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REMARKS CONCERNING UNIFORMLY BOUNDED OPERATORS ON HILBERT SPACE

BY
I. ISTRĂŢESCU

1. In [6] B. Sz.-Nagy has proved that every operator on a Hilbert space such that

$$\sup_n \|T^n\| < \infty \quad (n = \pm 1, \pm 2, \ldots)$$

is similar to a unitary operator.

The following problem is an extension of this result: If T and S are two operators such that

1. $\sup\{\|T^n\|, \|S^n\|\} < \infty \quad (n = 0, \pm 1, \pm 2, \ldots)$
2. $TS = ST$

then there exists a selfadjoint operator Q such that QTQ^{-1}, QSQ^{-1} are unitary operators?

Also, in [7] B. Sz.-Nagy has proved that every compact operator T such that

$$\sup \|T^n\| < \infty \quad (n = 1, 2, 3, \ldots)$$

is similar to a contraction.

Our aim in this note is to give a positive answer to the above question and to extend the result of B. Sz.-Nagy [7] to quasi-compact operators and to operators with compact imaginary part.

2. We use the concept of generalized limit for double sequences. The concept of generalized limit was considered by Mazur and many writers treated the problem of extending it to double sequences. We use here the results obtained by J. D. Hill [2].

Let $(m)_2$ be the Banach space of all bounded real or complex sequences $x = \{x_{i,j}\}$ with $\|x\| = \sup_{i,j} |x_{i,j}|$. The result of J. D. Hill which we need is the following:

THEOREM. *There exist continuous functionals L_2 over $(m)_2$ satisfying the conditions*:

1. $L_2(ax+by) = aL_2(x)+bL_2(y)$.
2. $L_2(x) \geq 0$ *if all* $x_{i,j} \geq 0$.
3. $L_2(x_{(m,n)}) = L_2(x)$, $x_{(m,n)i,j} = x_{i+m,j+n}$; *$m$, n are integers.*
4. $L_2(1) = 1$ *where* $1_{i,j} = 1$ *for all i, j.*

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Our result is the following:

THEOREM 1. *If T and S are two linear operators on a Hilbert space such that*

1. $\sup\{\|T^n\|, \|S^n\|\}<\infty \quad (n=0, \pm 1, \pm 2, \ldots)$
2. $TS=ST$

then there exists a selfadjoint operator Q such that QTQ^{-1}, QSQ^{-1} are unitary operators.

Proof. For every $x, y \in H$ we define

$$\langle\langle x, y\rangle\rangle = L_2\langle T^n S^m x, T^n S^m y\rangle.$$

By property 1 of the continuous functional L_2, we have

$$\langle\langle \alpha_1 x_1+\alpha_2 x_2, \beta_1 y_1+\beta_2 y_2\rangle\rangle = \alpha_1\bar{\beta}_1\langle\langle x_1, y_1\rangle\rangle+\alpha_1\bar{\beta}_2\langle\langle x_1, y_2\rangle\rangle+\alpha_2\bar{\beta}_1\langle\langle x_2, y_1\rangle\rangle+\alpha_2\bar{\beta}_2\langle\langle x_2, y_2\rangle\rangle$$

i.e. $\langle\langle x, y\rangle\rangle$ is a bounded hermitian bilinear form of the variable elements x and y. Also there exists a constant k such that

$$\frac{1}{k^2}\|x\|^2 \le \langle\langle x, x\rangle\rangle \le k^2\|x\|^2$$

for all $x \in H$. Thus we find a selfadjoint operator A such that

$$\langle\langle x, y\rangle\rangle = \langle Ax, y\rangle$$

which gives that

$$\frac{1}{k^2} I \le A \le k^2 I.$$

It is easy to see by the translation property of L_2 that

$$T^*AT = A$$
$$S^*AS = A.$$

From these we obtain that $U_T = QTQ^{-1}$ and $U_S = QSQ^{-1}$ are unitary operators, where Q is the positive square root of A. The theorem is proved.

REMARK. Is it possible to obtain a similar result for operators that do not commute?

3. We consider here the problem of similarity for quasicompact operators and for operators with compact imaginary part.

Recall that an operator T is said to be quasi-compact if there exists a compact operator S and an integer $k \ge 1$ such that

$$\|T^k - S\| < 1.$$

THEOREM 2. *If T is quasi-compact operator with the property $\|T^n\|=o(n)$ as $n\to\infty$, then T is similar to a contraction.*

Proof. Since $\|T^n\|=o(n)$ as $n\to\infty$, it follows that $\{n^{-1}T^n\}$ converges weakly to zero and by Lemma 1 [1, p. 709] we conclude that $\sigma(T)\subset\{z:|z|\le 1\}$ and every point $\lambda\in\sigma(T)$, $|\lambda|=1$ is a simple pole of the resolvent operator of T. From the quasi-compactness of T we have that $\sigma(T)\cap\{z:|z|=1\}$ is a finite set of eigenvalues of finite multiplicity. The rest of the proof is as in [7] and we omit it.

The class $\mathscr{R}$ of operators of Riesz type (which contains the compact operators) was introduced in [5] by A. F. Ruston. Also in [5] is given a characterization of $\mathscr{R}$ as the class of asymptotically quasi-compact operators, i.e. $T\in\mathscr{R}$ if T is a bounded linear operator and

$$\lim_{n\to\infty}\left\{\inf_{C\in I(H)}\|T^n-C\|\right\}^{1/n}=0$$

where $I(H)$ is the ideal of compact operators on H.

COROLLARY. [4] *If $T\in\mathscr{R}$ and $\|T^n\|\le M$, $n=1,2,\ldots$, then T is similar to a contraction.*

It is easy to see that this corollary is true for an operator T for which

$$\lim_{n\to\infty}\left\{\inf_{C\in I(H)}\|T^n-C\|\right\}^{1/n}<1$$

and $\|T^n\|=o(n)$ as $n\to\infty$.

THEOREM 3. *Let $T=A+iB$ with B compact, $\sigma(A)$ disjoint from the set $\{-1,1\}$ and $\|T^n\|\le M$, $n=1,2,\ldots$. Then T is similar to a contraction.*

Proof. Since A is a selfadjoint operator with uniformly bounded iterates we have that $\sigma(A)\subset(-1,1)$ and $\|A\|<1$. From this we have that $\|T-iB\|<1$ and therefore T is quasi-compact operator and by Theorem 2, T is similar to a contraction.

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POLITECHNIC INSTITUTE TIMIŞOARA,
TIMIŞOARA, ROUMANIA

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