COMPLETION IN THE BIDUAL

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Let E, $\hat{E}$, and E' denote a locally convex linear Hausdorff space, the completion of E and the dual of E, respectively. It has been observed that $\hat{E}$ is a subspace of E'' under certain conditions on E. It is the primary goal of this paper to give necessary and sufficient conditions for the inclusion $\hat{E} \subset E''$ to be valid. Such conditions are found and are given in Theorem 4. With a variation of the technique used, several equivalent characterizations of semi-reflexive spaces are given in Theorem 5. The notation throughout will follow that in [2].

The simplest example of a space for which $\hat{E} \subset E''$ is when E is a normed space. In this case, $\hat{E}$ can be taken to be the closure of E in E''. More generally, if E is infra-barrelled and E' is bornological for the $B(E', E)$-topology it is again true that $\hat{E} \subset E''$. This follows from the fact that E'' is then $B(E'', E')$-complete and that $B(E'', E')$ induces the original topology on E. Thus, if E'' is complete for the topology $\varepsilon(E'', E')$ (natural topology) then $\hat{E}$ can always be taken as $\bar{E}$ in E''. In this paper, E'' will always be considered with the natural topology. The approach used is based on the concept of convergence on filters introduced by J. Brace [1] and the following definition and theorem are fundamental to the results.

1. DEFINITION [1]. *A filter $\mathscr{F}$ in E' converges to a functional $f_0 \in E^*$ on a filter $\mathscr{G}$ in E if for every $\varepsilon > 0$ there is an $F \in \mathscr{F}$ with the property that for each $f \in F$ a $G_f \in \mathscr{G}$ can be found such that*

$$|f(x) - f_0(x)| < \varepsilon \text{ for all } x \in G_f.$$

2. THEOREM [1]. *A filter $\mathscr{F}$ in E' converges at a point $x \in E$ to a function f_0 which is continuous at x if and only if $\mathscr{F}$ converges to f_0 on the filter $\mathscr{N}(x)$, of neighborhoods of x.*

The following duality theorem also due to Brace, is often useful.

3. THEOREM [1]. *Let $\mathscr{F}$ and $\mathscr{G}$ be as in 1, $f \in E^*$, $X_0 \in -E'^*$, and suppose that*

i) $\mathscr{F}(x) \to f(x)$ *for each $x \in E$, and*
ii) $\mathscr{G}(x') \to x_0(x')$ *for each $x' \in E'$.*

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Then, $\mathscr{F} \to f$ on $\mathscr{G}$ if and only if $\mathscr{G} \to x_0$ on $\mathscr{F}$.

With these preliminaries out of the way, the first result can be established. It is observed that each Cauchy filter $\mathscr{F}$ in E converges to a functional $x \in E'^*$ uniformly on every equicontinuous subset of E' [2]. In all that follows, $\mathscr{N}_\beta(\theta)$ will denote the filter of $\beta(E', E)$ neighborhoods of θ in E'.

4. THEOREM. *The following statements are equivalent.*

(a) *The completion $\hat{E}$ is a subspace of E'' when E'' has the $\varepsilon(E'', E')$-topology.*

(b) *Every Cauchy filter in E converges to a point of E'^* on $\mathscr{N}_\beta(\theta)$.*

(b′) *For every Cauchy filter $\mathscr{F}$ in E and $\varepsilon > 0$, there is an $F \in \mathscr{F}$ such that for each $x \in F$ a $V_x \in \mathscr{N}_\beta(\theta)$ can be found, with the property that $|x(x')-x_0(x')| < \varepsilon$ for all $x' \in V_x$, where x_0 is the limit of $\mathscr{F}$ in E'^*.*

(c) *The filter $\mathscr{N}_\beta(\theta)$ in E' converges to θ on every Cauchy filter in E.*

(c′) *For every Cauchy filter $\mathscr{F}$ in E and $\epsilon > 0$, there is a $V \in \mathscr{N}_\beta(\theta)$ such that for every $x' \in V$, an $F_{x'} \in \mathscr{F}$ can be found, having the property that*

$$|x'(x)| < \varepsilon \text{ for all } x \in F_{x'}.$$

(d) *For every Cauchy filter $\mathscr{F}$ in E there is a bounded subset B of E such that $B^0 \subset \cup \{F^0 : F \in \mathscr{F}\}$.*

(e) *Every element of $\hat{E}$ is the limit of a Cauchy filter having a base composed of bounded subsets of E.*

PROOF. That (a) and (e) are equivalent follows from the fact that E'' is the union of the $\sigma(E'^*, E')$-closures in E'^* of all bounded subsets of E [**3**, **143**].

(a)$\Leftrightarrow$(b). Let $\mathscr{F}$ be a Cauchy filter in E and x_0 its limit E'^*. Statement (a) implies that $x_0 \in E''$, i.e., x_0 is a $B(E', E)$-continuous functional. By Theorem 1, $x_0 \in E''$ if and only if $\mathscr{F}$ converges to x_0 on every $\mathscr{N}_\beta(x')$.

(b)$\Leftrightarrow$(b′). Statement (b′) is simply the analytic formulation of (b).

(b′)$\Leftrightarrow$(c). It must be verified that the conditions of Theorem 3 are fulfilled. That $\mathscr{N}_\beta(\theta) \to \theta$ at each $x \in E$ is merely the statement that each x in E is $B(E', E)$-continuous as a functional on E', i.e., that $E \subset E''$. The second condition demands that $\mathscr{F}$ must converge to $x_0 \in E'^*$ at each $x' \in E$, which it clearly does. Thus, applying the duality theorem, the result follows.

(c)$\Leftrightarrow$(c′). Statement (c′) is the analytic form of (c).

(c)$\Leftrightarrow$(d). Since $\mathscr{N}_\beta(\theta)$ has a base composed of polars of bounded subsets of E, if V in the member of $\mathscr{N}_\beta(\theta)$ of (c′) there is a bounded set B in E such that $B^0 \subset V$. Thus, in particular, for each $x' \in B^0$ there is an $F_{x'} \in \mathscr{F}$ with the property that $|x'(x)| < \varepsilon \leqq 1$ for all $x \in F_{x'}$. That is $B^0 \subset \cup \{F^0 : F \in \mathscr{F}\}$. Conversely, suppose condition (d) is satisfied, and let $\varepsilon > 0$ be given. Then we note that $\varepsilon/2\, B^0 \in \mathscr{N}_\beta(\theta)$ and for each $x' \in \varepsilon/2\, B^0$, $2/\varepsilon x' \in B^0$ and hence there exists $F_{x'} \in \mathscr{F}$ such that $|2/\varepsilon x'(x)| < 1$ for all $x \in F_{x'}$. But, this is equivalent to $|x'(x)| < \varepsilon/2 < \varepsilon$ for all $x \in F_{x'}$.

We conclude, with the following theorem concerning semi-reflexivity. Here $\mathscr{N}_\sigma(\theta)$ denote the filter base of $\sigma(E', E)$-neighborhood of θ in E'.

5. THEOREM. *The following statements are equivalent.*

(a) *A locally convex space E is semi-reflexive.*

(b) *Every bounded, weak Cauchy filter in E converges to a point of E'^* on $\mathscr{N}_\sigma(\theta)$.*

(b′) *For every bounded, weak Cauchy filter $\mathscr{F}$ in E and $\varepsilon > 0$, there is an $F \in \mathscr{F}$ such that for each $x \in F$, $V_x \in \sigma(\theta)$ can be found, with the property, that $|x(x') - x_0(x')| < \varepsilon$ for all $x' \in V_x$, where x_0 is the limit of $\mathscr{F}$ in E'^*.*

(c) *The filter $\mathscr{N}_\sigma(\theta)$ in E' converges to θ on every bounded, weak Cauchy filter in E.*

(c′) *For every bounded, weak Cauchy filter $\mathscr{F}$ in E, and $\varepsilon > 0$, there is a $V \in \mathscr{N}_\sigma(\theta)$ such that for every $x' \in V$, and $F_{x'} \in \mathscr{F}$ can be found, having the property that $|x'(x)| < \varepsilon$ for all $x \in F_{x'}$.*

(d) *For every bounded weak Cauchy filter $\mathscr{F}$ in E there is a finite set $S \subset E$ such that*

$$S^0 \subset \cup \{F^0 : F \in \mathscr{F}\}.$$

(e) *Every bounded weak Cauchy filter in E converges in the $\sigma(E'', E')$-topology to a point in E.*

PROOF: That (a) and (e) are equivalent follows for the same reason that (a) and (e) of Theorem 4 are equivalent.

(a)⇔(b). If E is semi-reflexive, then each $x_0 \in E''$ is a $\sigma(E', E)$-continuous functional on E'. Let $\mathscr{F}$ be the Cauchy filter with a base of bounded subsets of E converging to x_0. The duality theorem shows that $\mathscr{F}$ converges to x_0 on $\mathscr{N}_\sigma(\theta)$. Conversely let x_0 be an arbitrary point in E'' and $\mathscr{F}$ the bounded weak Cauchy filter in E converging to x_0 in the $\sigma(E'', E')$-topology. If $\mathscr{F}$ converges to x_0 on $\mathscr{N}_\sigma(\theta)$, then Theorem 2 states that x_0 is $\sigma(E', E)$-continuous.

(b)⇔(b′). Statement (b′) is the analytic form of (b).

(b′)⇔(c). As in theorem 4, all that must be verified here, is that the conditions of the duality theorem hold. $\mathscr{N}_\sigma(\theta) \to \theta$ at each $x \in E$ simply states that every $x \in E$ is $\sigma(E', E)$-continuous on E'. If is a bounded weak Cauchy filter then the filter base $\mathscr{F}(x')$ is a Cauchy filter of real numbers for each $x' \in E'$ and hence $\mathscr{F}$ converges at each x' to an $x_0 \in E'^*$.

(c)⇔(c′). Statement (c′) is the analytic form of (c).

(c′)⇔(d). See part (c′)⇔(d) of Theorem 4.

6. REMARKS. First, it should be noted that in both of the examples given in the introduction, the fact that $\hat{E} \subset E''$ follows from the com-

pleteness of E''. Thus, it would be of interest to find an example of a locally convex space whose bidual is not complete for the $\varepsilon(E'', E')$-topology, but satisfies the conditions of Theorem 4. Second, the utility of Theorem 4 and Theorem 5 would be greatly enhanced if one could find equivalent formulations which involved only the space E. Third, there is a related problem which deserves mention, namely: When is $\hat{E} = E''$? (as for example when E is a pre-Hilbert Space). We note that spaces for which this is true are "bidual dense" and should share some of the properties of reflexive spaces. We plan to discuss these questions in a subsequent paper.

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References

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