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SUBCLASSES OF STARLIKE FUNCTIONS SUBORDINATE TO CONVEX FUNCTIONS

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1. Introduction. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the unit disk $\Delta = \{z:|z| < 1\}$, with $S^*(\alpha)$ and $K(\alpha)$ designating the subclasses of S that are, respectively, starlike of order α and convex of order α, $0 \leqq \alpha < 1$. If $f(z)$ and $g(z)$ are analytic in Δ, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$, if there exists a Schwarz function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ in Δ, such that $f(z) = g(w(z))$. A function $f(z) = z + \ldots$ is said to be in $S^*[A, B]$ if

$$(1) \quad \frac{zf'}{f} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \Delta, -1 \leqq B < A \leqq 1)$$

and in $K[A, B]$ if

$$(2) \quad 1 + \frac{zf''}{f'} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \Delta, -1 \leqq B < A \leqq 1).$$

The family $S^*[A, B]$ was investigated in [**2**], [**3**], and [**5**]. We say that $f(z) = z + \ldots$ is in $S^*(a, b)$ if

$$(3) \quad \left|\frac{zf'}{f} - a\right| < b \quad (z \in \Delta, a \geqq b)$$

and in $K(a, b)$ if

$$(4) \quad \left|\left(1 + \frac{zf''}{f'}\right) - a\right| < b \quad (z \in \Delta, a \geqq b).$$

The family $S^*(a, b)$ was introduced in [**10**]. In addition to the condition $a \geqq b$ for the families $S^*(a, b)$ and $K(a, b)$, at the origin we have

$$(5) \quad |1 - a| < b.$$

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Observe that $(1 + z)/(1 - z)$ is mapped by Δ onto the right half plane so that $S^*[-1, 1]$ and $K[-1, 1]$ are, respectively, the families of starlike and convex functions. Note that functions in $S^*[A, B]$ and $S^*(a, b)$ are starlike, that functions in $K[A, B]$ and $K(a, b)$ are convex, and that $f \in K[A, B]$ ($f \in K(a, b)$) if and only if $zf' \in S^*[A, B]$ ($zf' \in S^*(a, b)$).

In Section 2 we investigate relationships between the various classes. In Section 3 we find the largest disk in which functions in S satisfy (2) or (4). In Section 4 we determine extremal functions for the order of starlikeness of the families $K[A, B]$ and $K(a, b)$.

The convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

In the final section, we give necessary and sufficient conditions in terms of convolution operators for functions to be in each of the four classes. It is also shown that these classes are invariant under particular integral operators.

2. Containment properties. We begin by showing the relationship between the families $S^*(a, b)$ and $S^*[A, B]$.

THEOREM 1. (i) *If* $-1 < B < A \leqq 1$, *then*

$$S^*[A, B] \equiv S^*\left(\frac{1 - AB}{1 - B^2}, \frac{A - B}{1 - B^2}\right).$$

(ii) *If* $a \geqq b$, *then*

$$S^*(a, b) \equiv S^*\left[\frac{b^2 - a^2 + a}{b}, \frac{1 - a}{b}\right].$$

Proof. Since $|z| = 1$ is mapped by $(1 + Az)/(1 + Bz)$ onto a circle centered at

$$a = \frac{1 - AB}{1 - B^2}$$

with radius

$$b = \frac{A - B}{1 - B^2},$$

(i) follows because $a \geqq b$ is equivalent to $(1 + B)(1 - A) \geqq 0$.

To prove (ii), we must find A, $B(-1 < B < A \leqq 1)$ such that

$$a = \frac{1 - AB}{1 - B^2} \quad \text{and} \quad b = \frac{A - B}{1 - B^2}.$$

Substituting $A = B + b(1 - B^2)$ into the first expression, we get the cubic polynomial equation

$$bB^3 + (a - 1)B^2 - bB + 1 - a = 0$$

whose three solutions are $B = \pm 1, (1 - a)/b$. The values $B = \pm 1$ are not admissible, so we set $B = (1 - a)/b$ and obtain

$$A = \frac{b^2 - a^2 + a}{b}.$$

The inequalities $-1 < B < A \leqq 1$ now follow from (5) when $a \geqq b$, and the proof is complete.

Note that

$$S^*[A, -1] = S^*((1 - A)/2)$$

and that

$$S^*\left(\frac{1 - AB}{1 - B^2}, \frac{A - B}{1 - B^2}\right) \subset S^*\left(\frac{1 - A}{1 - B}\right).$$

Letting $B \to -1^+$, we also obtain the family of functions starlike of order $(1 - A)/2$.

Since the proof of Theorem 1 involved only the relationship between pairs (A, B) and (a, b), we may replace (1) and (3) with (2) and (4) to obtain the following

COROLLARY. *Theorem* 1 *is valid with* $S^*[A, B]$ *and* $S^*(a, b)$ *replaced with* $K[A, B]$ *and* $K(a, b)$, *respectively.*

Remark. Since $S^*[A, B]$ may always be expressed as $S^*(a, b)$ for appropriate a and b, the coefficient bounds in [**2**] and the distortion bounds in [**3**] may be obtained from the coefficient and distortion bounds found in [**10**].

THEOREM 2. $S^*(c, d) \subset S^*(a, b)$ *if and only if*

$$|a - c| \leqq b - d \quad \textit{and} \quad S^*[C, D] \subset S^*[A, B]$$

if and only if

$$|AD - BC| \leqq (A - B) - (C - D).$$

Proof. We have $S^*(c, d) \subset S^*(a, b)$ if and only if

$$\{w : |w - c| < d\} \subset \{w : |w - a| < b\}.$$

The diameters along the real axis are, respectively, the line segments from $(c - d)$ to $(c + d)$ and from $(a - b)$ to $(a + b)$. Thus, the containment follows if and only if $a - b \leqq c - d$ and $c + d \leqq a + b$, or equivalently, $|a - c| \leqq b - d$. Similarly, we have $S^*[C, D] \subset S^*[A, B]$ if and only if

$$\left\{w: \left|w - \frac{1 - CD}{1 - D^2}\right| < \frac{C - D}{1 - D^2}\right\} \subset \left\{w: \left|w - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2}\right\},$$

which is equivalent to the inequalities

$$\frac{1 - A}{1 - B} \leqq \frac{1 - C}{1 - D} \quad \text{and} \quad \frac{1 + C}{1 + D} \leqq \frac{1 + A}{1 + B},$$

or

$$|AD - BC| \leqq (A - B) - (C - D).$$

This completes the proof.

The proof of Theorem 2 also furnishes us with a proof of the following

COROLLARY. *$K(c, d) \subset K(a, b)$ if and only if $|a - c| \leqq b - d$ and $K[C, D] \subset K[A, B]$ if and only if $|AD - BC| \leqq (A - B) - (C - D)$.*

To prove our next containment result, we will need the following lemma that was given in [7].

LEMMA 1. *If G is analytic and H is analytic, univalent and convex in Δ with the range of G'/H' contained in some convex set $\mathscr{D}$, then the range of numbers*

$$(G(z_2) - G(z_1))/(H(z_2) - H(z_1))$$

for $|z_1| < 1$ and $|z_2| < 1$ is also contained in $\mathscr{D}$.

THEOREM. 3. *$K(a, b) \subset S^*(a, b)$ and $K[A, B] \subset S^*[A, B]$.*

Proof. Set $\mathscr{D} = \{w:|w - a| < b\}$. If $f \in K(a, b)$, then $(zf')'/f' \subset \mathscr{D}$. By Lemma 1,

$$\frac{z_2 f'(z_2) - z_1 f'(z_1)}{f(z_2) - f(z_1)} \subset \mathscr{D} \quad \text{for } |z_1| < 1, |z_2| < 1.$$

Setting $z_1 = 0$, we have

$$z_2 f'(z_2)/f(z_2) \subset \mathscr{D} \quad \text{for all } z_2, |z_2| < 1,$$

so that $f \in S^*(a, b)$. The proof that $K[A, B] \subset S^*[A, B]$ is identical, with

$\mathcal{D}$ replaced by the convex set

$$\mathcal{D}' = \left\{ w : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$

3. Radii problems. It is well known that the disk $\{z : |z| < 2 - \sqrt{3}\}$ is mapped by all functions in S onto a convex domain. Our next theorem is a generalization of this result.

Definition. The *$K(a, b)$ radius of S*, denoted $\rho(a, b)$ is the radius of the largest disk $|z| < \rho(a, b)$ in which the inequality (4) holds for all $f \in S$. The *$K[A, B]$ radius of S*, denoted $\rho[A, B]$, is the radius of the largest disk $|z| < \rho[A, B]$ for which the subordination (2) holds for all $f \in S$.

THEOREM 4. *With the notation above,*

$$\rho(a, b) = \min\left\{ \frac{2 - \sqrt{3 + (a - b)^2}}{1 + a - b}, \frac{-2 + \sqrt{3 + (a + b)^2}}{1 + a + b} \right\}$$

and

$$\rho[A, B] = \min\left\{ \frac{2(1 - B) - \sqrt{3(1 - B)^2 + (1 - A)^2}}{2 - A - B}, \frac{-2(1 + B) + \sqrt{3(1 + B)^2 + (1 + A)^2}}{2 + A + B} \right\}.$$

Equality in both cases occurs for $f(z) = z/(1 - z)^2$.

Proof. For $f \in S$ and $|z_0| = \rho < 1$, it is known [**4**] that

$$\left| \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) - \frac{1 + \rho^2}{1 - \rho^2} \right| < \frac{4\rho}{1 - \rho^2}.$$

Thus, inequality (4) is true for $|z| < \rho$ if

$$a - b \leqq (1 - 4\rho + \rho^2)/(1 - \rho^2) \quad \text{and} \quad (1 + 4\rho + \rho^2)/(1 - \rho^2) \leqq a + b.$$

These two inequalities are equivalent to

$$(7) \qquad (1 + a - b)\rho^2 - 4\rho + (1 - a + b) \geqq 0 \quad \text{and}$$

$$(8) \qquad (1 + a + b)\rho^2 + 4\rho + (1 - a - b) \leqq 0.$$

But (7) is true if

$$\rho \leqq \frac{2 - \sqrt{3 + (a - b)^2}}{1 + a - b}$$

and (8) is true if

$$\rho \leqq \frac{-2 + \sqrt{3 + (a + b)^2}}{1 + a + b}.$$

This gives the result for $\rho(a, b)$. The result for $\rho[A, B]$ follows from the Corollary to Theorem 1, upon noting that

$$\rho[A, B] = \rho\left(\frac{1 - AB}{1 - B^2}, \frac{A - B}{1 - B^2}\right).$$

COROLLARY 1.

$$\rho(\alpha, \alpha) = \frac{-2 + \sqrt{3 + 4\alpha^2}}{1 + 2\alpha}$$

for $1/2 < \alpha < 2\sqrt{3}/3$ *and* $\rho(\alpha, \alpha) = 2 - \sqrt{3}$, *the radius of convexity for* S, *when* $\alpha \geqq 2\sqrt{3}/3$.

Proof. Substituting $a = b = \alpha$ in Theorem 4, we get

$$\rho(\alpha, \alpha) = \min\left\{2 - \sqrt{3}, \frac{-2 + \sqrt{3 + 4\alpha^2}}{1 + 2\alpha}\right\}.$$

But $(-2 + \sqrt{3 + 4\alpha^2})/(1 + 2\alpha)$ is an increasing function of α and is equal to $2 - \sqrt{3}$ when $\alpha = 2\sqrt{3}/3$.

COROLLARY 2. *If* $f \in S$, *then* f *is convex of order* α, $0 \leqq \alpha < 1$, *for*

$$|z| < \frac{2 - \sqrt{3 + \alpha^2}}{1 + \alpha}.$$

Proof. Since $K[A, -1] = K((1 - A)/2)$, or equivalently $K[1 - 2\alpha, -1] = K(\alpha)$,

$$\rho[1 - 2\alpha, -1] = \min\left\{\frac{2 - \sqrt{3 + \alpha^2}}{1 + \alpha}, 1\right\}.$$

4. Orders of starlikeness. MacGregor has shown [7] that for $f \in K(\alpha)$, z in Δ,

$$zf'/f \prec zF'_\alpha/F_\alpha,$$

where

$$F_\alpha(z) = \begin{cases} \dfrac{1 - (1 - z)^{2\alpha - 1}}{2\alpha - 1}, & \alpha \neq 1/2 \\ -\log(1 - z), & \alpha = 1/2. \end{cases}$$

We will need the following lemmas to determine the extremal functions for the orders of starlikeness of $K(a, b)$ and $K[A, B]$.

LEMMA 2. *Suppose* $h(z) = 1 + \dots$ *is analytic and convex in* Δ *with* $\operatorname{Re} h(z) > 0$ *for* $z \in \Delta$, *and* $p(z) = 1 + \dots$ *is analytic in* Δ *with*

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z).$$

If the differential equation

$$q(z) + \frac{zq'(z)}{q(z)} = h(z)$$

has a univalent solution $q(z)$ *in* Δ, *then* $p(z) \prec q(z)$ *in* Δ.

LEMMA 3. *If* $f(z) = z + \dots$ *is analytic and convex in* Δ, *then so is*

$$F(z) = \frac{2}{z}\int_0^z f(t)dt.$$

Lemma 2 was proved in [**1**] and Lemma 3 was proved in [**6**].

THEOREM 5. *If* $f \in K(a, b)$ *and* $c = b^2 - (1 - a)^2$, *then* $(zf'/f) \prec (zF'/F)$ *for* $z \in \Delta$ *where*

$$F(z) = \begin{cases} \dfrac{b}{c + 1 - a}\left[\left(1 + \dfrac{1 - a}{b}z\right)^{(1+c-a)/(1-a)} - 1\right], & (1 - a)(c + 1 - a) \neq 0, \\ (e^{bz} - 1)/b & , a = 1, \\ \dfrac{b}{1 - a}\log\left(1 + \dfrac{1 - a}{b}z\right) & , c + 1 - a = 0. \end{cases}$$

Proof. If $f \in K(a, b)$ then

$$f_1(z) = \left(\left(1 + \frac{zf''}{f'}\right) - a\right)/b$$

has modulus less than one in Δ. Thus

$$f_2(z) = (f_1(z) - f_1(0))/(1 - f_1(0)f_1(z))$$

is a Schwarz function. Hence, by Schwarz Lemma $f_2(z) \prec z$, or equivalently,

$$(9) \quad 1 + \frac{zf''}{f'} \prec a + b\left\{\frac{\dfrac{1 - a}{b} + z}{1 + \dfrac{1 - a}{b}z}\right\}.$$

Note that F defined above is the solution to

$$(10)\quad 1 + \frac{zF''}{F'} = a + b\left\{\frac{\frac{1-a}{b} + z}{1 + \frac{1-a}{b}z}\right\}.$$

Setting

$$p(z) = zf'/f, \quad q(z) = zF'/F, \quad \text{and}$$

$$h(z) = a + b\left\{\frac{\frac{1-a}{b} + z}{1 + \frac{1-a}{b}z}\right\},$$

we observe that

$$p + \frac{zp'}{p} = 1 + \frac{zf''}{f'} \quad \text{and} \quad q + \frac{zq'}{q} = 1 + \frac{zF''}{F'}.$$

In view of (9), (10), and the fact that h is convex in Δ, we see that our conclusion will follow from Lemma 2 if we can show that $q(z)$ is univalent in Δ. For $(1 - a)(c + 1 - a) \neq 0$, we have

$$q(z) = \frac{c + 1 - a}{b}z\left[\frac{\left(1 + \frac{1-a}{b}z\right)^{c/(1-a)}}{\left(1 + \frac{1-a}{b}z\right)^{(c+1-a)/(1-a)} - 1}\right],$$

and

$$\frac{1}{q(z)} = \frac{b}{c + 1 - a}\frac{1}{z}\left[\left(1 + \frac{1-a}{b}z\right) - \left(1 + \frac{1-a}{b}z\right)^{-c/(1-a)}\right]$$

$$= \frac{b}{c + 1 - a}\frac{1}{z}\int_0^z\left[\frac{1-a}{b}\right.$$

$$\left. + \frac{c}{b}\left(1 + \frac{1-a}{b}t\right)^{-((c/(1-a))+1)}\right]dt.$$

As a consequence of Lemma 3, $1/q(z)$ is univalent if

$$k(z) = \left(1 + \frac{1-a}{b}z\right)^{-((c/(1-a))+1)}$$

$$= \left(1 + \frac{1-a}{b}z\right)^{((a^2-b^2-a)/(1-a))}$$

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is convex. (Note that the normalization is not crucial because $k(z)$ is convex if and only if $(k(z) - A_1)/A_2$ is convex for any constants A_1 and A_2, $A_2 \neq 0$.) We have

$$1 + \frac{zk''(z)}{k'(z)} = \frac{1 + \left(\frac{a^2 - b^2 - a}{b}\right)z}{1 + \left(\frac{1 - a}{b}\right)z},$$

so that for $z = e^{i\theta}$,

$$\operatorname{Re}\left\{1 + \frac{zk''(z)}{k'(z)}\right\}$$
$$= \frac{1 - \left(\frac{1 - a}{b}\right)^2 a - (1 - a) + \left[\frac{(1 - a)^2 - b^2}{b}\right]\cos\theta}{\left|1 + \frac{1 - a}{b}z\right|^2}.$$

Since from (5) we see that $|(1 - a)/b| < 1$, the numerator of this last expression is bounded below by

$$1 - \left(\frac{1 - a}{b}\right)^2 a - (1 - a) + \frac{(1 - a)^2 - b^2}{b}$$
$$= (a - b)\left(1 - \left(\frac{1 - a}{b}\right)^2\right) \geqq 0.$$

Thus $k(z)$ is convex, and consequently $q(z)$ is univalent in Δ for

$$(1 - a)(c + 1 - a) \neq 0.$$

If $a = 1$, then the univalence of $q(z)$ follows from the convexity of e^{-bz}, since

$$\frac{1}{q(z)} = \frac{1}{z}\int_0^z e^{-bt}dt.$$

Finally, if $c + 1 - a = 0$, we must show that

$$q(z) = \frac{\left(\frac{1 - a}{b}\right)z}{\left(1 + \frac{1 - a}{b}z\right)\log\left(1 + \frac{1 - a}{b}z\right)}$$

is univalent in Δ.

Since $|(1 - a)/b| < 1$, it suffices to show that

$$q_1(z) = \frac{z}{(1 + z)\log(1 + z)}$$

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is univalent in Δ. But

$$\frac{1}{q_1(z)} = \frac{1}{z}\int_0^z (1 + \log(1 + t))dt,$$

and the univalence of $q(z)$ follows from the convexity of $\log(1 + z)$, $z \in \Delta$. This completes the proof.

From the relationship between $K(a, b)$ and $K[A, B]$ we immediately obtain the following

COROLLARY. *If $f \in K[A, B]$, then $(zf'/f) \prec (zF'/F)$ for z in Δ and*

$$F(z) = \begin{cases} \dfrac{(1 + Bz)^{A/B} - 1}{A} & , \ AB \neq 0 \\ (e^{Az} - 1)/A & , \ B = 0 \\ \dfrac{1}{B}\log(1 + Bz) & , \ A = 0. \end{cases}$$

Proof. In view of the corollary to Theorem 1, the proof follows from Theorem 5 upon setting $a = (1 - AB)/(1 - B^2)$ and $b = (A - B)/(1 - B^2)$.

Remarks 1. In [**1**] the corollary was proved for the special case $-1 \leqq B < 0$ and $B < A \leqq -B$.

2. One could prove this corollary directly by showing that the solution $q(z) = zF'(z)/F(z)$ to the equation

$$q(z) + zq'(z)/q(z) = (1 + Az)/(1 + Bz), \ -1 \leqq B < A \leqq 1$$

is univalent in Δ.

3. Since $K[A, -1] = K((1 - A)/2)$, the result of MacGregor [**7**] is a special case of the corollary.

5. Convolution properties. In [**11**], necessary and sufficient conditions are given in terms of convolution operators for a function to be in $S^*(\alpha)$ or $K(\alpha)$. We now do this for the classes $S^*(a, b)$, $K(a, b)$, $S^*[A, B]$, and $K[A, B]$. We assume, in Theorems 6 and 7 and their corollaries, that $f(z) = z + \ldots$ is analytic in Δ and that (5) is satisfied.

THEOREM 6. *$f \in S^*(a, b)$ if and only if for all z in Δ and all ζ, $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \dfrac{a + b\zeta}{1 - a - b\zeta}z^2}{(1 - z)^2}\right] \neq 0.$$

Proof. A function f is in $S^*(a, b)$ if and only if $(zf'/f) - a \neq b\zeta$ for z in

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Δ and $|\zeta| = 1$, which, from the normalization of f, is equivalent to

$$(11) \quad \frac{1}{z}[zf' - (a + b\zeta)f] \neq 0, \quad z \in \Delta.$$

Since $zf' = f * z/(1 - z)^2$ and $f = f * z/(1 - z)$, we may write (11) as

$$\frac{1}{z}\left[f * \left(\frac{z}{(1-z)^2} - \frac{(a+b\zeta)z}{1-z}\right)\right] = (1 - a - b\zeta)\frac{1}{z}\left[f * \frac{z + \frac{a+b\zeta}{1-a-b\zeta}z^2}{(1-z)^2}\right] \neq 0.$$

This completes the proof.

COROLLARY. *$f \in K(a, b)$ if and only if for all z in Δ and all ζ, $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \left(\frac{1+a+b\zeta}{1-a-b\zeta}\right)z^2}{(1-z)^3}\right] \neq 0.$$

Proof. Set

$$g(z) = \frac{z + \frac{a+b\zeta}{1-a-b\zeta}z^2}{(1-z)^2},$$

and note that

$$zg'(z) = \frac{z + \left(\frac{1+a+b\zeta}{1-a-b\zeta}\right)z^2}{(1-z)^3}.$$

From the identity $zf' * g = f * zg'$ and the fact that $f \in K(a, b)$ if and only if $zf' \in S^*(a, b)$, the result follows from Theorem 6.

THEOREM 7. *$f \in S^*[A, B]$ if and only if for all z in Δ and all ζ, $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \frac{\zeta - A}{A - B}z^2}{(1-z)^2}\right] \neq 0.$$

Proof. A function f is in $S^*[A, B]$ if and only if

$$zf'/f \neq (1 + A\zeta)/(1 + B\zeta) \quad \text{for } z \text{ in } \Delta \text{ and } |\zeta| = 1,$$

which is equivalent to

$$\frac{1}{z}[(1 + B\zeta)zf' - (1 + A\zeta)f]$$

$$= \frac{1}{z}\left[f * \left(\frac{(1 + B\zeta)z}{(1 - z)^2} - \frac{(1 + A\zeta)z}{1 - z}\right)\right] \neq 0,$$

and the result follows.

The next corollary follows from Theorem 7 just as the previous corollary followed from Theorem 6.

COROLLARY. *$f \in K[A, B]$ if and only if for all z in Δ and all ζ, $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \left(\frac{2\zeta - A - B}{A - B}\right)z^2}{(1 - z)^3}\right] \neq 0.$$

Remark. For $A = 1 - 2\alpha$ and $B = -1$ in Theorem 7 and its corollary, we obtain the results found in [**11**].

We next use a version of a lemma due to Ruscheweyh and Sheil-Small [**9**], which will enable us to show that the four classes we have been considering remain invariant under particular convolution operators.

LEMMA 4. *If $\varphi \in K(0)$ and $g \in S^*(0)$, then for each function F, analytic in Δ, the image of Δ under $(\varphi * Fg)/(\varphi * g)$ is a subset of the convex hull of $F(\Delta)$.*

THEOREM 8. *If $f \in S^*(a, b)$, $S^*[A, B]$, $K(a, b)$, or $K[A, B]$, then so is $f * \varphi$ for any function $\varphi(z) = z + \dots$, analytic and convex in Δ.*

Proof. We have $f_1 \in S^*(a, b)$ if and only if, for z in Δ,

$$F_1 = \frac{zf_1'}{f_1} \prec a + b\left\{\frac{\frac{1 - a}{b} + z}{1 + \frac{1 - a}{b}z}\right\} = G_1,$$

and $f_2 \in S^*[A, B]$ if and only if

$$F_2 = \frac{zf_2'}{f_2} \prec \frac{1 + Az}{1 + Bz} = G_2.$$

Since G_1 and G_2 are convex, an application of Lemma 4 yields

$$\frac{\varphi * F_1 f_1}{\varphi * f_1} = \frac{z(\varphi * f_1)'}{\varphi * f_1} \prec G_1$$

and

$$\frac{\varphi * F_2 f_2}{\varphi * f_2} = \frac{z(\varphi * f_2)'}{\varphi * f_2} \prec G_2,$$

so that $\varphi * f_1 \in S^*(a, b)$ and $\varphi * f_2 \in S^*[A,B]$. The result for $K(a, b)$ and $K[A, B]$ now follows from the relationship $f \in K(a, b)(f \in K[A, B])$ if and only if $zf' \in S^*(a, b)(zf' \in S^*[A, B])$.

COROLLARY 1. *If $f \in S^*(a, b)$, $S^*[A, B]$, $K(a, b)$, or $K[A, B]$, then so is*

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \quad \operatorname{Re}\gamma \geqq -1/2.$$

Proof. We may write

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt = f * \sum_{n=1}^{\infty}\frac{1+\gamma}{n+\gamma}z^n.$$

The result follows from Theorem 8 upon noting that $\sum_{n=1}^{\infty}\frac{1+\gamma}{n+\gamma}z^n$ is convex in Δ. See [**8**].

COROLLARY 2. *If $f \in S^*(a, b)$, $S^*[A, B]$, $K(a, b)$ or $K[A, B]$, then so is*

$$\int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta}d\zeta, \quad |x| \leqq 1, x \neq 1.$$

Proof. We may write

$$\int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta}d\zeta = f * h$$

where

$$h(z) = \sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)n}z^n = \frac{1}{1-x}\log\left[\frac{1-xz}{1-z}\right], \quad |x| \leqq 1, x \neq 1.$$

Since h is convex, the result follows from Theorem 8.

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