq y \in L_p(\mathcal{N})$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$$\|x\|_p = \sup_{\substack{z \in L_{p'}(\mathcal{N})_+ \\ \|z\|_{p'} \leq 1}} \operatorname{tr}(zx) \leq \sup_{\substack{z \in L_{p'}(\mathcal{N})_+ \\ \|z\|_{p'} \leq 1}} \operatorname{tr}(zy) = \|y\|_p.$$

The operator space structure of Haagerup's L_p-spaces is slightly more difficult to define because interpolation is not immediately applicable for Haagerup L_p-spaces. Indeed, we first consider the case $p = 1$. Then the mapping $\beta\colon L_1(\mathcal{N}) \to \mathcal{N}_*^{op}$ defined by

$$\beta(D)(x) = \operatorname{tr}(Dx)$$

is well-defined since $\mathcal{N}$ and $\mathcal{N}^{op}$ coincide as (dual) Banach spaces (not as algebras of course). Given a matrix $[D_{ij}]$ of elements in $L_1(\mathcal{N})$ we observe that

$$\begin{aligned}
\|[\beta(D_{ij})]\|_{S_1^n \widehat{\otimes} \mathcal{N}_*^{op}} &= \sup_{\|[x_{ij}]\|_{M_n(\mathcal{N}^{op})} \leq 1} \Big| \sum_{ij} \operatorname{tr}(D_{ij} x_{ij}) \Big| \\
&= \sup_{\|[x_{ij}]\|_{M_n(\mathcal{N})} \leq 1} \Big| \sum_{ij} \operatorname{tr}(D_{ij} x_{ji}) \Big| \\
&= \|[D_{ij}]\|_{L_1(M_n \otimes \mathcal{N}, tr_n \otimes w)}.
\end{aligned}$$

Here tr_n is the non-normalized trace on M_n and, $\operatorname{tr}_n \otimes w$ the n.s.f. weight on $M_n \otimes \mathcal{N}$. Therefore, the operator space structure induced by β satisfies

$$S_1^n \widehat{\otimes} L_1(\mathcal{N}) = L_1(M_n \otimes \mathcal{N}).$$

We will call this the *natural operator space structure on* $L_1(\mathcal{N})$. In order to obtain the operator space structure for $L_p(\mathcal{N})$, we first assume that ϕ is a normal faithful state with density $D \in L_1(\mathcal{N})$. Then, we may define the symmetric embedding $I\colon \mathcal{N} \to L_1(N)$ by $I(x) = D^{\frac{1}{2}} x D^{\frac{1}{2}}$ and by interpolation (see [BL] for general information) the operator space

$$E_p(\mathcal{N}) = [I(\mathcal{N}), L_1(\mathcal{N})]_{\frac{1}{p}}.$$

According to Kosaki's results [Ko, Theorem 9.1] there is a natural isometric isomorphism between $E_p(\mathcal{N})$ and $L_p(\mathcal{N})$ which sends $x \in L_p(\mathcal{N})$ to $D^{\frac{1}{2p'}} x D^{\frac{1}{2p'}} \in E_p(\mathcal{N}) \subset L_1(\mathcal{N})$. This induces the *natural operator space structure* on $L_p(\mathcal{N})$. Indeed, according to [P7, Corollary 1.4], we obtain by complex interpolation

$$S_p^n[L_p(\mathcal{N})] = L_p(M_n \otimes \mathcal{N}). \tag{3.5}$$

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In the general case, we use a strictly semifinite normal weight w. This means there is an increasing family of projection e_i with $w(e_i) < \infty$ such that e_i converges strongly to 1 and $\sigma_t^w(e_i) = e_i$. Then, we obtain the natural operator space structure on $e_i L_p(\mathcal{N}) e_i = L_p(e_i \mathcal{N} e_i)$ using the state $\phi_i = w(e_i)^{-1} e_i w e_i$. It is easily checked that we obtain compatible operator space structures for $e_i \le e_j$. By density of $\bigcup_i e_i L_p(\mathcal{N}) e_i$ in $L_p(\mathcal{N})$, we finally obtain the natural operator space structure on $L_p(\mathcal{N})$ which still satisfies (3.5). Thus (3.5) completely determines the operator space structure by

$$\|x\|_{M_n(L_p(\mathcal{M}))} = \sup_{\|a\|_{S_{2p}^n} \|b\|_{S_{2p}^n} \le 1} \|a.x.b\|_{L_p(M_n \otimes \mathcal{N}, tr_n \otimes w)}.$$

Note that we could have used this formula as a definition. But then it is not clear that Ruan's axiom (R2) is verified. Note, moreover, that (3.5) is also compatible with the isomorphisms obtained by a change of weight (see [Te1]) and hence the natural operator space structure is indeed defined for Haagerup's abstract $L_p(\mathcal{N})$ space. In this paper, we will sometimes indicate that certain isometries are indeed complete isometries. However, using (3.5) this often follows automatically and therefore the proof of these statements will be omitted. We refer to [KR, section 11] for tensor products of unbounded operators and [KR, proposition 13.1.12] for the tensor product of modular groups. Haagerup L_p-spaces are also compatible with normal conditional expectations. In the context of a faithful normal state, we will use the following observation from [JX, Proposition 2.3].

Proposition 3.1 *Let $\mathcal{N}$ be a von Neumann algebra, $\mathcal{M}$ a von Neumann subalgebra with a normal, faithful state ϕ and $\mathcal{E}\colon \mathcal{N} \to \mathcal{M}$ a faithful normal conditional expectation. For $0 < p \le \infty$ there is a natural (completely) isometric embedding $i_p\colon L_p(\mathcal{M}) \to L_p(\mathcal{N})$. For $1 \le p \le \infty$, there is a contraction $\mathcal{E}_p\colon L_p(\mathcal{N}) \to L_p(\mathcal{N})$ such that*

$$\mathcal{E}_p(i_{2p}(a) x i_{2p}(b)) = i_{2p}(a)\mathcal{E}(x) i_{2p}(b)$$

for all $x \in \mathcal{M}$, $a \in L_r(\mathcal{M}), b \in L_s(\mathcal{M})$ with $\frac{1}{r} + \frac{1}{s} = \frac{1}{p} \le 1$. Moreover, if $D_{\mathcal{M}}$ denotes the density of $\phi \in L_1(M)$ and $D_{\mathcal{N}}$ denotes the density of $\phi \circ \mathcal{E}$ in $L_1(\mathcal{N})$, then

$$(3.6) \qquad i_p(D_{\mathcal{M}}^{\frac{1-\theta}{p}} x D_{\mathcal{M}}^{\frac{\theta}{p}}) = D_{\mathcal{N}}^{\frac{1-\theta}{p}} x D_{\mathcal{N}}^{\frac{\theta}{p}} \text{ and } \mathcal{E}_p(D_{\mathcal{N}}^{\frac{1-\theta}{p}} y D_{\mathcal{N}}^{\frac{\theta}{p}}) = D_{\mathcal{M}}^{\frac{1-\theta}{p}} \mathcal{E}(y) D_{\mathcal{M}}^{\frac{\theta}{p}}$$

for all $1 \le p \le \infty$, $0 \le \theta \le 1$ and $x \in \mathcal{M}$, $y \in N$.

Remark 3.2 It is easily checked that $\mathcal{E}_p^* = i_{p'}$ for conjugate indices $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof Indeed, let $\mathrm{tr}_{\mathcal{N}}$ be the trace on $L_1(\mathcal{N})$ and $\mathrm{tr}_{\mathcal{M}}$ the trace on $L_1(\mathcal{M})$. Let $a = D_{\mathcal{N}}^{\frac{1}{p}} x \in L_p(\mathcal{N})$ and $b = y D_{\mathcal{M}}^{\frac{1}{p'}} \in L_{p'}(\mathcal{M})$, then we deduce from (3.6) ($\theta = 1$)

$$\begin{aligned} \mathrm{tr}_{\mathcal{M}}(\mathcal{E}_p(a)b) &= \mathrm{tr}_{\mathcal{M}}(D_{\mathcal{M}}^{\frac{1}{p}} \mathcal{E}(x) y D_{\mathcal{M}}^{\frac{1}{p'}}) = \mathrm{tr}_{\mathcal{M}}(\mathcal{E}(xy) D_{\mathcal{M}}) = \phi \circ \mathcal{E}(xy) \\ &= \mathrm{tr}_{\mathcal{N}}(xy D_{\mathcal{N}}) = \mathrm{tr}_{\mathcal{N}}(D_{\mathcal{N}}^{\frac{1}{p}} xy D_{\mathcal{N}}^{\frac{1}{p'}}) \\ &= \mathrm{tr}_{\mathcal{N}}(a i_{p'}(b)). \end{aligned}$$

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Therefore the assertion follows by density (see [JX, Lemma 1.1]) using the duality brackets $\langle a, b\rangle = \mathrm{tr}(ab)$ for $\mathcal{M}, \mathcal{N}$, respectively. ■

The following well-known observation allows us to identify the support of the composition state.

Lemma 3.3 *Let $\mathcal{E}\colon \mathcal{N} \to \mathcal{M}$ be a faithful, normal conditional expectation and ϕ a normal state on $\mathcal{M}$. Then the support $s(\phi)$ of ϕ and the support of $\phi \circ \mathcal{E}$ coincide.*

Proof Let $s = s(\phi)$ and $\tilde{s} = s(\phi \circ \mathcal{E})$ and $x \in \mathcal{N}$, then

$$\phi \circ \mathcal{E}(sx) = \phi(s\mathcal{E}(x)) = \phi(\mathcal{E}(x)) = \phi \circ \mathcal{E}(x).$$

Hence $\tilde{s} \leq s$ and thus $s\tilde{s}s = \tilde{s}$ and $\mathcal{E}(s\tilde{s}s) \leq \mathcal{E}(s) = s$. However,

$$\begin{aligned}\phi(s - s\mathcal{E}(\tilde{s})s) &= \phi(1) - \phi(s\mathcal{E}(\tilde{s}s)) = 1 - (\phi \circ \mathcal{E})(\tilde{s})\\ &= \phi(1) - (\phi \circ \mathcal{E})(1) = \phi(1) - \phi(1) = 0.\end{aligned}$$

Since ϕ is faithful on $s\mathcal{M}s$, we deduce

$$\mathcal{E}(s) = s = s\mathcal{E}(\tilde{s})s = \mathcal{E}(s\tilde{s}s) = \mathcal{E}(\tilde{s}).$$

Since $\mathcal{E}$ is faithful, we deduce $s = \tilde{s}$. ■

Let $(\mathcal{M}_i)$, $(\mathcal{N}_j)$ be families of von Neumann algebras and $\mathcal{U}$, $\mathcal{U}'$ be ultrafilters on the index sets I and J, respectively. As in the first part, we will consider their corresponding von Neumann algebras

$$\mathcal{M}_{\mathcal{U}} = \Big(\prod\nolimits_{\mathcal{U}} L_1(\mathcal{M}_i)\Big)^* \text{ and } \mathcal{N}_{\mathcal{U}'} = \Big(\prod\nolimits_{\mathcal{U}'} L_1(\mathcal{N}_j)\Big)^*,$$

which can be identified with $z_{\mathcal{U}}(\prod \mathcal{M}_i)^{**}$, $z_{\mathcal{U}'}(\prod \mathcal{N}_j)^{**}$, respectively. In the following it will be useful to work with the tracial map $Tr_{\mathcal{U}}\colon \prod_{\mathcal{U}} L_1(\mathcal{M}_i) \to \mathbb{C}$, defined by

$$\mathrm{Tr}_{\mathcal{U}}((x_i)) = \lim_{i,\mathcal{U}} \mathrm{tr}_i(x_i).$$

Let $\frac{1}{p} + \frac{1}{p'} = 1$. Then, we note that for every $x \in \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$ and every $y \in \prod_{\mathcal{U}} L_{p'}(\mathcal{M}_i)$ we have

$$\mathrm{Tr}_{\mathcal{U}}(xy) = \lim_{i,\mathcal{U}} \mathrm{tr}_i(x(i)y(i)) = \lim_{i,\mathcal{U}} \mathrm{tr}_i(y(i)x(i)) = \mathrm{Tr}_{\mathcal{U}}(yx). \tag{3.7}$$

In particular, for every $x \in \mathcal{M}_{\mathcal{U}}$ and $y \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$, we deduce that

$$\mathrm{Tr}_{\mathcal{U}}(xy) = \mathrm{Tr}_{\mathcal{U}}(yx)$$

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is well-defined by approximation with a bounded net x_α converging in the strong* topology to x. Every (component wise) positive element $d \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$ with $Tr_{\mathcal{U}}(d) = 1$ defines a positive, normal state on $(\prod \mathcal{M}_i)^{**}$ defined by

$$\phi_d(x) = Tr_{\mathcal{U}}(dx).$$

Then ϕ_d extends to a normal functional on $(\prod \mathcal{M}_j)^{**}$ such that $z_{\mathcal{U}}\phi_d = \phi_d$. Hence, ϕ_d is a normal state in the predual $(\mathcal{M}_{\mathcal{U}})_*$ and we obtain an isometric map $V\colon \prod_{\mathcal{U}} L_1(\mathcal{M}_i) \to (\mathcal{M}_{\mathcal{U}})_*$ satisfying

$$V(d) = \phi_d.$$

Raynaud [Ra2] extends this map to other values of p. In our context, we will use the inverse of that map, namely a family of linear maps $(T_p)_{0<p<\infty}$ such that

$$T_p\colon L_p(\mathcal{M}_{\mathcal{U}}) \to \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$$

and

$$T_p(ab) = T_r(a)T_s(b), \tag{3.8}$$

holds for all $\frac{1}{r} + \frac{1}{s} = \frac{1}{p}$ and $a \in \prod_{\mathcal{U}} L_r(\mathcal{M}_i)$, $b \in \prod_{\mathcal{U}} L_s(\mathcal{M}_i)$, see [Ra2, Theorem 5.1]. Moreover, using the identification between $(\mathcal{M}_{\mathcal{U}})_*$ and $L_1(\mathcal{M}_{\mathcal{U}})$, we have

$$V \circ T_1 = id = T_1 \circ V.$$

We deduce for conjugate indices ($\frac{1}{p} + \frac{1}{p'} = 1$) that

$$\mathrm{Tr}_{\mathcal{U}}(T_p(a)T_{p'}(b)) = \mathrm{Tr}_{\mathcal{U}}(T_1(ab)) = V_1(T_1(ab))(1) = \mathrm{tr}_{\mathcal{M}_{\mathcal{U}}}(ab). \tag{3.9}$$

Therefore using the duality bracket $L_p(\mathcal{M}_{\mathcal{U}})^* = L_{p'}(\mathcal{M}_{\mathcal{U}})$ given by $\mathrm{tr}_{\mathcal{M}_{\mathcal{U}}}$ and the duality bracket on $(\prod_{\mathcal{U}} L_p(\mathcal{M}_i))^* = \prod_{\mathcal{U}} L_{p'}(\mathcal{M}_i)$ given by $\mathrm{Tr}_{\mathcal{U}}$, we deduce

$$T_p^* = T_{p'}. \tag{3.10}$$

Moreover these maps preserve the Mazur map, *i.e.,* for a positive element $a \in \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$ we have

$$T_p(a)^p = T_1(a^p). \tag{3.11}$$

We refer to [Ra2] for more details on this map. (Raynaud actually considers the ultrapower of a given von Neumann algebra but his arguments carry over verbatim to this more general situation.) Let us note that in view of (3.5), the map T_p is a completely isometric isomorphism.

Fact 3.4 $\prod_{\mathcal{U}} L_p(\mathcal{M}_i)$ is a left and right $\mathcal{M}_{\mathcal{U}}$ module. More precisely, if (x_α) is a bounded net in $\prod \mathcal{M}_i$ converging in the strong* topology to $x \in (\prod_i \mathcal{M}_i)^{**}$ and if $a \in \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$, then $x_\alpha a$ is converging in norm to the element called xa.

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Proof Using Raynaud's Theorem [Ra2, Theorem 3.6.] which identifies $L_p(\mathcal{M}_\mathcal{U})$ with $\prod_\mathcal{U} L_p(\mathcal{M}_i)$ (see (3.8) below), it is clear that $\prod_\mathcal{U} L_p(\mathcal{M}_i)$ is a left and right $\mathcal{M}_\mathcal{U}$ module. We can also argue as in [J1, Lemma 2.3] that for every bounded, strongly convergent net $(x_\alpha) \subset \prod \mathcal{M}_i$ and for every $a \in \prod_\mathcal{U} L_p(\mathcal{M}_i)$, we obtain a norm convergent net $x_\alpha a$, where $(x_\alpha a)(i) = x_\alpha(i)a(i)$ is defined pointwise. ■

The next lemma ensures that the support of the state is compatible with its density.

Lemma 3.5 *Let $d \in \prod_\mathcal{U} L_1(\mathcal{M}_i)_+$ and ϕ_d be its associated state. For $0 < p < \infty$ let $d^{\frac{1}{p}} = (d(i)^{\frac{1}{p}})$. Then*

$$s(\phi_d)d^{\frac{1}{p}} = d^{\frac{1}{p}} = d^{\frac{1}{p}}s(\phi_d).$$

Similarly, for every $a \in \prod_\mathcal{U} L_p(\mathcal{M}_i)$

$$z_\mathcal{U} a = a.$$

Proof We note that for every $x \in \prod_i \mathcal{M}_i$

$$\begin{aligned}\mathrm{Tr}_\mathcal{U}(dx) &= \phi_d(x) = s(\phi_d).\phi_d(x) = \phi_d(xs(\phi_d)) \\ &= \mathrm{Tr}_\mathcal{U}(dxs(\phi_d)) = \mathrm{Tr}_\mathcal{U}(s(\phi_d)dx).\end{aligned}$$

Hence $s(\phi_d)d = d$. Similarly, we deduce $z_\mathcal{U} a = a$ for every $a \in \prod_\mathcal{U} L_1(\mathcal{M}_i)$. Since the proof for both assertions is very similar, we will only show the first assertion. Let S be the set of all p's such that $s(\phi_d)d^{\frac{1}{p}} = d^{\frac{1}{p}}$. We have just proved $1 \in S$. Let us show $p \in S$ implies $2p \in S$. Indeed,

$$\begin{aligned}\|(1-s(\phi_d))d^{\frac{1}{2p}}\|_{2p}^2 &= \|(1-s(\phi_d))d^{\frac{1}{p}}(1-s(\phi_d))\|_p \\ &\le \|(1-s(\phi_d))d^{\frac{1}{p}}\|_p\|(1-s(\phi_d))\|_\infty = 0.\end{aligned}$$

Therefore, by induction we deduce that $\{2^k | k \in \mathbb{N}\} \subset S$. However, for $q < p \in S$, we deduce from Hölder's inequality and using $0 < r < \infty$ defined by $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ that

$$\|(1-s(\phi_d))d^{\frac{1}{q}}\|_q \le \|(1-s(\phi_d))d^{\frac{1}{p}}\|_p\|d^{\frac{1}{r}}\|_r = 0.$$

Hence, we find $(0,\infty) \subset S$ and the assertion is proved. ■

Before we prove Fubini's theorem, we shall first provide a natural embedding of $L_p(\mathcal{M}) \otimes L_p(\mathcal{N})$ in $L_p(\mathcal{M} \otimes \mathcal{N})$.

Proposition 3.6 *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Then there is a family of linear maps $(I_p)_{1 \le p < \infty}$, $I_p \colon L_p(\mathcal{M}) \otimes L_p(\mathcal{N}) \to L_p(\mathcal{M} \otimes \mathcal{N})$ all with dense range such that*

(3.12) $\quad I_p(ma \otimes nb) = (m \otimes n)I_p(a \otimes b)$ *and* $I_p(am \otimes bn) = I_p(a \otimes b)(m \otimes n)$,

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holds for all $a \in L_p(\mathcal{M})$, $b \in L_p(\mathcal{N})$, $m \in \mathcal{M}$, $n \in \mathcal{N}$. *Moreover,*

$$I_p(a_1a_2 \otimes b_1b_2) = I_{p_1}(a_1 \otimes b_1)I_{p_2}(a_2 \otimes b_2) \tag{3.13}$$

holds for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $a_1 \in L_{p_1}(\mathcal{M})$, $a_2 \in L_{p_2}(\mathcal{M})$, $b_1 \in L_{p_1}(\mathcal{N})$ *and* $b_2 \in L_{p_2}(\mathcal{N})$. *Finally for positive elements* $a \in L_1(\mathcal{M})$ *and* $b \in L_1(\mathcal{N})$

$$I_p(a^{\frac{1}{p}} \otimes b^{\frac{1}{p}}) = I_1(a \otimes b)^{\frac{1}{p}}. \tag{3.14}$$

Proof Let us first assume that $\mathcal{M}$ and $\mathcal{N}$ are σ-finite, *i.e.*, there exist a normal faithful state ϕ on $\mathcal{M}$ and ψ in $\mathcal{N}$. Let $C \in L_1(\mathcal{M})$ and $C' \in L_1(\mathcal{N})$ be the densities of ϕ, ϕ, respectively. Then $\phi \otimes \psi$ defines a normal faithful state and we denote by $D \in L_1(\mathcal{M} \otimes \mathcal{N})$ its density. Then, we may define for $m \in \mathcal{M}$ and $n \in \mathcal{N}$

$$I_p(mC^{\frac{1}{p}} \otimes nC'^{\frac{1}{p}}) = (m \otimes n)D^{\frac{1}{p}}$$

and extend I_p by linearity to the dense subspace $\mathcal{M}C^{\frac{1}{p}} \otimes \mathcal{N}C'^{\frac{1}{p}} \subset L_p(\mathcal{M}) \otimes_\pi L_p(\mathcal{N})$. (Here π denotes the Banach space projective tensor product.) Now, we want to show that I_p extends by continuity to $L_p(\mathcal{M}) \otimes L_p(\mathcal{N})$. Since this embedding is compatible with Kosaki's interpolation [Ko, Theorem 9.1] and invoking the bilinear complex interpolation, it suffices to show the continuity for $p = \infty$ and $p = 1$. This is obvious for $p = \infty$. For $p = 1$ it follows from Fact 2.5 combined with the canonical isometric identification between $L_1(\mathcal{M})$ and $\mathcal{M}_*$, $L_1(\mathcal{N})$ and $\mathcal{N}_*$, respectively. We denote by $I_p \colon L_p(\mathcal{M}) \otimes_\pi L_p(\mathcal{N}) \to L_p(\mathcal{M} \otimes \mathcal{N})$ the uniquely determined continuous extension. According to Kaplansky's density theorem the unit ball of $\mathcal{M} \otimes_{\min} \mathcal{N}$ is strong* dense in the unit ball of $\mathcal{M}\bar{\otimes}\mathcal{N}$ and then [J1, Lemma 2.3.] implies that I_p has a dense range for all $1 \le p < \infty$. The first equation in (3.12) is obvious by definition. For the second we assume that m, n are analytic elements, *i.e.*, $t \mapsto \sigma_t^\phi(m)$ and $t \mapsto \sigma_t^\psi(n)$ extends to an analytic function on $\mathbb{C}$. Since $\sigma_t^{\phi\otimes\psi} = \sigma_t^\psi \otimes \sigma_t^\psi$, see [KR, Volume II, section 9], we see that $x \otimes y$ is analytic as well. It follows from [Te1, Lemma 19] that the right hand side of

$$\sigma_{-iz}^\phi(m)C = C^z mC^{1-z}$$

is analytic in $L_1(\mathcal{M})$. Since the left hand side is analytic too, and both functions coincide on the boundary of the strip $\{0 \le Re(z) \le 1\}$, we have equality for all $\{0 \le Re(z) \le 1\}$. This argument applied to C, C' and D yields

$$\sigma^\psi_{\frac{-iz}{p}}(m)C = C^{\frac{z}{p}} mC^{\frac{1-z}{p}}, \; \sigma^\psi_{\frac{-iz}{p}}(n)C' = C'^{\frac{z}{p}} nC'^{\frac{1-z}{p}}$$

and

$$\sigma^{\phi\otimes\psi}_{\frac{-iz}{p}}(m \otimes n)D = D^{\frac{z}{p}}(m \otimes n)D^{\frac{1-z}{p}}.$$

Therefore, we deduce

$$\begin{aligned} I_p(C^{\frac{1}{p}} m \otimes C'^{\frac{1}{p}} n) &= I_p(\sigma^\phi_{\frac{-i}{p}}(m)C^{\frac{1}{p}} \otimes \sigma^\psi_{\frac{-i}{p}}(n)C'^{\frac{1}{p}}) = \sigma^\phi_{\frac{-i}{p}}(m) \otimes \sigma^\psi_{\frac{-i}{p}}(n)D^{\frac{1}{p}} \\ &= \sigma^{\phi\otimes\psi}_{\frac{-i}{p}}(m \otimes n)D^{\frac{1}{p}} = D^{\frac{1}{p}}(m \otimes n). \end{aligned}$$

By continuity and strong density of the analytic elements, we obtain the second part of (3.12). The same argument shows that for analytic elements m_1, m_2 and n_1, n_2 and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, we have

$$\begin{aligned} I_p(C^{\frac{1}{p_1}} m_1 m_2 C^{\frac{1}{p_2}} \otimes C'^{\frac{1}{p_1}} n_1 n_2 C'^{\frac{1}{p_2}}) &= D^{\frac{1}{p_1}} (m_1 m_2 \otimes n_1 n_2) D^{\frac{1}{p_2}} \\ &= D^{\frac{1}{p_1}} (m_1 \otimes n_1) l (m_2 \otimes n_2) D^{\frac{1}{p_2}} \\ &= I_{p_1}(C^{\frac{1}{p_1}} m_1 \otimes C'^{\frac{1}{p_1}} n_1) l I_{p_2}(m_2 C^{\frac{1}{p_2}} \otimes n_2 C'^{\frac{1}{p_2}}). \end{aligned}$$

Again by continuity and density, we deduce (3.13). Now, we establish (3.14) for certain values of p. We assume that $a \le C$ and $b \le C'$. Then $a^{\frac{1}{p}} C^{-\frac{1}{p}} \in \mathcal{M}$, $b^{\frac{1}{p}} C'^{-\frac{1}{p}} \in \mathcal{N}$ for all $2 \le p < \infty$ (see [J1, Lemma 1.1]). Let $1 \le m \in \mathbb{N}$. Then, we deduce by induction from (3.13) that

$$I_1(a \otimes b) = I_1(a^{\frac{m}{m}} \otimes b^{\frac{m}{m}}) = I_m(a^{\frac{1}{m}} \otimes b^{\frac{1}{m}})^m.$$

Hence for $k, m \in \mathbb{N}$ with $m \ge k$, we have

$$I_{\frac{m}{k}}(a^{\frac{k}{m}} \otimes b^{\frac{k}{m}}) = I_m(a^{\frac{1}{m}} \otimes b^{\frac{1}{m}})^k = I_1(a \otimes b)^{\frac{k}{m}}.$$

Therefore the module property (3.12) implies

$$I_1(a \otimes b)^{\frac{k}{m}} = I_{\frac{m}{k}}(a^{\frac{k}{m}} \otimes b^{\frac{k}{m}}) = (a^{\frac{k}{m}} C^{-\frac{k}{m}} \otimes b^{\frac{k}{m}} C'^{-\frac{k}{m}}) D^{\frac{k}{m}}.$$

Now, we want to show that

$$I_p(a^{\frac{1}{p}} \otimes b^{\frac{1}{p}}) D^{1-\frac{1}{p}} = (a^{\frac{1}{p}} C^{-\frac{1}{p}} \otimes b^{\frac{1}{p}} C'^{-\frac{1}{p}} D$$

holds for all $2 \le p \le \infty$. By the above this is true for rational values p and it suffices to have continuity. Following the argument in [JX, Lemm 7.4], we see that $a^z C^{-z} C^{\frac{1}{2}}$ is even complex differentiable in $L_2(\mathcal{M})$ for $0 \le Re(z) \le \frac{1}{2}$. Using $\|a^z C^{-z}\| \le 1$, we deduce by density that for all $h \in L_2(\mathcal{M})$ the function $z \mapsto a^z C^{-z} h$ is analytic. Let us say that $a^z C^{-z}$ is strongly analytic. Similarly, $z \mapsto b^z C'^{-z}$ is strongly analytic. Using $\|a^z C^{-z} \otimes b^z C'^{-z}\| \le 1$, we deduce that $z \mapsto a^z C^{-z} \otimes b^z C'^{-z}$ is strongly analytic on $L_2(\mathcal{M}) \otimes L_2(\mathcal{N})$, and hence for the canonical representation $\pi \colon \mathcal{M} \otimes \mathcal{N} \to (\mathcal{M} \otimes \mathcal{N}) \rtimes \mathbb{R}$, we see that $z \mapsto a^z C^{-z} \otimes b^z C'^{-z}$ is strongly analytic. Therefore, for every $u \in \mathcal{M} \bar{\otimes} \mathcal{N}$ the function

$$g(z) = tr_{\mathcal{M} \otimes \mathcal{N}}((a^z C^{-z} \otimes b^z C'^{-z}) D u)$$

is analytic on $0 \le Re(z) \le \frac{1}{2}$ and in particular continuous. We observe that for $z = \frac{m}{k}$ we have

$$g(\frac{m}{k}) = \mathrm{tr}_{\mathcal{M} \otimes \mathcal{N}}((a^{\frac{m}{k}} C^{-\frac{m}{k}} \otimes b^{\frac{m}{k}} C'^{-\frac{m}{k}}) D^{\frac{m}{k}} D^{1-\frac{m}{k}} u) = \mathrm{tr}_{\mathcal{M} \otimes \mathcal{N}}(I_1(a \otimes b)^{\frac{m}{k}} D^{1-\frac{m}{k}} u).$$

As above we deduce from the proof of [JX, Lemm 7.4] that $I_1(a\otimes b)^z D^{-z}$ is strongly analytic because

$$\begin{aligned} I_1(a\otimes b) &= I_1(C^{\frac12}C^{-\frac12}aC^{-\frac12}\otimes C'^{\frac12}C'^{-\frac12}bC'^{-\frac12}C'^{\frac12}) \\ &= I_1(C\otimes C')^{\frac12}(C^{-\frac12}aC^{-\frac12}\otimes C'^{-\frac12}bC'^{-\frac12})I_1(C\otimes C')^{\frac12} \\ &\le I_1(C\otimes C') = D. \end{aligned}$$

Hence, for a dense subset of $[0,\frac12]$ the continuous functions g and

$$\tilde g(z) = \mathrm{tr}_{\mathcal{M}\otimes\mathcal{N}}(I_1(a\otimes b)^z D^{1-z}u)$$

coincide and therefore $g=\tilde g$. Since $u\in\mathcal{M}\bar\otimes\mathcal{N}$ was arbitrary and the map $x\mapsto xD^{1-\frac1p}$ is injective, we deduce

$$I_p(a^{\frac1p}\otimes b^{\frac1p}) = (a^{\frac1p}C^{-\frac1p}\otimes b^{\frac1p}C^{-\frac1p})D^{\frac1p} = I_1(a\otimes b)^{\frac1p}$$

for all $p\ge 2$. However, for $1\le p\le 2$, we have

$$I_p(a^{\frac1p}\otimes b^{\frac1p}) = I_{2p}(a^{\frac1{2p}}\otimes b^{\frac1{2p}})^2 = I_1(a\otimes b)^{\frac2{2p}} = I_1(a\otimes b)^{\frac1p}.$$

Since $C^{\frac12}\mathcal{M}_+C^{\frac12}$, $C'^{\frac12}\mathcal{N}_+C'^{\frac12}$ is dense in $L_1(\mathcal{M})_+$, $L_1(\mathcal{N})_+$, respectively and the inverse of the Mazur map $D\mapsto D^{\frac1p}$ is continuous (see [Ra2, Lemma 3.2]), we obtain (3.14) in the σ-finite case. This also shows that the family of maps $(I_p)_{p\ge1}$ is uniquely determined by

$$I_1(D_\phi\otimes D_\psi) = D_{\phi\otimes\psi}$$

which holds for all $\phi\in\mathcal{N}_*$ and $\psi\in\mathcal{M}_*$ and operators D_ϕ, D_ψ, $D_{\phi\otimes\psi}$ associated to ϕ, ψ and $\phi\otimes\psi$, respectively. Indeed, given C_1, C_2 and C_1' and C_2' with full support, we have

$$\begin{aligned} I_p^{C_2,C_2'}(mC_1^{\frac1p}\otimes nC_1'^{\frac1p}) &= (m\otimes n)I_1^{C_2,C_2'}(C_1\otimes C_1')^{\frac1p} = (m\otimes n)D^{\frac1p}_{\phi_{C_1}\otimes\psi_{C_2}} \\ &= (m\otimes n)I_1^{C_1,C_1'}(C_1\otimes C_1')^{\frac1p} = I_p^{C_1,C_1'}(mC_1\otimes nC_1'). \end{aligned}$$

Here, we used the states $\phi_{C_1}(x) = \mathrm{tr}(C_1x)$, $\psi_{C_1'}(x) = \mathrm{tr}(C_1'x)$. With the help of this uniqueness property, we are now able to obtain the same result in the general case. Let w, w' be strictly semifinite weights on $\mathcal{M}$, $\mathcal{N}$ respectively and (e_i), (f_j) be an increasing net of projections converging to 1 such that $\sigma_t^w(e_i)=e_i$, $w_i = e_iwe_i$ is finite and $\sigma_t^{w'}(f_j)=f_j$ and $w_j' = f_jw'e_j$ is finite. By compatibility, we deduce that $\bigcup_{i,j} I_p^{w_i\otimes w_j'}$ is well-defined. Since $p<\infty$, we deduce from $\lim_{i,j} e_i\otimes f_j = 1$ the norm density of I_p. Let us conclude this proof by showing that this construction is independent of the weight w, w'. Indeed, if $\tilde w$ and $\tilde w'$ is another pair of normal faithful weights, then the equation $I_1(D_\phi\otimes D_\psi) = D_{\phi\otimes\psi}$ is preserved by

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the canonical isomorphism (induced by the cocyle) of the corresponding Haagerup L_1 spaces. Moreover, the canonical isomorphisms

$$\alpha_p^{\mathcal{N}} \colon L_p(\mathcal{N}, w) \to L_p(\mathcal{N}, \tilde{w}'),$$
$$\alpha_p^{\mathcal{M}} \colon L_p(\mathcal{M}, w') \to L_p(\mathcal{N}, \tilde{w}'),$$
$$\alpha_p^{\mathcal{N}\otimes\mathcal{M}} \colon L_p(\mathcal{N}\otimes\mathcal{M}, w\otimes w') \to L_p(\mathcal{N}\otimes\mathcal{M}, \tilde{w}\otimes\tilde{w}')$$

preserve the Mazur map. Therefore, we deduce that

$$\begin{aligned}\alpha_p^{\mathcal{N}\otimes\mathcal{M}} I_p((\alpha_p^{\mathcal{N}})^{-1}(C^{\frac{1}{p}}) \otimes (\alpha_p^{\mathcal{M}})^{-1}(C'^{\frac{1}{p}}))& \\ &= \alpha_p^{\mathcal{N}\otimes\mathcal{M}} I_p((\alpha_1^{\mathcal{N}})^{-1}(C)^{\frac{1}{p}} \otimes (\alpha_1^{\mathcal{M}})^{-1}(C')^{\frac{1}{p}}) \\ &= \alpha_p^{\mathcal{N}\otimes\mathcal{M}}\big((D^{w\otimes w'}_{\phi_C\otimes\psi_{C'}})^{\frac{1}{p}}\big) = (D^{\tilde{w}\otimes\tilde{w}'}_{\phi_C\otimes\psi_{C'}})^{\frac{1}{p}}.\end{aligned}$$

for all norm one elements $C \in L_1(\mathcal{N})_+$, $C' \in L_1(\mathcal{M})_+$ with states $\phi_C(x) = tr(Cx)$, $\psi_{C'}(y) = tr(C'y)$ and associated operator $D^{w\otimes w'}_{\phi_C\otimes\psi_{C'}} \in L_1(\mathcal{N}\otimes\mathcal{M}, w\otimes w')$, $D^{\tilde{w}\otimes\tilde{w}'}_{\phi_C\otimes\psi_{C'}} \in L_1(\mathcal{N}\otimes\mathcal{M}, \tilde{w}\otimes\tilde{w}')$, respectively. Using the fact that the α_ps preserve the module action of $\mathcal{N}$, $\mathcal{M}$, $\mathcal{N}\otimes\mathcal{M}$, respectively, we deduce that the family of maps

$$\tilde{I}_p = \alpha_p^{\mathcal{N}\otimes\mathcal{M}} \circ I_p \circ (\alpha_p^{\mathcal{N}})^{-1} \otimes (\alpha_p^{\mathcal{M}})^{-1}$$

satisfies (3.12)–(3.14) and does not dependent on w and w'. ■

Remark 3.7 This construction may be extended to $0 < p < 1$. Again, we assume that $\mathcal{M}$ and $\mathcal{N}$ are σ-finite, ϕ and ψ are normal faithful states with densities C and C' and joint density D. Given analytic elements m, n, we observe that for $\frac{1}{2} \le p \le 1$

$$\begin{aligned}\big\|(m\otimes n)D^{\frac{1}{p}}\big\|_p^2 &= \big\|D^{\frac{1}{p}}(m^*m\otimes n^*n)D^{\frac{1}{p}}\big\|_{2p} = \big\|\sigma_{-\frac{i}{p}}(m^*m\otimes n^*n)D^{\frac{2}{p}}\big\|_{2p} \\ &\le \big\|\sigma^{\phi}_{-\frac{i}{p}}(m^*m)C^{\frac{2}{p}}\big\|_{2p}\big\|\sigma^{\psi}_{-\frac{i}{p}}(n^*n)C'^{\frac{2}{p}}\big\|_{2p} \\ &= \big\|mC^{\frac{1}{p}}\big\|_p^2\big\|nC'^{\frac{1}{p}}\big\|_p^2.\end{aligned}$$

Thus, we still have continuity and hence a Cauchy-sequence argument (using (3.4)) provides a unique continuous extension to the algebraic tensor product $L_p(\mathcal{M}) \otimes L_p(\mathcal{N})$. We still have the multiplicative property (3.13) and thus

$$I_p(a^{\frac{1}{p}} \otimes b^{\frac{1}{p}}) = I_{2p}(a^{\frac{1}{2p}} \otimes b^{\frac{1}{2p}})^2 = I_1(a\otimes b)^{\frac{1}{p}}.$$

Hence, the properties (3.12)–(3.14) remain true for $\frac{1}{2} \le p \le 1$ and then induction yields them for all $0 < p \le 1$.

For the proof of Fubini's theorem, we will consider two different ultrafilters on the index set $I \times J$ defined by

$$S \in \mathcal{U}_1 \iff_{df} \{i \in I | \{j \in J | (i,j) \in S\} \in \mathcal{U}'\} \in \mathcal{U}$$

and

$$S \in \mathcal{U}_2 \iff_{df} \{j \in J | \{i \in I | (i,j) \in S\} \in \mathcal{U}\} \in \mathcal{U}'.$$

Given densities $d \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$ and $d' \in \prod_{\mathcal{U}'} L_1(\mathcal{N}_j)$, we will consider two different densities

$$d_1(i,j) = I_1(d(i) \otimes d'(j)) \in \prod\nolimits_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j)$$

and

$$d_2(i,j) = I_1(d(i) \otimes d'(j)) \in \prod\nolimits_{\mathcal{U}_2} L_1(\mathcal{M}_i \otimes \mathcal{N}_j).$$

Throughout the rest of the paper, we will use the following notation:

$$\mathcal{M}_0 = \prod \mathcal{M}_i, \quad \mathcal{N}_0 = \prod \mathcal{N}_j, \quad C_0 = \mathcal{M}_0 \otimes_{\min} \mathcal{N}_0.$$

Clearly, $d \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$, $d' \in \prod_{\mathcal{U}'} L_1(\mathcal{N}_j)$ define normal states ϕ_d, $\phi_{d'}$ on $\mathcal{M}_{\mathcal{U}}$, $\mathcal{N}_{\mathcal{U}'}$ respectively. Let us denote by $s(d) = s(\phi_d)$, $s(d') = s(\phi_{d'})$ their corresponding support projections. We will also use the normal functional ϕ_{d_1} on

$$\mathcal{A}_{\mathcal{U}_1} = (\prod\nolimits_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j))^* = z_{\mathcal{U}_1}(\prod \mathcal{M}_i \bar{\otimes} \mathcal{N}_j)^{**}.$$

In the following, we will denote by $\pi_0 \colon C_0 \to \mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$ the natural *-homomorphism given by $\pi_0(n \otimes m) = n \otimes m$ and by $\pi \colon C_0 \to \mathcal{A}_{\mathcal{U}_1}$ the *-homomorphism given by

$$\langle \pi\Big(\sum_{k=1}^{l} n_k \otimes m_k\Big), y\rangle = \sum_{k=1}^{l} \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \langle n_k(i) \otimes m_k(j), y(i,j)\rangle$$

for all $x = \sum_{k=1}^{l} n_k \otimes m_k \in C_0$ and all $y \in \prod_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j)$.

Proposition 3.8

(i) *There is a normal, completely positive map*

$$E \colon \mathcal{A}_{\mathcal{U}_1} \to \mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$$

such that

$$\pi_0 = E \circ \pi.$$

(ii) *Let $s(E)$ be the support projection of E, then there is a von Neumann subalgebra C of $s(E)\mathcal{A}_{\mathcal{U}_1}s(E)$ such that the restriction $\rho = E|_C$ is a von Neumann algebra isomorphism between C and $\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$ satisfying*

$$\rho\big(s(E)\pi(x)s(E)\big) = \pi_0(x) \text{ and } E\big(\pi(x_1)z\pi(x_2)\big) = \pi_0(x_1)E(z)\pi_0(x_2)$$

for all $x, x_1, x_2 \in C_0$ and $z \in \mathcal{A}_{\mathcal{U}_1}$.

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(iii) $\mathcal{E} = \rho^{-1}E\colon s(E)\mathcal{A}_{\mathcal{U}_1}s(E) \to C$ *is a normal faithful conditional expectation onto* C.

(iv) *Let* $d \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$, $d' \in \prod_{\mathcal{U}'} L_1(\mathcal{N}_j)$ *and* d_1 *defined as above. Let* $\phi_{T_1^1(d_1)}$ *be the normal state induced by* d_1. *Then*

$$E_*(\phi_d \otimes \phi_{d'}) = \phi_{T_1^1(d_1)}$$

and $s(\phi_{d_1}) \leq s(E)$.

Proof Let (P_i) be the family of completely contractive projections onto the normal part $P_i\colon \mathcal{M}_i^* \to L_1(\mathcal{M}_i)$ and (Q_j) be the corresponding family $Q_j\colon \mathcal{N}_j^* \to L_1(\mathcal{N}_j)$. Then $P = (P_i)$ and $Q = (Q_j)$ define completely contractive projections. We consider

$$\widehat{i_1 \otimes i_1'}\colon \Big(\prod\nolimits_{\mathcal{U}} L_1(\mathcal{M}_i)\Big) \mathbin{\widehat{\otimes}} \Big(\prod\nolimits_{\mathcal{U}'} L_1(\mathcal{N}_j)\Big) \to \prod\nolimits_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j)$$

defined by

$$\widehat{i_1 \otimes i_1'}(a \otimes b)(i, j) = I_1(a(i) \otimes b(j)).$$

We will use the map Φ_1 from the proof of Proposition 2.11. We deduce from Proposition 2.11 that the embedding

$$\Phi_1\widehat{i_1 \otimes i_1'}\colon \Big(\prod\nolimits_{\mathcal{U}} L_1(\mathcal{M}_i)\Big) \mathbin{\widehat{\otimes}} \Big(\prod\nolimits_{\mathcal{U}'} L_1(\mathcal{N}_j)\Big) \to \Big(\prod\nolimits_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j\Big)^*$$

given by

$$\begin{aligned}\Phi_1\widehat{i_1 \otimes i_1'}(a \otimes b)(x) &= \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} tr_{\mathcal{M}_i \otimes \mathcal{N}_j}(I_1(a(i) \otimes b(j))x(i, j)) \\ &= \mathrm{Tr}_{\mathcal{U}_1}(\widehat{i_1 \otimes i_1'}(a \otimes b)x)\end{aligned} \tag{3.15}$$

is well-defined and isometric. Since Φ_1 is isometric, we deduce that $\widehat{i_1 \otimes i_1'}$ is isometric as well. Tensoring with $S_1^n = M_n^*$, it is easily seen that $\widehat{i_1 \otimes i_1'}$ is indeed completely isometric. Let $E_1 = (\widehat{i_1 \otimes i_1'})^*\colon (\prod_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j))^* \to \mathcal{M}_{\mathcal{U}}\bar{\otimes}\mathcal{N}_{\mathcal{U}'}$ be the dual map. Then $E = E_1(T_1^1)^*\colon \mathcal{A}_{\mathcal{U}_1} \to \mathcal{M}_{\mathcal{U}}\bar{\otimes}\mathcal{N}_{\mathcal{U}'}$ is completely contractive and unital. According to [Tk, Lemma 3.2] (see also [Pa]), we deduce that E is completely positive. Consider $x = \sum_{k=1}^{l} n_k \otimes m_k \in C_0$ and $z \in \prod \mathcal{M}_i\bar{\otimes}\mathcal{N}_j$. Then we get, for every $v \otimes w \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i) \mathbin{\widehat{\otimes}} \prod_{\mathcal{U}'} L_1(\mathcal{N}_j)$,

$$\begin{aligned}\langle E_1(\pi(x)z), \widehat{i_1 \otimes i_1'}(v \otimes w)\rangle &= \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \big\langle x(i, j)z(i, j), I_1(v(i) \otimes w(j))\big\rangle \\ &= \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \sum_{k=1}^{l} \big\langle (n_k(i) \otimes m_k(j))z(i, j), I_1(v(i) \otimes w(j))\big\rangle \\ &= \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \big\langle z(i, j), \sum_{k=1}^{n} I_1(a_k(i).v(i) \otimes b_k(j).w(j))\big\rangle \\ &= \langle E_1(z), \pi_0(x).(v \otimes w)\rangle \\ &= \langle \pi_0(x)E_1(z), v \otimes w\rangle.\end{aligned}$$

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Since T_1^1 is compatible with the natural inclusion of

$$\prod\nolimits_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j) \subset \Big(\prod\nolimits_{I\times J} \mathcal{M}_i \bar{\otimes} \mathcal{N}_j \Big)^*,$$

and using $E(1) = 1$ we deduce $\pi_0 = E \circ \pi$ and the proof of (i) is completed. Note that a similar argument yields $E(z\pi(x)) = E(z)\pi_0(x)$. Let $s = s(E)$ be the support projection of E. Then we deduce for every $x \in C_0$ and $z = \pi(x)$,

$$\begin{aligned} E(sz^*sszs) = E(z^*sz) &= \pi_0(x)^*E(s)\pi_0(x) \\ &= \pi_0(x)^*\pi_0(x) = E(z^*)E(z). \end{aligned}$$

Hence $s\pi(C_0)s$, and by strong* continuity also its weak closure C, is in the multiplicative domain of E. See [Ch] for the definition and properties of the multiplicative domain. Since E is obviously faithful on $s\mathcal{A}_{\mathcal{U}_1}s$, we deduce that E yields an injective von Neumann algebra homomorphism $\rho = E|_C : C \to \mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$. Let us show that ρ is surjective. Given $x \in \mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$, we can find $\tilde{x} \in C_0^{**}$ such that the normalization π_0^{nor} of π_0 satisfies $\pi_0^{nor}(\tilde{x}) = x$. According to Kaplansky's density Theorem [Tk, III. 3.4.], we can find a bounded net $x_\lambda \in C_0$ converging weakly to $\tilde{x}$. Let us denote by $\pi^{nor} : C_0^{**} \to \mathcal{A}_{\mathcal{U}_1}$ the normalization of π. Then $\pi^{nor}(x_\lambda)$ is weakly convergent to $\pi^{nor}(\tilde{x})$ and we deduce

$$E(\pi^{nor}(\tilde{x})) = \lim_\lambda E(\pi^{nor}(x_\lambda)) = \lim_\lambda \pi_0(x_\lambda) = x.$$

Hence ρ is an isomorphism between sCs and $\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$. Moreover, $\mathcal{E} = \rho^{-1}E$ is a conditional expectation onto C. For the last assertion (iv), we first observe that by definition $\widehat{i_1 \otimes i_1'}(d \otimes d') = d_1$. Let $\phi_{T_1^1(d_1)}$ be the state on $\mathcal{A}_{\mathcal{U}_1}$ induced by d_1. We consider the family of states $\psi(i, j) = \phi_{d(i)} \otimes \phi_{d'(j)}$ and deduce from (3.15) that

$$\phi_{T_1^1(d_1)} = \Phi_1(\psi) = E_*(\phi_d \otimes \phi_{d'}) = (\phi_d \otimes \phi_{d'}) \circ E. \tag{3.16}$$

Moreover, for every $x \in \mathcal{A}_{\mathcal{U}_1}$

$$\phi_{T_1^1(d_1)}(x) = (\phi_d \otimes \phi_{d'})(E(x)) = (\phi_d \otimes \phi_{d'})(E(sxs)) = \phi_{d_1}(sxs).$$

The assertion is proved. ■

The next Lemma enables us to reduce the study of finite sets in the ultraproduct to spaces generated by one state and thus to 'localize' the proof of the Fubini theorem.

Lemma 3.9 *Let $0 < p < \infty$ and $x_1, \dots, x_n \in \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$. Then there exists a positive element $d \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$ and elements $b_1, \dots, b_n \in \prod \mathcal{M}_i$ such that for all $k = 1, \dots, n$ and $i \in I$*

$$x_k(i) = d^{\frac{1}{2p}}(i) b_k(i) d^{\frac{1}{2p}}(i).$$

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Proof We can decompose $x_k(i) = \sum_{r=1}^{4} (-1)^{\frac{r}{2}} x_{k,r}(i)$ where all the $x_{k,r}(i)$'s are positive elements and $\|x_{k,r}(i)\|_p \leq \|x_k(i)\|_p$. Therefore it suffices to show the assertion for positive elements $x_1, \dots, x_n$. Define

$$a(i) = \sum_{k=1}^{n} x_k(i)$$

and let $s(i)$ be its support projection. For every index $i \in I$ and every $k = 1, \dots, n$, we have $x_k(i) \leq a(i)$. According to [J1, Lemma 1.1], we deduce that the contractions

$$v_k(i) = x_k(i)^{\frac{1}{2}} a(i)^{-\frac{1}{2}} s(i) \in \mathcal{M}_i$$

Hence, we get

$$x_k(i) = a(i)^{\frac{1}{2}} v_k(i)^* v_k(i) a(i)^{\frac{1}{2}}.$$

We obtain contractions $b_k(i) = v_k(i)^* v_k(i)$ and the positive elements $d(i) = a(i)^p \in L_1(\mathcal{M}_i)$. Then, we get $d^{\frac{1}{2p}}(i) = a(i)^{\frac{1}{2}}$ such that $x_k(i) = d^{\frac{1}{2p}}(i) b_k(i) d^{\frac{1}{2p}}(i)$ for all $k = 1, \dots, n$ and for all $i \in I$. ■

Proof of Theorem 0.1 Let $x_1, \dots, x_l \in \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$ and $y_1, \dots, y_l \in \prod_{\mathcal{U}'} L_p(\mathcal{N}_j)$. We apply Lemma 3.9 and an obvious normalization in order to find positive states $d \in \prod_{\mathcal{U}} L_1(\mathcal{M}_i)$ and $d' \in \prod_{\mathcal{U}'} L_1(\mathcal{N}_j)$ and bounded elements $n_k \in \prod \mathcal{M}_i$, $m_k \in \prod \mathcal{N}_j$ such that for $k = 1, \dots, l$,

$$x_k = d^{\frac{1}{2p}} m_k d^{\frac{1}{2p}} \text{ and } y_k = d'^{\frac{1}{2p}} n_k d'^{\frac{1}{2p}}.$$

Let us define

$$x = \sum_{k=1}^{l} n_k \otimes m_k \in C_0.$$

We consider

$$d_1(i, j) = I_1(d(i) \otimes d(j)) \in \prod\nolimits_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j)$$

and the states

$$\phi = \phi_d \otimes \phi_{d'} \in (\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'})_* \text{ and } \phi_{T_1^1(d_1)} = \phi \circ E \in (\mathcal{A}_{\mathcal{U}_1})_*.$$

Here E is the completely positive map from Proposition 3.8 and the last equality is (3.16). We denote the densities of ϕ and $\phi_{T_1^1(d_1)}$ by

$$D \in L_1(\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}) \text{ and } T_1^1(d_1) = D_1 \in L_1(\mathcal{A}_{\mathcal{U}_1}), \tag{3.17}$$

respectively. Let $s(E)$ be the support projection of E. We recall that

$$C \subset s(E) \mathcal{A}_{\mathcal{U}_1} s(E)$$

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is isomorphic to $\mathcal{M}_{\mathcal{U}}\bar{\otimes}\mathcal{N}_{\mathcal{U}'}$ and the isomorphism is given by the restriction of E to C. Since $\mathcal{E} = \rho^{-1}E\colon s(E)\mathcal{A}_{\mathcal{U}_1}s(E) \to C$ is faithful, we deduce from Lemma 3.3 that

$$\rho^{-1}(s(\phi_d \otimes \phi_{d'})) = s(D_1) = s(\phi_{d_1}).$$

We will use the notation $s^{\otimes}$ for the support of $\phi_d \otimes \phi_{d'}$ and $s_1 = s(\phi_{d_1})$. The restriction $\mathcal{E}_{s_1}\colon s_1\mathcal{A}_{\mathcal{U}_1}s_1 \to s_1Cs_1$ of $\mathcal{E}$ to $s_1\mathcal{A}_{\mathcal{U}_1}s_1$ is a normal faithful conditional expectation satisfying $\phi_{d_1} = (\phi_d \otimes \phi_{d'}) \circ \mathcal{E}_{s_1}$. Hence, we can apply Proposition 3.1 and find an isometric embedding

$$i_p\colon L_p(s^{\otimes}(\mathcal{M}_{\mathcal{U}}\bar{\otimes}\mathcal{N}_{\mathcal{U}'})s^{\otimes}) \to L_p(s_1\mathcal{A}_{\mathcal{U}_1}s_1)$$

such that (using Lemma 3.5)

$$i_p(D^{\frac{1}{2p}}\pi_0(x)D^{\frac{1}{2p}}) = i_p(D^{\frac{1}{2p}}s^{\otimes}\pi_0(x)s^{\otimes}D^{\frac{1}{2p}}) = D_1^{\frac{1}{2p}}s_1\pi(x)s_1D_1^{\frac{1}{2p}} = D_1^{\frac{1}{2p}}\pi(x)D_1^{\frac{1}{2p}}.$$

Therefore, we have

$$\|D^{\frac{1}{2p}}\pi_0(x)D^{\frac{1}{2p}}\|_{L_p(\mathcal{M}_{\mathcal{U}}\otimes\mathcal{N}_{\mathcal{U}'})} = \|D_1^{\frac{1}{2p}}\pi(x)D_1^{\frac{1}{2p}}\|_{L_p(\mathcal{A}_{\mathcal{U}_1})}. \tag{3.18}$$

Now, we apply Raynaud's isomorphism T_p^1 to $\mathcal{A}_{\mathcal{U}_1}$ and deduce from (3.11) and (3.17) that

$$\begin{aligned} T_p^1(D_1^{\frac{1}{2p}}\pi(x)D_1^{\frac{1}{2p}}) &= T_{2p}^1(D_1^{\frac{1}{2p}})\pi(x)T_{2p}^1(D_1^{\frac{1}{2p}}) = T_1^1(D_1)^{\frac{1}{2p}}\pi(x)T_1^1(D_1)^{\frac{1}{2p}} \\ &= d_1^{\frac{1}{2p}}\pi(x)d_1^{\frac{1}{2p}}. \end{aligned}$$

Combining this with (3.18) we deduce from [Ra2, Theorem 4.3]

$$\|d_1^{\frac{1}{2p}}\pi(x)d_1^{\frac{1}{2p}}\|_{\prod_{\mathcal{U}_1} L_p(\mathcal{M}_i\otimes\mathcal{N}_j)} = \|D^{\frac{1}{2p}}\pi_0(x)D^{\frac{1}{2p}}\|_{L_p(\mathcal{M}_{\mathcal{U}}\otimes\mathcal{N}_{\mathcal{U}'})}. \tag{3.19}$$

Moreover, the module properties (3.12)–(3.14) imply

$$(d_1^{\frac{1}{2p}}\pi(x)d_1^{\frac{1}{2p}})(i,j) = \sum_{k=1}^{l} I_p(x_k(i) \otimes y_k(j)).$$

By definition of $\mathcal{U}_1$, we deduce with (3.19)

$$\begin{aligned} \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^{l} I_p(x_k(i) \otimes y_k(j)) \Big\|_{L_p(\mathcal{M}_i\otimes\mathcal{N}_j)} \\ &= \Big\| \Big(\sum_{k=1}^{l} I_p(x_k(i) \otimes y_k(j)) \Big)_{ij} \Big\|_{\prod_{\mathcal{U}_1} L_p(\mathcal{M}_i\otimes\mathcal{N}_j)} \\ &= \|T_p^1(D_1^{\frac{1}{2p}}\pi(x)D_1^{\frac{1}{2p}})\|_{\prod_{\mathcal{U}_1} L_p(\mathcal{M}_i\otimes\mathcal{N}_j)} \\ &= \|D^{\frac{1}{2p}}\pi_0(x)D^{\frac{1}{2p}}\|_{L_p(\mathcal{M}_{\mathcal{U}}\otimes\mathcal{N}_{\mathcal{U}'})}. \end{aligned}$$

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Changing $\mathcal{U}_1$ to $\mathcal{U}_2$ and d_1 to d_2, we obtain the same result and hence

$$\begin{aligned}\lim_{i,\mathcal{U}}\lim_{j,\mathcal{U}'}\Big\|\sum_{k=1}^{l} I_p(x_k(i)\otimes y_k(j))\Big\|_{L_p(\mathcal{M}_i\otimes\mathcal{N}_j)} \\ = \|D^{\frac{1}{2p}}\pi_0(x)D^{\frac{1}{2p}}\|_{L_p(\mathcal{M}_{\mathcal{U}}\otimes\mathcal{N}_{\mathcal{U}'})} \\ = \lim_{j,\mathcal{U}'}\lim_{i,\mathcal{U}}\Big\|\sum_{k=1}^{l} I_p(x_k(i)\otimes y_k(j))\Big\|_{L_p(\mathcal{M}_i\otimes\mathcal{N}_j)}.\end{aligned}$$

The assertion is proved. ■

Hidden in Proposition 3.8 is the fact that for $p = 1$ the embedding

$$\widehat{i_1\otimes i_1'}\colon \Big(\prod\nolimits_{\mathcal{U}} L_1(\mathcal{M}_i)\Big)\,\widehat{\otimes}\,\Big(\prod\nolimits_{\mathcal{U}'} L_1(\mathcal{N}_j)\Big) \to \prod\nolimits_{\mathcal{U}_1} L_1(\mathcal{M}_i\otimes\mathcal{N}_j)$$

given by

$$(\widehat{i_1\otimes i_1'})(a\otimes b)(i,j) = I_1(a(i)\otimes b(j))$$

is (completely) isometric. We want to study the image of the corresponding map

$$\widehat{i_p\otimes i_p'}\colon \Big(\prod\nolimits_{\mathcal{U}} L_p(\mathcal{M}_i)\Big)\otimes\Big(\prod\nolimits_{\mathcal{U}'} L_p(\mathcal{N}_j)\Big) \to \prod\nolimits_{\mathcal{U}_1} L_p(\mathcal{M}_i\otimes\mathcal{N}_j)$$

given by

$$(\widehat{i_p\otimes i_p'})(a\otimes b)(i,j) = I_p(a(i)\otimes b(j))$$

and show that the closure of the image is (completely) complemented and (completely) isometrically isomorphic to $L_p(\mathcal{M}_{\mathcal{U}}\otimes\mathcal{N}_{\mathcal{U}'})$.

We will need the following extension of Proposition 3.1 for non-faithful conditional expectations which occur naturally in our context.

Lemma 3.10 *Let $\mathcal{E}\colon\mathcal{N}\to\mathcal{M}$ be a normal conditional expectation and $1\le p<\infty$. Then there is an isometric embedding $\iota_p\colon L_p(\mathcal{M})\to L_p(\mathcal{N})$ and a contraction $\mathcal{E}_p\colon L_p(\mathcal{M})\to L_p(\mathcal{N})$ such that*

$$\iota_p(D_{\mathcal{M}}^{\frac{1-\theta}{p}}xD_{\mathcal{M}}^{\frac{\theta}{p}}) = D_{\mathcal{N}}^{\frac{1-\theta}{p}}xD_{\mathcal{N}}^{\frac{\theta}{p}} \text{ and } \mathcal{E}_p(D_{\mathcal{N}}^{\frac{1-\theta}{p}}yD_{\mathcal{N}}^{\frac{\theta}{p}}) = D_{\mathcal{M}}^{\frac{1-\theta}{p}}\mathcal{E}(y)D_{\mathcal{M}}^{\frac{\theta}{p}}$$

hold for all $0\le\theta\le1$, $x\in\mathcal{M}$, $y\in\mathcal{N}$ and all $\phi\in\mathcal{M}_*$ with density $\mathcal{D}_{\mathcal{M}}\in L_1(\mathcal{M})$ and density $\mathcal{D}_{\mathcal{N}}$ of $\phi\circ\mathcal{E}$. For conjugated indices $\frac{1}{p}+\frac{1}{p'}=1$ the duality $\mathcal{E}_p^*=\iota_{p'}$ holds with respect to the duality bracket given by the trace.

Proof Let $s = s(\mathcal{E})$ be the support projection of $\mathcal{E}$. As in the proof of Lemma 3.8, we observe that for every $x\in\mathcal{M}$

$$\mathcal{E}(sx^*ssx) = x^*\mathcal{E}(s)x = x^*x = \mathcal{E}(x^*)\mathcal{E}(x).$$

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Hence, the restriction of $\mathcal{E}$ to $s\mathcal{M}s$ yields a von Neumann algebra homomorphism. In particular, we deduce from Lemma 3.3 applied to the faithful unital, normal conditional expectation $\mathcal{E}\colon s\mathcal{N}s \to \mathcal{M}$ that for every normal $\phi \in \mathcal{M}_*$, we have $s(\phi) = s(E \circ \phi)$. In view of

$$D_{\mathcal{N}}^{\frac{1}{p}} s(\phi) = D_{\mathcal{N}}^{\frac{1}{p}} = s(\phi) D_{\mathcal{N}}^{\frac{1}{p}},$$

we conclude that it suffices to prove the assertion under the additional assumption that $\mathcal{E}$ is faithful. Now, we consider a normal strictly semifinite weight w on $\mathcal{M}$ with increasing family of projections (e_i) converging to 1. For $p < \infty$ it is easy to show that $\bigcup_i e_i L_p(\mathcal{M}) s_i$, $\bigcup_i e_i L_p(\mathcal{N}) e_i$ is norm dense in $L_p(\mathcal{M})$, $L_p(\mathcal{N})$, respectively. For $i \le i'$ and associated positive functions $w_i = e_i w e_i$ and $w_{i'} = e_{i'} w e_{i'}$, we can find a bounded operator $x \in \mathcal{M}$ such that the corresponding densities satisfy

$$D_i^{\frac{1}{2p}} = x D_{i'}^{\frac{1}{2p}}.$$

Therefore, the contractions i_p and $\mathcal{E}_p$ from Proposition 3.1 constructed for $w_{i'}(1)^{-1} w_{i'}$ extend the corresponding maps for $w_i(1)^{-1} w_i$. By density, we can then extend these maps to $L_p(\mathcal{M})$, $L_p(\mathcal{N})$, respectively, and obtain the assertion by approximation. The last assertion follows again by approximation from Remark 3.2. ■

In our next theorem we combine these general observations with the proof of Theorem 0.1. This provides the following description for the closure of the algebraic tensor product of two ultraproducts with respect to the natural L_p norm.

Theorem 3.11 *Let $1 \le p < \infty$ and $T_p\colon L_p(\mathcal{M}_{\mathcal{U}}) \to \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$, $T_p'\colon L_p(\mathcal{N}_{\mathcal{U}'}) \to \prod_{\mathcal{U}'} L_p(\mathcal{N}_j)$, $T_p^1\colon L_p(\mathcal{A}_{\mathcal{U}_1}) \to \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j)$ be Raynaud's isomorphisms. Then there exists an isometric embedding $\iota_p\colon L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'}) \to L_p(\mathcal{A}_{\mathcal{U}_1})$ making the following diagram commutative*

$$\begin{array}{ccc}
L_p(\mathcal{M}_{\mathcal{U}}) \otimes L_p(\mathcal{N}_{\mathcal{U}'}) & \xrightarrow{T_p \otimes T_p'} & \left(\prod_{\mathcal{U}} L_p(\mathcal{M}_i)\right) \otimes \left(\prod_{\mathcal{U}'} L_p(\mathcal{N}_j)\right) \\
\Big\downarrow I_p & & \\
L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'}) & & \Big\downarrow \widehat{i_p \otimes i_p'} V \\
\Big\downarrow \iota_p & & \\
L_p(\mathcal{A}_{\mathcal{U}_1}) & \xrightarrow{T_p^1} & \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j)
\end{array}$$

For every state $\phi \in (\mathcal{M}_{\mathcal{U}})_* \otimes (\mathcal{N}_{\mathcal{U}'})_*$ with density $D \in L_1(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$ and density $D_1 \in L_1(\mathcal{A}_{\mathcal{U}_1})$ of $\iota_1(\phi)$ and for every $x \in \mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$, we have

$$T_p^1 \iota_p (D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = T_1(D_1)^{\frac{1}{2p}} x T_1(D_1)^{\frac{1}{2p}}.$$

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Proof We will use the notation from Lemma 3.8. Since $\mathcal{M}_{\mathcal{U}}\bar{\otimes}\mathcal{N}_{\mathcal{U}'}$ is isomorphic to $C \subset s(E)\mathcal{A}_{\mathcal{U}_1}s(E)$, we deduce from Lemma 3.10 the existence of the map

$$\iota_p \colon L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'}) \to L_p(\mathcal{A}_{\mathcal{U}_1})$$

satisfying

$$\iota_p(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = D_1^{\frac{1}{2p}} x D_1^{\frac{1}{2p}}$$

for all positive densities $D \in L_1(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$ and all $x \in \mathcal{M}_{\mathcal{U}}\bar{\otimes}\mathcal{N}_{\mathcal{U}'}$. According to (3.8) we deduce that ι_p satisfies the equality stated in the second part of the assertion. In order to prove that the diagram is commutative it suffices, by linearity, to consider positive elements $d^{\frac{1}{p}} \in \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$, $d'^{\frac{1}{p}} \in \prod_{\mathcal{U}'} L_p(\mathcal{N}_j)$ and $d_1 = \widehat{i_1 \otimes i_1'}(d \otimes d') \in \prod_{\mathcal{U}_1} L_1(\mathcal{M}_i \otimes \mathcal{N}_j)$. Then the proof of Theorem 0.1 shows that for the corresponding density $D_1 = \widehat{i_1 \otimes i_1'}(d \otimes d') \in L_1(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$, we have

$$\iota_1(D) = D_1$$

and hence by (3.15)

$$T_p^1 \iota_p(D^{\frac{1}{p}}) = T_p^1(D_1^{\frac{1}{p}}) = (T_1(D_1))^{\frac{1}{p}} = d_1^{\frac{1}{p}} = \widehat{i_p \otimes i_p'}(d^{\frac{1}{p}} \otimes d'^{\frac{1}{p}}).$$

The assertion follows by linearity since L_p is spanned by the positive elements. ■

In order to show that the image of ι_p is complemented for $1 \le p < \infty$, we will use the following procedure which associates with every $x \in \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j)$ a linear map $T_x \colon \prod_{\mathcal{U}'} L_{p'}(\mathcal{N}_j) \to \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$ defined by duality as

$$\mathrm{Tr}_{\mathcal{U}}(T_x(b)a) = \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \mathrm{tr}_{\mathcal{M}_i \otimes \mathcal{N}_j} \left(x(i,j) I_{p'}(a(i) \otimes b(j)) \right),$$

where $b \in \prod_{\mathcal{U}'} L_{p'}(\mathcal{N}_j)$ and $a \in \prod_{\mathcal{U}} L_{p'}(\mathcal{M}_i)$. Note that by uniform convexity of $L_p(\mathcal{M}_i)$ we still have $(\prod_{\mathcal{U}} L_{p'}(\mathcal{M}_i))^* = \prod_{\mathcal{U}} L_p(\mathcal{M}_i)$. This definition is motivated by the investigation of completely p-summing maps introduced by Pisier, see [P7].

Corollary 3.12 *Let $1 \le p < \infty$. The closure of $\widehat{i_p \otimes i_p'}$ is (completely) contractively complemented. More precisely*

(i) *There is a (complete) contraction $\mathcal{E}_p^1 \colon \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j) \to \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j)$ onto the closure Y_p of the image of $\widehat{i_p \otimes i_p'}$.*
(ii) *Y_p is (completely) isometrically isomorphic to $L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$.*
(iii) *The kernel J_p of $\mathcal{E}_p^1$ consists of the elements $x \in \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j)$ such that $T_x = 0$.*

Proof Let $1 \le p < \infty$ and

$$\iota_p \colon L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'}) \to L_p(\mathcal{A}_{\mathcal{U}_1})$$

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be the complete contraction from Theorem 3.11. Let E be the completely positive map from Lemma 3.8. Using the obvious isomorphism between $L_p(C)$ and $L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$, we deduce from Lemma 3.10 that there is a complete contraction

$$E_p \colon L_p(\mathcal{A}_{\mathcal{U}_1}) \to L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$$

satisfying

$$E_p(D_1^{\frac{1}{2p}} \pi(x) D_1^{\frac{1}{2p}}) = D^{\frac{1}{2p}} x D^{\frac{1}{2p}}$$

for $x \in \mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$ and all normal states $\phi \in (\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'})_*$ with density D and corresponding density D_1 of $\phi \circ E$. In particular, we deduce from the density of the span of positive elements and Theorem 3.11 that $E_p \iota_p = id_{L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})}$. Since T_p^1 is (completely) isometric, we deduce from the density of $L_p(\mathcal{M}_{\mathcal{U}}) \otimes L_p(\mathcal{N}_{\mathcal{U}'})$ in $L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$ that Y_p coincides with

$$T_p^1 \iota_p(L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})).$$

In particular, Y_p is (completely) isometrically isomorphic to $L_p(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$. Since $E_p \colon L_p(\mathcal{A}_{\mathcal{U}_1}) \to L_p(C)$ is a (complete) contraction onto $L_p(C)$, we deduce that $\mathcal{E}_p^1 = T_p^1 \mathcal{E}_p (T_p^1)^{-1}$ is a (complete) contraction onto Y_p. Hence, (i) and (ii) are proved. Let us consider $x \in \prod_{\mathcal{U}_1} L_p(\mathcal{M}_i \otimes \mathcal{N}_j)$. If $p' < \infty$, we can use duality and the density of $I_{p'}(L_{p'}(\mathcal{M}_{\mathcal{U}}) \otimes L_{p'}(\mathcal{N}_{\mathcal{U}'}))$ in $L_{p'}(\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'})$ to deduce that $\mathcal{E}_p^1(x) = 0$ if and only if

$$\mathrm{tr}_{\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'}}(E_p((T_p^1)^{-1}(x)) I_{p'}(a \otimes b)) = 0$$

for all $a \in L_{p'}(\mathcal{M}_{\mathcal{U}})$ and all $b \in L_{p'}(\mathcal{N}_{\mathcal{U}'})$. If $p' = \infty$, we recall that by (a variation of) Fact 2.5, we have

$$(\mathcal{M}_{\mathcal{U}}^{op} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}^{op})_* = L_1(\mathcal{M}_{\mathcal{U}}) \widehat{\otimes} L_1(\mathcal{N}_{\mathcal{U}'}) = L_1(\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}).$$

Clearly, $\mathcal{M}_{\mathcal{U}}^{op} \otimes \mathcal{N}_{\mathcal{U}'}^{op}$ is σ-weakly dense and hence a normal functional vanishes if it vanishes on $\mathcal{M}_{\mathcal{U}}^{op} \otimes \mathcal{N}_{\mathcal{U}'}^{op}$ and we have the same conclusion. Let $a \in L_{p'}(\mathcal{M}_{\mathcal{U}})$ and $b \in L_{p'}(\mathcal{N}_{\mathcal{U}'})$ and consider $\tilde{a} = T_{p'}^{-1}(a)$, $\tilde{b} = T_{p'}^{-1}(b)$, respectively. Using the isomorphism between C and $\mathcal{M}_{\mathcal{U}} \bar{\otimes} \mathcal{N}_{\mathcal{U}'}$ and the second part of Lemma 3.10, we deduce that

$$\begin{aligned}
\mathrm{tr}_{\mathcal{M}_{\mathcal{U}} \otimes \mathcal{N}_{\mathcal{U}'}}(E_p((T_p^1)^{-1}(x)) I_{p'}(a \otimes b)) &= \langle E_p((T_p^1)^{-1}(x)), I_{p'}(a \otimes b) \rangle \\
&= \mathrm{tr}_{\mathcal{A}_{\mathcal{U}_1}} \big((T_p^1)^{-1}(x) (T_{p'}^1 I_{p'}(a \otimes b)) \big) \\
&= \mathrm{Tr}_{\mathcal{U}_1}(x \widehat{i_{p'} \otimes i'_{p'}}(\tilde{a} \otimes \tilde{b})) \\
&= \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} tr_{\mathcal{M}_i \otimes \mathcal{N}_j} \big(x(i,j) I_{p'}(\tilde{a}(i) \otimes \tilde{b}(j)) \big) \\
&= \mathrm{Tr}_{\mathcal{U}}(T_x(\tilde{b})\tilde{a}),
\end{aligned}$$

We deduce $\mathcal{E}_p^1(x) = 0$ if and only if $T_x = 0$. ■

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4 Appendix

Following a suggestion by G. Elliott, we will show that the Fubini theorem fails for $p = \infty$. Let us start with an estimate for the Fubini constant with respect to n vectors.

Proposition 4.1 *Let $(x_1(i)), \dots, (x_n(i))$ and $(y_1(j)), \dots, (y_n(j))$ be bounded families of operators in $B(H)$, $B(K)$, respectively. Then*

$$\lim_{j,\mathcal{U}'} \lim_{i,\mathcal{U}} \Big\| \sum_{k=1}^{n} x_k(i) \otimes y_k(j) \Big\|_{B(H)\otimes_{\min} B(K)} \le \sqrt{n} \lim_{i,\mathcal{U}} \lim_{j,\mathcal{U}'} \Big\| \sum_{k=1}^{n} x_k(i) \otimes y_k(j) \Big\|_{B(H)\otimes_{\min} B(K)}.$$

holds for all ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ on I, J, respectively.

Proof We will apply Pisier's theorem about the *cb*-distance of an n dimensional operator space to the operator space OH_n (see [P6]). Indeed, OH_n is a subspace of $B(\ell_2)$ spanned by operators $T_1, \dots, T_n$ such that

$$\Big\| \sum_{l=1}^{n} T_l \otimes a_l \Big\|_{B(\ell_2 \otimes K)} = \Big\| \sum_{l=1}^{n} a_l \otimes \bar{a}_l \Big\|_{B(K\otimes K)}^{\frac{1}{2}}$$

for all $a_1, \dots, a_n \in B(K)$. In particular, we have

$$\sup_{l=1,..,n} \|a_l\| \le \Big\| \sum_{l=1}^{n} T_l \otimes a_l \Big\| \le \sum_{l=1}^{n} \|a_l\| \tag{4.1}$$

We refer to [P6] for basic properties of OH_n. Let us fix an index $i \in I$ and consider an n-dimensional subspace $E(i) \subset B(H)$ containing the $x_1(i), \dots, x_n(i)$. According to Pisier's theorem (see [P6]), we can find linearly independent elements $e_1(i), \dots, e_n(i) \in E(i)$ such that

$$\Big\| \sum_{l=1}^{n} T_l \otimes a_l \Big\| \le \Big\| \sum_{k=1}^{n} e_l(i) \otimes a_l \Big\| \le \sqrt{n} \Big\| \sum_{l=1}^{n} T_l \otimes a_l \Big\| \tag{4.2}$$

for all operators $a_l \in B(K)$. Then the $x_k(i)$s are linear combinations of the $e_l(i)$'s and thus there are coefficients $a_{k,l}(i) \in \mathbb{C}$ such that

$$x_k(i) = \sum_{l=1}^{n} a_{kl}(i) e_l(i).$$

It follows from (4.1) and (4.2) that

$$|a_{kl}(i)| \le \Big\| \sum_{l=1}^{n} a_{kl}(i) T_l \Big\| \le \Big\| \sum_{l=1}^{n} a_{kl}(i) e_l(i) \Big\| = \|x_k(i)\|.$$

Recall that we assume $\|x_k(i)\| \le C$ and $\|y_k(j)\| \le C$ for some constant $C > 0$. Then we can find coefficients $a_{kl} \in \mathbb{C}$ such that

$$U_\varepsilon = \Big\{ i \;\Big|\; \forall_{1\le k,l\le n} |a_{kl} - a_{kl}(i)| < \frac{\varepsilon}{Cn^{\frac{5}{2}}} \Big\} \in \mathfrak{U}.$$

For such an element $i \in U_\varepsilon$ we deduce for all j that

$$\begin{aligned}
&\Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) - \sum_{k,l=1}^n a_{kl} e_l(i) \otimes y_k(j) \Big\| \\
&\qquad = \Big\| \sum_{k,l=1}^n (a_{kl}(i) - a_{kl}) e_l(i) \otimes y_k(j) \Big\| \\
&\qquad \le \sqrt{n} \Big\| \sum_{k,l=1}^n (a_{kl}(i) - a_{kl}) T_l(i) \otimes y_k(j) \Big\| \\
&\qquad \le \sqrt{n} \sum_{l=1}^n \Big\| \sum_{k=1}^n (a_{kl}(i) - a_{kl}) y_k(j) \Big\| \\
&\qquad \le \frac{\varepsilon}{Cn^{\frac{5}{2}}} n^{\frac{5}{2}} \sup_{k,j} \|y_k(j)\| < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce

$$\begin{aligned}
\lim_{i,\mathfrak{U}} \lim_{j,\mathfrak{U}'} \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) \Big\| &= \lim_{i,\mathfrak{U}} \lim_{j,\mathfrak{U}'} \Big\| \sum_{k,l=1}^n a_{kl} e_l(i) \otimes y_k(j) \Big\|, \\
\lim_{j,\mathfrak{U}'} \lim_{i,\mathfrak{U}} \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) \Big\| &= \lim_{j,\mathfrak{U}'} \lim_{i,\mathfrak{U}} \Big\| \sum_{k,l=1}^n a_{kl} e_l(i) \otimes y_k(j) \Big\|.
\end{aligned}$$

In particular, we deduce from (4.2)

$$\begin{aligned}
\lim_{i,\mathfrak{U}} \lim_{j,\mathfrak{U}'} \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) \Big\| &= \lim_{i,\mathfrak{U}} \lim_{j,\mathfrak{U}'} \Big\| \sum_{l=1}^n e_l(i) \otimes \Big(\sum_{k=1}^n a_{kl} y_k(j) \Big) \Big\| \\
&\le \sqrt{n} \lim_{j,\mathfrak{U}'} \Big\| \sum_{l=1}^n T_l \otimes \Big(\sum_{k=1}^n a_{kl} y_k(j) \Big) \Big\| \\
&\le \sqrt{n} \lim_{j,\mathfrak{U}'} \lim_{i,\mathfrak{U}} \Big\| \sum_{l=1}^n e_l(i) \otimes \Big(\sum_{k=1}^n a_{kl} y_k(j) \Big) \Big\| \\
&= \sqrt{n} \lim_{j,\mathfrak{U}'} \lim_{i,\mathfrak{U}} \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) \Big\|.
\end{aligned}$$

This yields the upper estimate $\sqrt{n}$ for the Fubini constant with n-vectors. ■

Proposition 4.2 *There are bounded families of operators $(x_1(i)), \dots, (x_n(i))$ and $(y_1(j)), \dots, (y_n(j))$ in $B(\ell_2)$ such that*

$$\lim_j \lim_i \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) \Big\|_{B(\ell_2)\otimes_{\min} B(\ell_2)} = c_n \lim_i \lim_j \Big\| \sum_{k=1}^n x_k(i) \otimes y_k(j) \Big\|_{B(\ell_2)\otimes_{\min} B(\ell_2)}$$

and

$$c_n \geq \frac{\sqrt{n}}{2}.$$

Proof This result is an application of Pisier's estimate of the exactness constant for n-dimensional operator spaces. We will use some operator space terminology as in [P8]. Indeed, given an n-dimensional operator space $E \subset B(\ell_2)$, we consider the operator space E^i defined with E as underlying Banach space and the matrix norms

$$\|x\|_{M_m(E^i)} = \sup_{\|u\colon E \to M_i\|_{cb} \leq 1} \|(id \otimes u)(x)\|_{M_m(M_i)}.$$

Let us collect some obvious facts. If $i < j$, then

$$\|id\colon E^j \to E^i\|_{cb} \leq 1 \tag{4.3}$$

because the $i \times i$ matrices M_i sit as a corner in the $j \times j$ matrices M_j. Thus every complete contraction $u\colon E \to M_i$ induces a contraction $u\colon E \to M_j$. In particular,

$$\lim_i \lim_j \|id\colon E^j \to E^i\|_{cb} = 1.$$

On the other hand, given a matrix $x \in M_m(E)$ we have

$$\|x\|_{M_m(E)} = \sup_i \sup_{i, rk(p_i) \leq i} \|(1 \otimes p_i)x(1 \otimes p_i)\|_{M_m(B(\ell_2))} = \sup_i \|x\|_{M_m(E^i)}.$$

Hence, we deduce from (4.3)

$$\begin{aligned} \lim_j \lim_i \|id\colon E^j \to E^i\|_{cb} &= \inf_j \sup_i \|id\colon E^j \to E^i\|_{cb} \\ &= \inf_j \sup_i \sup_{\|x\|_{M_m(E^j)} \leq 1} \|x\|_{M_m(E^i)} \\ &= \inf_j \|id\colon E^j \to E\|_{cb}. \end{aligned}$$

Let us recall

$$ex(E) = \inf_j \|id\colon E^j \to E\|_{cb}.$$

We refer to [P8] for the fact that there is an n-dimensional operator space $E = E_n$ such that

$$ex(E) \geq \frac{\sqrt{n}}{2}. \tag{4.4}$$

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Finally, let $(x_k) \subset E$ $(x_k^*) \subset E^*$ be an Auerbach basis, *i.e.*, $\|x_k\| = \|x_k^*\| = 1$ and $x_k^*(x_l) = \delta_{kl}$. Denote by $v_i \colon E \to E^i$ the formal identity map and define

$$x_k(i) = v_i(x_k) \in E^i \text{ and } y_k(j) = v_j^{*\,-1}(x_k^*) \in (E^j)^*.$$

Note that $(x_k(i))_{k=1}^n$, $(y_k(i))_{k=1}^n$ still form a biorthogonal basis for $E(i)$. By operator space duality, we may assume $(E^j)^* \subset B(\ell_2)$ and thus

$$\|id\colon E^j \to E^i\|_{cb} = \Big\| \sum_{k=1}^n y_k(j) \otimes x_k(i) \Big\|_{B(\ell_2)\otimes_{\min} B(\ell_2)}.$$

Therefore

$$\lim_i \lim_j \Big\| \sum_{k=1}^n y_k(j) \otimes x_k(i) \Big\|_{B(\ell_2)\otimes_{\min} B(\ell_2)} = 1$$

and

$$\lim_j \lim_i \Big\| \sum_{k=1}^n y_k(j) \otimes x_k(i) \Big\|_{B(\ell_2)\otimes_{\min} B(\ell_2)} = ex(E) \geq \frac{\sqrt{n}}{2}.$$

The assertion is proved. ∎

Remark 4.3 Note that for all $i \in \mathbb{N}$, the space E^i is a subspace of $\ell_\infty(I; M_i)$, a nuclear C^*-algebra. Hence, we may assume $E^i \subset \mathcal{K}$. Similarly, we can consider $((E^j)^*)^l$ as a subspace of $\mathcal{K}$ and thus, we find $x_k(i) \in \mathcal{K}$ and $y_k(j,l) \in \mathcal{K}$ such that

$$\lim_i \lim_{j\leq l} \Big\| \sum_{k=1}^n y_k(j,l) \otimes x_k(i) \Big\| = 1 \quad \text{and} \quad \lim_{j\leq l} \lim_i \Big\| \sum_{k=1}^n y_k(j,l) \otimes x_k(i) \Big\| = c_n.$$

Thus the Fubini theorem for arbitrary ultrafilters does not even hold with a constant for the nuclear C^*-algebra $\mathcal{K}$.

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Department of Mathematics
University of Illinois
Urbana, IL 61801
U.S.A.
e-mail: junge@math.uiuc.edu