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UNIQUENESS OF MASSEY PRODUCTS ON THE STABLE HOMOTOPY OF SPHERES

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1. Introduction. The product on the stable homotopy ring of spheres π_*^S can be defined by composing, smashing or joining maps. Each of these three points of view is used in Section 2 to define Massey products on π_*^S. In fact we define composition and smash Massey products $\langle x_1, \ldots, x_t \rangle$ where $x_1, \ldots, x_{t-1} \in \pi_*^S$, $x_t \in \pi^*(E)$ and E is a spectrum. In Theorem 3.2, we prove that these three types of Massey products are equal. Consequently, a theorem which is easy to prove for one of these Massey products is also valid for the other two. For example, [**3**, Theorem 8.1] which relates algebraic Massey products in the Adams spectral sequence to Massey smash products in π_*^S is now also valid for Massey composition products in π_*^S. This paper generalizes to the case of matrix Massey products. (See [**3**, § 7].) However, we will not work in that generality to keep the ideas from becoming lost in a morass of notation.

Our Massey composition product agrees with that of Toda [**8**] and Spanier [**7**]. It is not clear whether it agrees with the higher composition product of Gershenson [**2**]. Our Massey smash product agrees with that of Porter [**5**] and corresponds under the Pontrjagin Thom isomorphism to the Massey product of manifolds defined in [**3**]. In Theorem 3.3 we prove that our Massey product is a subset of the Toda bracket of J. Cohen [**1**]. Moreover, the threefold product and bracket are equal. I conjecture that for $n \geqq 4$ the n-fold bracket may be larger than the n-fold product. These remarks relate to the following problem which J. Cohen raises at the end of [**1**, § 4]. Given a Thom spectrum MG, is there a geometric Toda bracket of stably almost G-manifolds which corresponds to Cohen's Toda bracket under the Pontrjagin Thom isomorphism? This is a significant question because J. Cohen proves in [**3**, Theorem 4.5] that his Toda brackets decompose the elements of the kernel of the Hurewicz homomorphism of MG.

We will work in the following coordinate free setting inspired by the one of May [**4**, Ch. II]. Let $\mathscr{R}^\infty$ be the real inner product space with orthonormal basis $\mathscr{B} = \{b_1, b_2, \ldots\}$. We consider only finite dimensional subspaces of $\mathscr{R}^\infty$ which have a subset of $\mathscr{B}$ as a basis. Internal direct sum is denoted by $+$, and if W' is a subspace of W then $W'^\perp$

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denotes the orthogonal complement of W' in W. All spaces in this paper are based CW complexes, all maps are based and all homotopies, cones and suspensions are reduced. Let S denote one-point compactification. In particular, $S(\mathscr{R}^n) \equiv S^n$. The isomorphism from V to $\mathscr{R}^{\dim V}$ which preserves the ordered standard bases induces a canonical homeomorphism from SV to $S^{\dim V}$. Thus a map from SV to SW determines an element of $\pi_{\dim V}(S^{\dim W})$. If $i_1 < \ldots < i_t$, then define $D(\mathscr{R}b_{i_1} + \ldots + \mathscr{R}b_{i_t})$ as

$$S(\mathscr{R}b_{i_1} + \ldots + \mathscr{R}b_{i_{t-1}}) \wedge CS(\mathscr{R}b_{i_t})$$

where $C(\ ) = (I, \{1\}) \wedge (\ )$. If

$$f:SU_1 \wedge \ldots \wedge SU_t \wedge X \rightarrow SU_1 \wedge \ldots \wedge SU_t \wedge Y \text{ and } 1 \leqq j_1 < \ldots < j_k \leqq t$$

then define $C_{j_1,\ldots j_k}(f)$ as the canonical map from $X \wedge SU_1 \wedge \ldots \wedge DU_{j_1} \wedge \ldots \wedge DU_{j_k} \wedge \ldots \wedge SU_t$ to $Y \wedge SU_1 \ldots \wedge DU_{j_1} \wedge \ldots \wedge DU_{j_k} \wedge \ldots \wedge SU_t$ induced by f. Define an equivalence relation on ∂I^{t-1} by $(a_1, \ldots, a_{t-1}) \sim (b_1, \ldots, b_{t-1})$ if $\max(a_1, \ldots, a_{t-1}) = 1$ and $\max(b_1, \ldots, b_{t-1}) = 1$. For $t \geqq 3$ choose homeomorphisms

$$h_t:S^{t-2} \rightarrow (\partial I^{t-1})/\sim.$$

Then the maps $T \circ (h_t \wedge 1_{SV_1 \wedge \ldots \wedge SV_t})$ define homeomorphisms

$$h:S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \rightarrow \partial[DV_1 \wedge \ldots \wedge DV_{t-1} \wedge SV_t].$$

Here and throughout this paper we let T denote a canonical interchange map. We let ϵ denote the structure map $E \wedge S \rightarrow E$ or $S \wedge S \rightarrow S$ of the appropriate spectrum.

2. Definitions. In 2.1, 2.2 and 2.3 we define the Massey composition, smash and join products of unstable maps between spheres as subsets of $\pi_*(E)$. In Corollary 2.5 we show that these Massey products depend only on the stable homotopy classes of the original maps. Thus there are induced Massey products which are defined on stable homotopy classes.

Definition 2.1. Let E be a spectrum. Let

$$g_{i-1,i}:SV_i \wedge \ldots \wedge SV_t \wedge SU \rightarrow SV_{i+1} \wedge \ldots \wedge SV_t \wedge E_iU$$

be given, $1 \leqq i \leqq t$, such that

$$\mathscr{R}^{t-2} \perp V_1 \perp \ldots \perp V_t \perp U, E_i = S \text{ for } 1 \leqq i \leqq t-1 \text{ and } E_t = E.$$

A defining system for $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_0'$ consists of maps

$$g_{i,j}:DV_{i+1} \wedge \ldots \wedge DV_{j-1} \wedge SV_j \wedge \ldots \wedge SV_t \wedge SU \rightarrow SV_{j+1} \wedge \ldots \wedge SV_t \wedge E_jU$$

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for $0 \leqq i < j - 1 < t$, $(i, j) \neq (0, t)$, such that

$$g_{i,j}|\partial(DV_{i+1} \wedge \ldots \wedge DV_{j-1} \wedge SV_j \wedge \ldots \wedge SV_t \wedge SU)$$
$$= \bigcup_{k=i+1}^{j-1} g_{i,j}{}^k$$

where $g_{i,j}{}^k$ is the composite map

$$DV_{i+1} \wedge \ldots \wedge DV_{k-1} \wedge SV_k \wedge DV_{k+1} \wedge \ldots$$
$$\wedge DV_{j-1} \wedge SV_j \wedge \ldots \wedge SV_t \wedge SU \xrightarrow{C_{k+1,\ldots,j-1}(g_{i,k})}$$
$$DV_{k+1} \wedge \ldots \wedge DV_{j-1} \wedge SV_j \wedge \ldots \wedge SV_t \wedge SU$$
$$\xrightarrow{g_{k,j}} SV_{j+1} \wedge \ldots \wedge SV_t \wedge E_j U.$$

If $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_0'$ is defined then define $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_0'$ as the set of homotopy classes of the maps

$$\tilde{g}_{0,t} \equiv \bigcup_{k=1}^{t-1} g_{0,t}{}^k \circ (h \wedge 1_{SU}) : S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU \rightarrow EU$$

for all defining systems $\{g_{i,j}\}$ of $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_0'$. Define

$$\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_0 = \lim_{\rightarrow W} \langle g_{0,1} \wedge 1_{SW}, \ldots, \epsilon \circ (g_{t-1,t} \wedge 1_{SW})\rangle_0'.$$

This direct limit is taken over all W with $W \perp (\mathscr{R}^{t-2} + V_1 + \ldots + V_t + U)$. If $W' \subset W$ then the map $- \wedge 1_{S(W'^{\perp})}$ sends a defining system of

$$\langle g_{0,1} \wedge 1_{SW'}, \ldots, \epsilon \circ (g_{t-1,t} \wedge 1_{SW'})\rangle_0'$$

to a defining system of

$$\langle g_{0,1} \wedge 1_{SW}, \ldots, \epsilon \circ (g_{t-1,t} \wedge 1_{SW})\rangle_0'.$$

Definition 2.2. Let E be a spectrum. Let

$$g_{i-1,i}: SV_i \wedge SU_i \rightarrow E_i U_i$$

be given, $1 \leqq i \leqq t$, such that $\mathscr{R}^{t-2} \perp V_1 \perp U_1 \perp \ldots \perp V_t \perp U_t$, $E_i = S$ for $1 \leqq i \leqq t - 1$ and $E_t = E$. A defining system for $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\wedge'$ consists of maps

$$g_{i,j}: DV_{i+1} \wedge SU_{i+1} \wedge \ldots \wedge DV_{j-1} \wedge SU_{j-1} \wedge SV_j \wedge SU_j$$
$$\rightarrow E_j(U_{i+1} + \ldots + U_j)$$

for $0 \leqq i < j - 1 < t$, $(i, j) \neq (0, t)$, such that

$$g_{i,j}|\partial(DV_{i+1} \wedge SU_{i+1} \wedge \ldots \wedge DV_{j-1} \wedge SU_{j-1} \wedge SV_j \wedge SU_j)$$
$$= \bigcup_{k=i+1}^{j-1} g_{i,j}^k$$

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where $g_{i,j}{}^k$ is the map $\epsilon \circ T \circ (g_{i,k} \wedge g_{k,j})$. If $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\wedge{}'$ is defined, then define $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\wedge{}'$ as the set of homotopy classes of the maps

$$\tilde{g}_{0,t} \equiv \epsilon \circ T \circ \left(\bigcup_{k=1}^{t-1} g_{0,t}^k\right) \circ T \circ (h \wedge 1_{SU_1 \wedge \ldots \wedge SU_t}) : S \times (\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU_1 \wedge \ldots \wedge SU_t \rightarrow E(U_1 + \ldots + U_t)$$

for all defining systems $\{g_{i,j}\}$ of $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\wedge{}'$. Define

$$\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\wedge = \lim_{W_1,\ldots,W_t} \langle g_{0,1} \wedge 1_{SW_1}, \ldots, \epsilon \circ (g_{t-1,t} \wedge 1_{SW_t})\rangle_\wedge{}'.$$

This direct limit is taken over all $W_1, \ldots, W_t$ with

$$W_1 \perp \ldots \perp W_t \perp (\mathscr{R}^{t-2} + V_1 + U_1 + \ldots + V_t + U_t).$$

If $W_i' \subset W_i$, $1 \leqq i \leqq t$, then the maps

$$\epsilon \circ T \circ (- \wedge 1_{S(W_{i+1}'{}^\perp) \wedge \ldots \wedge S(W_j'{}^\perp)}) \circ T$$

send a defining system of $\langle g_{0,1} \wedge 1_{SW_1'}, \ldots, \epsilon \circ (g_{t-1,t} \wedge 1_{SW_t'})\rangle_\wedge{}'$ to a defining system of $\langle g_{0,1} \wedge 1_{SW_1}, \ldots, \epsilon \circ (g_{t-1,t} \wedge 1_{SW_t})\rangle_\wedge{}'$. If $A \subset U$, $B \subset V$, $U \perp V \perp Rb_k$ then define the join $A *_k B$ as the canonical quotient of the appropriate subspace of $U + V + Rb_k$.

Definition 2.3. Let $g_{i-1,i}: SV_i \wedge SU_i \rightarrow SU_i$ be given, $1 \leqq i \leqq t$, such that

$$\mathscr{R}^{2t-3} \perp V_1 \perp U_1 \perp \ldots \perp V_t \perp U_t.$$

A defining system for $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_*{}'$ consists of maps

$$g_{i,j}: (DV_{i+1} \wedge SU_{i+1})*_{t+i-1} \cdots *_{t+j-4}(DV_{j-1} \wedge SU_{j-1})*_{t+j-3} \times (SV_j \wedge SU_j) \rightarrow SU_{i+1}*_{t+i-1} \cdots *_{t+j-3}SU_j$$

for $0 \leqq i < j - 1 < t$, $(i, j) \neq (0, t)$, such that

$$g_{i,j}|\partial[(DV_{i+1} \wedge SU_{i+1})*_{t+i-1} \cdots *_{t+j-4}(DV_{j-1} \wedge SU_{j-1})*_{t+j-3} \times (SV_j \wedge SU_j)] = \bigcup_{k=t+1}^{j-1} g_{tj}{}^k$$

where $g_{i,j}{}^k$ is the map $g_{i,k}*_{t+k-2}g_{k,j}$. If $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_*{}'$ is defined then define $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_*{}'$ as the set of homotopy classes of the maps

$$\tilde{g}_{0,t} \equiv \bigcup_{k=1}^{t-1} g_{0,t}^k \circ T \circ (h \wedge 1_{SU_1 \wedge \ldots \wedge SU_t}) : S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU_1*_{t-1} \cdots *_{2t-3}SU_t \rightarrow SU_1*_{t-1} \cdots *_{2t-8}SU_t$$

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for all defining systems $\{g_{i,j}\}$ of $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_*'$. Define

$$\langle g_{0,1}, \ldots, g_{t-1,t}\rangle^* = \varinjlim_{W_1,\ldots,W_t} \langle g_{0,1} \wedge 1_{SW_1}, \ldots, g_{t-1,t} \wedge 1_{SW_t}\rangle_*'.$$

This direct limit is taken over all $W_1, \ldots, W_t$ with

$$W_1 \perp \ldots \perp W_t \perp (\mathscr{R}^{2t-3} + V_1 + U_1 + \ldots + V_t + U_t).$$

If $W_i' \subset W_1$, $1 \leqq i \leqq t$, then the maps

$$T \circ (- \wedge 1_{S(W_{i+1}'^{\perp}) \wedge \ldots \wedge S(W_j'^{\perp})}) \circ T$$

send a defining system of

$$\langle g_{0,1} \wedge 1_{SW_1'}, \ldots, g_{t-1,t} \wedge 1_{SW_t'}\rangle_*'$$

to a defining system of

$$\langle g_{0,1} \wedge 1_{SW_1}, \ldots, g_{t-1,t} \wedge 1_{SW_t}\rangle_*'.$$

The following theorem will be used to show that the three Massey products defined above depend only on the stable homotopy classes of the original t maps.

THEOREM 2.4. *Let # denote* $\circ$, $\wedge$ *or* $*$, *and let* $\{g_{i,j}:A_{i,j} \rightarrow B_{i,j}\}$ *be a given defining system for* $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\#'$ *as in* 2.1, 2.2 *or* 2.3. *For a specific* $0 \leqq \alpha < \beta \leqq t$, $(\alpha, \beta) \neq (0, t)$, *let a map* $g_{\alpha,\beta}':A_{\alpha,\beta} \rightarrow B_{\alpha,\beta}$ *be given. Assume that*

$$g_{\alpha,\beta}'|\partial A_{\alpha,\beta} = g_{\alpha,\beta}|\partial A_{\alpha,\beta}$$

and that

$$H_{\alpha,\beta}:g_{\alpha,\beta} \simeq g_{\alpha,\beta}' \text{ rel } \partial A_{\alpha,\beta}.$$

Then there is a defining system $\{g_{i,j}'\}$ *for* $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\#'$ *which includes* $g_{\alpha,\beta}'$ *such that* $g_{i,j}' = g_{i,j}$ *if* $\alpha < i$ *or* $j < \beta$. *Moreover,* $\tilde{g}_{0,t} \simeq \tilde{g}_{0,t}'$.

Proof. We will define the $g_{i,j}'$ and a set of homotopies

$$\{H_{i,j}:I \ltimes A_{i,j} \rightarrow B_{i,j} | 0 \leqq i < \alpha < \beta \leqq j \leqq t,\ (i,j) \neq (0,t)\}$$

which includes $H_{\alpha,\beta}$ such that $H_{i,j}|I \ltimes \partial A_{i,j}$ is the union of maps $H_{i,j}^k$, $i < k < j$. If # is $\wedge$ or $*$ then $H_{i,j}^k$ is the composite map

$$I \ltimes (A_{i,k} \# A_{k,j}) \xrightarrow{\Delta \times 1} (I \times I) \ltimes (A_{i,k} \# A_{k,j}) \xrightarrow{T}$$
$$(I \ltimes A_{i,k}) \# (I \ltimes A_{k,j}) \xrightarrow{H_{i,k} \# H_{k,j}} B_{i,k} \# B_{k,j} \xrightarrow{\epsilon \circ T} B_{i,j}$$

where $\epsilon \circ T = 1$ if # is $*$. If # is $\circ$ then

$$H_{i,j}^k(t, a) = H_{k,j}[t, C_{k+1,\ldots,j-1}(H_{i,k})[t, a]].$$

If such $H_{i,j}$ can be defined, then $[\bigcup_{k=1}^{t-1} H_{0,t}^k]$ is a homotopy from $\tilde{g}_{0,t}$ to

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$\tilde{g}_{0,t}'$. Define $g_{i,j}'$ and $H_{i,j}$ by induction on $j - i \geqq \beta - \alpha$. $g_{\alpha,\beta}'$ and $H_{\alpha,\beta}$ are given. Inductively, $g_{i,j}'|\partial A_{i,j}$ and $H_{i,j}|I \ltimes \partial A_{i,j}$ have been defined. Observe that $\partial A_{i,j} \hookrightarrow A_{i,j}$ is the inclusion of a sphere in a disc which is a cofibration. Hence $g_{i,j}'$ and $H_{i,j}$ exist as required.

Let $x \in \pi_{k+n}(EU)$ where dim $U = n$. Then the *stable homotopy class of* x refers to the element of $\pi_k(E)$ determined by x. The special case of the following corollary for length three composition Massey products is given by E. Spanier [**6**, Theorem 4.4] and H. Toda [**8**, Proposition 1.3].

COROLLARY 2.5. *Let # denote* $\circ$, $\wedge$ *or* $*$, *and assume that* $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\#$ *is defined. Then* $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\#$ *depends only on the stable homotopy classes of* $g_{0,1}, \ldots, g_{t-1,t}$.

Proof. It suffices to prove the following. Let $\{g_{i,j}\}$ be a defining system for $\langle g_{0,1}, \ldots, g_{t-1,t}\rangle_\#'$, where $g_{i-1,i}: A_i \to E_i B_i$ and $E_i = S$ for all i if # is $*$. Let

$$f_{i-1,i}: A_i \wedge SW_i \to E_i(B_i + W_i),\ 1 \leqq i \leqq t,$$

be given such that $W_1 = \ldots = W_t$ if # is $\circ$. Let homotopies

$$\epsilon \circ (g_{i-1,i} \wedge 1_{SW_i}) \simeq f_{i-1,i}$$

be given, $1 \leqq i \leqq t$. Then there is a defining system $\{f_{i,j}\}$ of $\langle f_{0,1}, \ldots, f_{t-1,t}\rangle_\#'$ such that

$$\tilde{f}_{0,t} \simeq \begin{cases} \epsilon \circ (\tilde{g}_{0,t} \wedge 1_{SW_1}) & \text{if \# is } \circ \\ \epsilon \circ (\tilde{g}_{0,t} \wedge 1_{SW_1 \wedge \ldots \wedge SW_t}) & \text{if \# is } \wedge \text{ or} *. \end{cases}$$

This fact is a consequence of Theorem 2.4 applied t times with (α, β) equal to $(0, 1), (1, 2), \ldots, (t - 1, t)$.

3. Equality of the three Massey products. The following lemma will be used to prove that the join and composition Massey products are subsets of the smash Massey product. It says that the cofibration property and the Freudenthal Suspension Theorem can be applied simultaneously.

LEMMA 3.1. *Let* dim $X \leqq 2$ *connectivity* Y, $f: SU \wedge X \to SU \wedge Y$, $g: X \to Y$ *and* $h: f \simeq 1_{SU} \wedge g$. *Let* $F: CSU \wedge X \to SU \wedge Y$ *be an extension of* f. *Then there is an extension* $G: CX \to Y$ *of* g *and a homotopy*

$$H: I \ltimes CSU \wedge X \to SU \wedge Y$$

such that

$$H: F \simeq 1_{SU} \wedge G \text{ and } H|I \ltimes SU \wedge X = h.$$

Proof. We first prove this lemma in the case where $f = 1_{SU} \wedge g$ and

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$h = f \circ P_2$. We will construct an extension G of g and a homotopy

$$H: F \simeq SG \text{ rel } SU \wedge X.$$

Since F extends $1_{SU} \wedge g$ to the cone of its domain, this suspension of g is nullhomotopic. Hence g is nullhomotopic by the Freundenthal Suspension Theorem. Thus there is a homotopy

$$\alpha: [0, 1/3] \ltimes X \rightarrow Y$$

parameterized on the interval $[0, 1/3]$ with $\alpha: g \simeq *$. Let

$$\alpha': [1/3, 2/3] \ltimes X \rightarrow Y$$

by $\alpha'(t, x) = \alpha(2/3 - t, x)$. Let $H_1: F \simeq F_1$ rel $SU \wedge X$ such that

$$F_1(t, u \wedge x) = f(u \wedge x) = u \wedge g(x) \text{ for } 0 \leqq t \leqq 2/3.$$

Let $F_2: CSU \wedge X \rightarrow SU \wedge Y$ by:

$$F_2(t, u \wedge x) = \begin{cases} F_1(t, u \wedge x) & \text{if } 2/3 \leqq t \leqq 1 \\ u \wedge \alpha'(t, x) & \text{if } 1/3 \leqq t \leqq 2/3 \\ u \wedge \alpha(t, x) & \text{if } 0 \leqq t \leqq 1/3. \end{cases}$$

Then

$$F_1|[0, 2/3] \ltimes SU \wedge X = 1_{SU} \wedge (g \circ P_2) \text{ and}$$
$$F_2|[0, 2/3] \ltimes SU \wedge X = 1_{SU} \wedge (\alpha \cup \alpha').$$

Let $h: I \ltimes [0, 2/3] \ltimes X \rightarrow Y$ by:

$$h(s, t, x) = \begin{cases} \alpha(t - s/3, x) & \text{if } s/3 \leqq t \leqq 1/3 \\ \alpha'(s/3 + t, x) & \text{if } 1/3 \leqq t \leqq 2/3 - s/3 \\ g(x) & \text{if } 0 \leqq t \leqq s/3 \text{ or } 2/3 - s/3 \leqq t \leqq 2/3. \end{cases}$$

Thus $h: \alpha \cup \alpha' \simeq g \circ P_2$. If $H_2 = (1_{SU} \wedge h) \cup (F_1 \circ P_2)$ then $H_2: F_1 \simeq F_2$ rel $SU \wedge X$. Let $\langle 1/3, 1\rangle \ltimes SU \wedge X$ denote the image of $[1/3, 1] \ltimes SU \wedge X$ in $CSU \wedge X$. Observe that F_2 induces a map

$$\bar{F}_2: (\langle 1/3, 1\rangle \ltimes SU \wedge X)/(\{1/3\} \times SU \wedge X) \rightarrow SU \wedge Y.$$

The domain of $\bar{F}_2$ is homeomorphic to $S(SU \wedge X)$. Hence by the Freudenthal Suspension Theorem there is a homotopy

$$\bar{H}_3: \bar{F}_2 \rightarrow 1_{SU} \wedge \bar{F}_3.$$

Define $F_3: CSU \wedge X \rightarrow SU \wedge Y$ by:

$$F_3(t, u \wedge x) = \begin{cases} F_2(t, u \wedge x) & \text{if } 0 \leqq t \leqq 1/3 \\ u \wedge \bar{F}_3(t, x) & \text{if } 1/3 \leqq t \leqq 1. \end{cases}$$

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Define $H_3: F_2 \simeq F_3$ rel $SU \wedge X$ by:

$$H_3(s, t, u \wedge x) = \begin{cases} F_2(t, u \wedge x) & \text{if } 0 \leqq t \leqq 1/3 \\ \bar{H}_3(s, t, u \wedge x) & \text{if } 1/3 \leqq t \leqq 1. \end{cases}$$

Then $F_3 = 1_{SU} \wedge G$ where:

$$G(t, x) = \begin{cases} \alpha(t, x) & \text{if } 0 \leqq t \leqq 1/3 \\ \bar{F}_3(t, x) & \text{if } 1/3 \leqq t \leqq 1. \end{cases}$$

The map G and the homotopy H obtained by pasting together H_1, H_2, H_3 demonstrate that the lemma is valid in this special case.

In general H will be defined by pasting together three homotopies H', H'' and H''' as in Figure 1. Figure 1 illustrates the case $SU \wedge X = S^1$ when the triangle in the figure is rotated around the dotted line L.

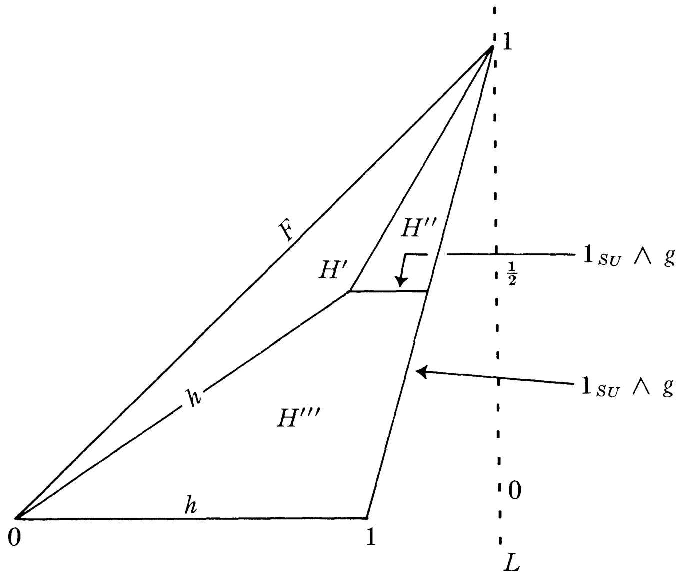

FIGURE 1. The homotopy **H**

Since $\{0\} \times SU \wedge X \hookrightarrow CSU \wedge X$ is a cofibration there is a homotopy

$$H': I \ltimes CSU \wedge X \to SU \wedge Y$$

such that $H'|I \ltimes \{0\} \times SU \wedge X = h$ and $H'|\{0\} \times CSU \wedge X = F$. Let H'' be the homotopy given by the special case of the lemma which we proved above applied to the map g and the extension $H'|\{1\} \times CSU \wedge X$ of $1_{SU} \wedge g$. Let

$$H''|\{1\} \times CSU \wedge X = 1_{SU} \wedge G''.$$

Let $H''': \{(t, s, u \wedge x) \in I \ltimes CSU \wedge X | 0 \leqq s \leqq 1/2 \text{ and } s \leqq t\} \to$

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$SU \wedge Y$ be defined by:

$$H'''(t, s, u \wedge x) = \begin{cases} h(2s, u \wedge x) & \text{if } 2s \geqq t \\ h(t, u \wedge x) & \text{if } 2s \leqq t. \end{cases}$$

Define $H: I \times CSU \wedge X \to SU \wedge Y$ by:

$$H(t, s, u \wedge x) = \begin{cases} H'\left(2t, \dfrac{s-t}{1-t}, u \wedge x\right) & \text{if } 0 \leqq t \leqq \frac{1}{2}, t \leqq s \\ H''(2t-1, 2s-1, u \wedge x) & \\ & \text{if } \frac{1}{2} \leqq t \leqq 1, \frac{1}{2} \leqq s \leqq 1 \\ H'''(t, s, u \wedge x) & \text{if } s \leqq t, 0 \leqq s \leqq \frac{1}{2}. \end{cases}$$

It is routine to check that H is well-defined, $H|I \rhd \{0\} \times SU \wedge X = h$ and $H: F \simeq 1_{SU} \wedge G$ where $G: CX \to Y$ is given by:

$$G(s, x) = \begin{cases} g(x) & \text{if } 0 \leqq s \leqq \frac{1}{2} \\ G''(2s-1, x) & \text{if } \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

Thus H and G meet the requirements of this lemma.

THEOREM 3.2. *Let E be a spectrum. Let $x_i \in \pi_*{}^S$, $1 \leqq i \leqq t-1$ and let $x_t \in \pi_*(E)$. Then the following conditions are equivalent.*

(a) *$\langle x_1, \ldots, x_t \rangle_0$ is defined.*

(b) *$\langle x_1, \ldots, x_t \rangle_\wedge$ is defined.*

If conditions (a), (b) *are true then $\langle x_1, \ldots, x_t \rangle_0 = \langle x_1, \ldots, x_t \rangle_\wedge$. If $E = S$ then condition* (c) *is equivalent to* (a) *and* (b).

(c) *$\langle x_1, \ldots, x_t \rangle_*$ is defined.*

If $E = S$ and conditions (a), (b), (c) *are true then*

$$\langle x_1, \ldots, x_t \rangle_0 = \langle x_1, \ldots, x_t \rangle_\wedge = \langle x_1, \ldots, x_t \rangle_*.$$

Proof. We begin by proving that if $\langle x_1, \ldots, x_t \rangle_\wedge$ is defined then $\langle x_1, \ldots, x_t \rangle_0$ is defined and $\langle x_1, \ldots, x_t \rangle_\wedge \subset \langle x_1, \ldots, x_t \rangle_0$. Let $x \in \langle x_1, \ldots, x_t \rangle_\wedge$ be represented by

$$\tilde{g}_{0,t} \in \langle g_{0,1}, \ldots, g_{t-1,t} \rangle_\wedge{}'$$

where $\{g_{i,j}\}$ is a defining system and

$$g_{i-1,i}: SV_i \wedge SU_i \to E_i U_i,\ 1 \leqq i \leqq t.$$

Let

$$G_{i-1,i}: SV_i \wedge \ldots \wedge SV_t \wedge S(U_1 + \ldots + U_t) \to SV_{i+1} \wedge \ldots \wedge SV_t \wedge E_i(U_1 + \ldots + U_t)$$

be given by

$$\epsilon \circ T \circ (g_{i-1,i} \wedge 1) \circ T.$$

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For $0 \leqq i < j - 1 < t$, $(i, j) \neq (0, t)$, define

$$G_{i,j}: DV_{i+1} \wedge \ldots \wedge DV_{j-1} \wedge SV_j \wedge \ldots \wedge SV_t \wedge S(U_1 + \ldots + U_t) \rightarrow SV_{j+1} \wedge \ldots \wedge SV_t \wedge E_j(U_1 + \ldots + U)$$

as

$$\epsilon \circ T \circ (g_{i,j} \wedge 1) \circ T.$$

Then $\{G_{i,j}\}$ is a defining system for $\langle G_{0,1}, \ldots, G_{t-1,t}\rangle_0'$ and $G_{i-1,i}$ represents x_i, $1 \leqq i \leqq t$. Thus $\langle x_1, \ldots, x_t\rangle_0$ is defined. Observe that the following diagram commutes:

$$\begin{array}{ccc} S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge S(U_1 + \ldots + U_t) & \xrightarrow{\tilde{G}_{0,t}} & E(U_1 + \ldots + U_t) \\ \downarrow \cong & \nearrow \tilde{g}_{0,t} & \\ S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU_1 \wedge \ldots \wedge SU_t & & \end{array}$$

Thus $G_{0,t}$ and $\tilde{g}_{0,t}$ represent the same element of $\pi_*(E)$ and

$$\langle x_1, \ldots, x_t\rangle_\wedge \subset \langle x_1, \ldots, x_t\rangle_0.$$

We prove next that if $\langle x_1, \ldots, x_t\rangle_0$ is defined, then $\langle x_1, \ldots, x_t\rangle_\wedge$ is defined and

$$\langle x_1, \ldots, x_t\rangle_0 \subset \langle x_1, \ldots, x_t\rangle_\wedge.$$

Let $x \in \langle x_1, \ldots, x_t\rangle_0$ be represented by $\tilde{g}_{0,t} \in \langle g_{0,1}, \ldots, g_{t-1,t}\rangle_0'$ where $\{g_{i,j}\}$ is a defining system and

$$g_{i-1,i}: SV_i \wedge \ldots \wedge SV_t \wedge SU \rightarrow SV_{i+1} \wedge \ldots \wedge SV_t \wedge E_i U.$$

Choose large dimensional subspaces U_i of $\mathscr{R}^\infty$, $1 \leqq i \leqq t - 1$, such that

$$U_1 \perp \ldots \perp U_{t-1} \perp (\mathscr{R}^{t-2} + V_1 + \ldots + V_t + U).$$

Let $U_t = U$. We construct a defining system $\{G_{i,j}\}$ for $\langle G_{0,1}, \ldots, G_{t-1,t}\rangle_\wedge'$ with the following properties:

(1) For $1 \leqq i \leqq t - 1$, $G_{i-1,i}: SV_i \wedge SU_i \rightarrow SU_i$ and $G_{t-1,t} = g_{t-1,t}$.
(2) For $1 \leqq i \leqq t - 1$, there is a homotopy $H_{i-1,i}$ from

$$G_{i-1,i} \wedge 1_{SV_{i+1} \wedge \ldots \wedge SV_t \wedge SU} \text{ to } T \circ (g_{i-1,i} \wedge 1_{SU_i}) \circ T.$$

(3a) For $0 \leqq i < j - 1 < t - 1$, there is a homotopy $H_{i,j}$ from

$$G_{i,j} \wedge 1_{SV_{j+1} \wedge \ldots \wedge SV_t \wedge SU} \text{ to } T \circ (g_{i,j} \wedge 1_{SU_{i+1} \wedge \ldots \wedge SU_j}) \circ T.$$

(3b) For $1 \leqq i \leqq t - 2$, there is a homotopy $H_{i,t}$ from

$$G_{i,t} \text{ to } \epsilon \circ (g_{i,t} \wedge 1_{SU_{i+1} \wedge \ldots \wedge SU_{t-1}}) \circ T.$$

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(4a) For $0 \leqq i < k < j < t$, $H_{i,j}|\{r\} \times DV_{i+1} \wedge \ldots \wedge SV_k \wedge \ldots \wedge SU$ is

$$G_{i,k} \wedge (H_{k,j}|\{2r\} \times DV_{k+1} \wedge \ldots \wedge SU) \text{ for } 0 \leqq r \leqq \tfrac{1}{2}$$

and is

$$T \circ (g_{k,j} \wedge 1_{SU_{i+1} \wedge \ldots \wedge SU_j}) \circ T \circ$$
$$[(C_{k+1,\ldots,j-1}(H_{i,k})|\{2r-1\} \times DV_{i+1} \wedge \ldots \wedge SU) \wedge 1_{SU_{k+1} \wedge \ldots \wedge SU_j}] \circ T$$

for $\frac{1}{2} \leqq r \leqq 1$.

(4b) For $1 \leqq i < k < t$, $H_{i,t}|\{r\} \times DV_{i+1} \wedge \ldots \wedge SV_k \wedge \ldots \wedge SU$ is

$$\epsilon \circ [G_{i,k} \wedge (H_{k,t}|\{2r\} \times DV_{k+1} \wedge \ldots \wedge SU)] \text{ for } 0 \leqq r \leqq \tfrac{1}{2}$$

and is

$$\epsilon \circ (g_{k,t} \wedge 1_{SU_{i+1} \wedge \ldots \circ SU_{t-1}}) \circ T \circ$$
$$[(C_{k+1,\ldots,t-1}(H_{i,k})|\{2r-1\} \times DV_{i+1} \wedge \ldots \wedge SU) \wedge 1_{SU_{k+1} \wedge \ldots \wedge SU_{t-1}}] \circ T$$

for $\frac{1}{2} \leqq r \leqq 1$.

We define the $G_{i,j}$ and $H_{i,j}$ by induction on $j - i \geqq 1$. By the Freudenthal Suspension Theorem we can find $G_{i-1,i}$ and $H_{i-1,i}$ which satisfy (1) and (2). Inductively,

$$G_{i,j}|\partial\,(\text{Domain } G_{i,j}) \text{ and}$$
$$H_{i,j}|I \ltimes \partial(DV_{i+1} \wedge \ldots \wedge SU)$$

have been defined. Note that $H_{i,j}|\{1\} \times \partial(DV_{i+1} \wedge \ldots \wedge SU)$ extends to $\{1\} \times DV_{i+1} \wedge \ldots \wedge SU$ as

$$\begin{cases} T \circ (g_{i,j} \wedge 1_{SU_{i+1} \wedge \ldots \wedge SU_j}) \circ T & \text{if } j < t \\ \epsilon \circ (g_{i,t} \wedge 1_{SU_{i+1} \wedge \ldots \wedge SU_{t-1}}) \circ T & \text{if } j = t. \end{cases}$$

If $j < t$ then we apply Lemma 3.1 to find $G_{i,j}$ and $H_{i,j}$ as in (3a) and (4a). If $j = t$ then we use the fact that the inclusion map of $\partial(DV_{i+1} \wedge \ldots \wedge SU)$ into $DV_{i+1} \wedge \ldots \wedge SU$ is a cofibration to find $G_{i,t}$ and $H_{i,t}$ as in (3b) and (4b). Thus $\{G_{i,j}\}$ is a defining system for $\langle G_{0,1}, \ldots, G_{t-1,t}\rangle_\wedge'$ and $G_{i-1,i}$ represents x_i, $1 \leqq i \leqq t$. Observe that the following diagram homotopy commutes.

$$\begin{array}{ccc} S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU_1 \wedge \ldots \wedge SU_t & \xrightarrow{\tilde{G}_{0,t}} & E(U_1 + \ldots + U_t) \\ \Big\downarrow T & & \Big\uparrow \epsilon \\ S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU \wedge SU_1 \wedge \ldots \wedge SU_{t-1} & \xrightarrow{\tilde{g}_{0,t} \wedge 1_{SU_1 \wedge \ldots \wedge SU_{t-1}}} & EU \wedge SU_1 \wedge \ldots \wedge SU_{t-1} \end{array}$$

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Thus $\tilde{G}_{0,t}$ and $\tilde{g}_{0,t}$ represent the same element of $\pi_*{}^S$, and

$$\langle x_1, \ldots, x_t \rangle_0 \subset \langle x_1, \ldots, x_t \rangle_\wedge.$$

Let $E = S$. Assume that $\langle x_1, \ldots, x_t \rangle_\wedge$ is defined. Let $x \in \langle x_1 \ldots . x_t \rangle_\wedge$ be represented by

$$\tilde{g}_{0,t} \in \langle g_{0,1}, \ldots, g_{t-1,t} \rangle_\wedge{}'$$

where $g_{i-1,i}:SV_i \wedge SU_i \rightarrow SU_i$ represents x_i and

$$\mathscr{R}^{2t-3} \perp V_1 \perp U_1 \perp \ldots \perp V_t \perp U_t.$$

Then the

$$G_{i,j} = T \circ [g_{i,j} \wedge 1_{S(\mathscr{R}b_{t+i-1} + \ldots + \mathscr{R}b_{t+j-3})}] \circ T$$

form a defining system for $\langle g_{0,1}, \ldots, g_{t-1,t} \rangle_*{}'$. Thus $\langle x_1, \ldots, x_t \rangle_*$ is defined. Observe that the following diagram commutes.

$$\begin{array}{ccc}
S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU_1 *_{t-1} \ldots *_{2t-3} SU_t & \xrightarrow{\tilde{G}_{0,t}} & SU_1 *_{t-1} \ldots *_{2t-3} SU_t \\
\downarrow T & & \downarrow T \\
[S(\mathscr{R}^{t-2} + V_1 + \ldots + V_t) \wedge SU_1 \wedge \ldots \wedge SU_t] & \xrightarrow{\tilde{g}_{0,t} \wedge 1} & (SU_1 \wedge \ldots \wedge SU_t) \\
\wedge\, S(\mathscr{R}b_{t-1} + \ldots + \mathscr{R}b_{2t-3}) & & \wedge\, S(\mathscr{R}b_{t-1} + \ldots + \mathscr{R}b_{2t-3})
\end{array}$$

Thus $\tilde{G}_{0,t}$ and $\tilde{g}_{0,t}$ represent the same element of $\pi_*{}^S$, and

$$\langle x_1, \ldots, x_t \rangle_\wedge \subset \langle x_1, \ldots, x_t \rangle_*.$$

Now assume that $\langle x_1, \ldots, x_t \rangle_*$ is defined. Let $x \in \langle x_1, \ldots, x_t \rangle_*$ be represented by

$$\tilde{G}_{0,t} \in \langle G_{0,1}, \ldots, G_{t-1,t} \rangle_*{}'$$

in the above notation. By Lemma 3.1 we can assume that each $G_{i,j}$, $i < j - 1$, is a $(j - i - 1)$-iterated suspension of a map $g_{i,j}$ as above. In particular

$$g_{i-1,i} = G_{i-1,i},\ 1 \leqq i \leqq t.$$

Thus $\langle x_1, \ldots, x_t \rangle_\wedge$ is defined, and

$$\langle x_1, \ldots, x_t \rangle_* \subset \langle x_1, \ldots, x_t \rangle_\wedge.$$

We conclude by proving that our Massey product is a subset of the Toda bracket $\langle x_1, \ldots, x_t \rangle_C$ of J. Cohen [**1**]. We will use the notation of [**1**, § 2] without explanation.

THEOREM 3.3. *Let E be a spectrum. Let $x_i \in \pi_*{}^S$ for $1 \leqq i \leqq t - 1$ and let $x_t \in \pi_*(E)$. If $\langle x_1, \ldots, x_t \rangle_0$ is defined then $\langle x_1, \ldots, x_t \rangle_C$ is defined and*

$$\langle x_1, \ldots, x_t \rangle_0 \subset \langle x_1, \ldots, x_t \rangle_C.$$

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Proof. Let $x \in \langle x_1, \ldots, x_t \rangle_0$ be represented by

$$\tilde{g}_{0,t} \in \langle g_{0,1}, \ldots, g_{t-1,t} \rangle_0{}'$$

where $\{g_{i,j}\}$ is a defining system and

$$g_{i-1,i}: SV_i \wedge \ldots \wedge SV_t \wedge SU \to SV_{i+1} \wedge \ldots \wedge SV_t \wedge E_i U$$

represents x_i, $1 \leqq i \leqq t$. Define $X \in \{g_{t-2,t-1}, \ldots, g_{1,2}\}$ by

$$X = \bigcup_{k=1}^{t-1} \text{Domain } g_{k,t}/\sim$$

where

$$y \sim C_{j+1,\ldots,t-1}(g_{i,j})(y) \text{ for } 1 \leqq i < j < t,$$
$$y \in DV_{i+1} \wedge \ldots \wedge SV_j \wedge \ldots \wedge SU \subset \partial\, (\text{Domain } g_{i,t}) \text{ and}$$
$$C_{j+1,\ldots,t-1}(g_{i,j})(y) \in \text{Domain } g_{j,t}.$$

Filter X by

$$F_n X = \text{Image}\left[\bigcup_{k=t-n}^{t-1} \text{Domain } g_{k,t} \to X\right] \quad \text{for} \quad 1 \leqq n \leqq t-1.$$

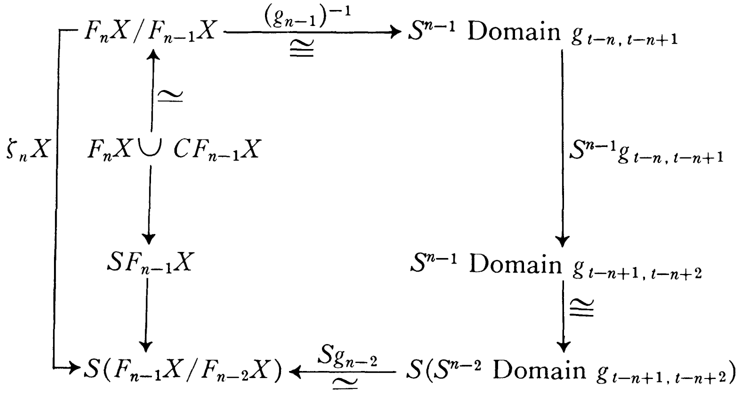
Then

$$F_n X / F_{n-1} X \cong \text{Domain } g_{t-n,t} \Big/ \left[\bigcup_{k=t-n+1}^{t-1} \text{Domain } C_{k+1,\ldots,t-1}(g_{t-n,k})\right]$$
$$= DV_{t-n+1} \wedge \ldots \wedge SV_t \wedge SU/\partial(DV_{t-n+1} \wedge \ldots \wedge SV_t \wedge SU)$$
$$\cong S^{t-(t-n+1)}(SV_{t-n+1} \wedge \ldots \wedge SV_t \wedge SU)$$
$$= S^{n-1} \text{ Domain } g_{t-n,-t-n+1}.$$

The required condition

$$\gamma_{n-1}[S^{n-1} g_{t-n,t-n+1}] = \zeta_n X$$

follows from the following homotopy commutative diagram. In this diagram the composite of the top map, the right map and the bottom map is $\gamma_{n-1}[S^{n-1} g_{t-n,t-n+1}]$.



Define $g: S^{t-2}$ Dom $g_{0,1} \to X$ as the composite map

$$S^{t-2}\text{Domain } g_{0,1} \cong \text{Domain } g_{0,t} \xrightarrow{G} X$$

where

$$G|DV_1 \wedge \ldots \wedge SV_k \wedge \ldots \wedge DV_{t-1} \wedge SV_t \wedge SU = C_{k+1,\ldots,t-1}(g_{0,k}).$$

Define $h: X \to EU$ by $h|$Domain $g_{k,t} = g_{k,t}$. It is routine to check that g and h are well-defined,

$$\text{hog} = \tilde{g}_{0,t},\ \sigma_X \circ g \simeq S^{t-2}g_{0,1} \text{ and } h \circ j_X = g_{t-1,t}.$$

Thus $x = [\tilde{g}_{0,t}] \in \langle x_1, \ldots, x_t \rangle_C$.

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