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On a conjecture of T. Lewis

D.N. Shanbhag

In this note, Conjecture C1 of Toby Lewis (1976), concerning a reciprocal pair of characteristic variables, is shown to be false.

Let $f(\cdot)$ and $g(\cdot)$ be two probability density functions of absolutely continuous distributions on the real line such that for some positive c_1 and c_2 the functions $c_1g(\cdot)$ and $c_2f(\cdot)$ are characteristic functions corresponding to $f(\cdot)$ and $g(\cdot)$ respectively. In what follows, we shall call the pair $\big(f(\cdot), g(\cdot)\big)$ a reciprocal pair of probability density functions. Recently Lewis [1] conjectured that for a reciprocal pair $\big(f(\cdot), g(\cdot)\big)$ if the distribution corresponding to $f(\cdot)$ is infinitely divisible, then the distribution corresponding to $g(\cdot)$ is also infinitely divisible. From the following theorem it is obvious that this conjecture of Lewis is false.

THEOREM. *There exist reciprocal pairs* $\big(f(\cdot), g(\cdot)\big)$ *of probability density functions such that the distribution corresponding to* $f(\cdot)$ *is infinitely divisible but the distribution corresponding to* $g(\cdot)$ *is not.*

Proof. Consider a real valued function $f^*(\cdot)$ defined on the real line such that

$$(1) \qquad f^*(t) = \exp\left\{\frac{-t^2\sigma^2}{2} + \int_R (\cos tx-1)d\mu(x)\right\} , \quad -\infty < t < \infty ,$$

where $\sigma^2 > 0$ and μ is a measure defined on the Borel σ-field of R such that $\mu(\{0\}) = 0$ and $\mu(R) \neq 0$, and

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$$\int_R \frac{x^2}{1+x^2}\, d\mu(x) < \infty .$$

It is obvious that the $f^*(\cdot)$ considered is the characteristic function of an infinitely divisible distribution. Denote this distribution by F. From Theorem 3.2.2 of Lukacs [2], it follows that $F'(x)$ exists for all x and that there exists a positive α such that $\left(F'(\cdot), \alpha f^*(\cdot)\right)$ is a reciprocal pair of probability density functions. If we denote the distribution function corresponding to probability density function $\alpha f^*(\cdot)$ by G, then it follows that

$$G(-x) + \{1-G(x)\} = o\left(\exp\left(-ax^2\right)\right) \quad \text{as} \quad x \to \infty , \tag{2}$$

where $o < a < \sigma^2/2$. Theorem 2 of Ruegg [3] then implies that if G is infinitely divisible, then it should be symmetric normal. (Note that for Ruegg's Theorem it is sufficient to have 'O' instead of 'o' in (2).) However, since we assume in (1) that $\mu(R) \neq 0$, it is immediate that we can not have G to be symmetric normal. Thus it follows that G is non-infinitely divisible and hence we have the theorem.

The reader may note that if $\left(f(\cdot), g(\cdot)\right)$ denotes a reciprocal pair of probability density functions such that the characteristic function corresponding to $f(\cdot)$ is infinitely divisible with a normal factor, then this characteristic function should be either normal or of the form (1). From the proof given above, it is then evident that if $\left(f(\cdot), g(\cdot)\right)$ is a reciprocal pair of probability density functions such that the corresponding characteristic functions are infinitely divisible with at least one of them with a normal factor, then both the probability density functions should be those corresponding to normal distributions.

References

[1] Toby Lewis, "Probability functions which are proportional to characteristic functions and the infinite divisibility of the von Moses distribution", *Perspectives in probability and statistics*, 19-28 (Academic Press, London, New York, San Francisco, 1976).

[2] Eugene Lukacs, *Characteristic functions*, second edition, revised and enlarged (Griffin, London, 1970).

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[3] Alan Ruegg, "A characterization of certain infinitely divisible laws", *Ann. Math. Statist.* **41** (1970), 1354-1356.

Department of Probability and Statistics,
University of Sheffield,
Sheffield,
England.