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PAR
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On the Minkowski-Hilbert Dialogue on Mathematization

Introduction. Il me fait un très grand plaisir de vous présenter aujourd'hui la conférence Jeffrey-Williams nommée ainsi en l'honeur des distingués fondateurs de la Société Mathématique du Canada.

Il y a beaucoup d'évidences indirectes d'un vif échange d'idées entre Herman Minkowski et David Hilbert, pendant la période de temps entre la soutenance de thèse de Hilbert (décembre 1884) et la mort prématurée de Minkowski (le 12 janvier 1909). Ces évidences se trouvent dans plusieurs travaux des deux grands mathématiciens, tel que le "Zahlbericht" et le discours de Hilbert à Paris sur les problèmes non résolus en mathématiques et la "Geometrie der Zahlen" et les "Diophantische Approximationen" par Minkowski.

Récemment on a publié les lettres de H. Minkowski à D. Hilbert écrites entre 1885 et 1908. Celles-ci donnent une évidence directe de l'interaction entre H. Minkowski et D. Hilbert et du rôle de A. Hurwitz qui encourageait cette interaction en intervenant amicalement.

Parmi les nombreux sujets de discussion souvent soulevés, on a porté particulièrement attention, en citant des détails contenus dans les lettres, à la relation entre Minkowski et Hilbert d'une part, et les mathématiciens contemporains français d'autre part, surtout Hermite, Jordan, et Poincaré, ainsi que Picard et Hadamard. On y remarque un intérêt soutenu pour Dedekind et Kronecker, pour la préparation de la "Geometrie der Zahlen", pour les questions des mathématiques pures et appliquées, et pour le discours bien connu de Hilbert à Paris (1900).

En accord avec C. Reid, on constate que Minkowski et Hilbert se considéraient en compétition avec les grands mathématiciens français, Poincaré en particulier. Cette attitude a motivé en grande partie la préparation et le contenu du discours à Paris. Cependant, on perçoit un ort courant d'admiration de la part de Minkowski pour les travaux de Hermite, Hadamard, C. Jordan et d'autres mathématiciens français et une satisfaction profonde à la suite de l'intérêt accordé à son propre travail par Hermite et Picard.

Il semble qu'il y ait eu de la part de Minkowski et Hilbert un certain désir de dévaloriser les travaux de Dedekind et Kronecker aux yeux de leurs contemporains allemands. En même temps, il y avait un échange d'idées entre Dedekind et Minkowski et toujours une appréciation du génie inventif de Kronecker de la part de Minkowski.

Minkowski a passé de nombreuses années à essayer de bâtir les fondements d'une théorie constructive des nombres. Les attitudes de Minkowski, Hilbert, et F. Klein en face des applications étaient complètement différentes, attitudes dont les conséquences sont encore apparentes de nos jours.

L'interprétation de Minkowski de "l'art pour l'art" dans les mathématiques etait tout-à-fait differente de celle de la premiére génération d'auteurs de Bourbaki.

There is abundant evidence of a lively and continuing discourse on basic ideas of the mathematical sciences between Hermann Minkowski (1864–1909) and David Hilbert (1862–1943), those two great mathematicians who laid the foundations for the mathematics of our century as many of us bear witness.

Not only do you see it, you can feel it in the warm afterglow of Hilbert's memorial speech on his dearest friend.[1]

You may remember from C. Reid's Hilbert biography[2] the three friends Adolf Hurwitz (1859–1919), Herman Minkowski, and David Hilbert walking 'under the apple tree' in deep discussion of basic theories of their subject.

Hurwitz, the senior of the three by a few years, initiated his two younger colleagues into what we may call now conceptual mathematics of a universalist direction. With the passing years the senior friend changed his role from an initiator to a sympathetic witness and commentator of the discussion that was going on between Minkowski and Hilbert.

What is it about?

Though there is much indirect evidence in the collected works of the three congenial friends and though we have seen and felt one of the two witnessing already, of *direct evidence* there are extant presently only the letters of Minkowski to Hilbert from 1885 to 1908, and of these mainly the correspondence from February 14, 1885 to July 28, 1900.[3]

In spite of all efforts to locate the letters of Hilbert to Minkowski their present existence and location remains uncertain. Minkowski's oldest daughter (Mrs. L. Rüdenberg) remembers having read them about 1930 and found them to be more businesslike in tone, less colorful than her father's letters. When Minkowski moved to Göttingen in 1902 the discourse was continued orally, probably often during one of their lovely walks in the woods near Göttingen; the need for elaboration of ideas in letter form had ceased more or less.

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(1) s. Minkowski [2] I, V–XXXI.

(2) s. Reid, p. 14.

(3) The letter of July 28, 1900 describes the impact of Hilbert's Paris address on Minkowski.

As anybody glancing through the letters will realize they are honest to goodness letters between friends (only one side still talking!) notwithstanding the courtesy and dignified friendliness prevailing in them which has moved one of my American colleagues to comment: "They sound as if they were written for posterity."

In fact they are a mélange of family correspondence, gossip, trivia, mathematical policy and mathematical discussion. There may be no signature at all, or as once happened, Minkowski signed with Hilbert's name.

It remains for the reader to discover ongoing lines of thought, he is *not* primed in any way.

For today's commentary I select passages referring to certain French mathematicians, to Dedekind and Kronecker, to the preparation and continuation of the "Geometrie der Zahlen", to "applied and pure mathematics", and to Hilbert's Paris address.

In the absence of a published translation of the letters, permit me to quote extensively in my English translation.

I. On French Mathematicians

February 14, 1885: I am resolved to publish some of my results already now. They will appear in the third issue of the current volume of Crelle.[(4)]

Indeed, suddenly I got afraid I could again be deprived of my joy. Poincaré of whose many-sided and swift working talents you certainly will have heard already, has started on a series of investigations not long ago which would find their brilliant completion by those very theorems of mine. It could happen indeed that he would find this completion now after the publication of my prize-essay.[(5)]

April 26, 1886 (written while Minkowski served with the Prussian army and Hilbert had gone on a postdoctoral trip to Paris): Please tell me in minutest detail what happened to you since last October. If one of the big men, Jordan or Hermite, would perhaps remember me, please give them my best recommendations, and explain to them that I am not lazy by nature.

December 29, 1887: For the New Year my best wishes. Let us hope that neither the French nor the Russians will disturb our circles.

June 2, 1893: The Poincaré prize winning memoir[(6)] did not impress me quite as much as it did impress Nöther, even though I believe to have it understood not

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[(4)] s. Minkowski [3].

[(5)] s. Minkowski [4].

[(6)] In 1885 King Oscar II of Sweden established a prize of 2500 Swedish crowns and a gold medal to reward an important discovery in the field of mathematical analysis on the occasion of his 60th birthday, January 21, 1889. The prize committee (K. Weierstrass, Charles Hermite, and G. Mittag-Leffler recommended unanimously the award of the prize to H. Poincaré for his long memoir on the three body problem published in 1892 and 1893 as Vol. I, II of "Les Méthodes nouvelles de la Mécanique céleste." S. Poincaré [1].

any less. I also had to referee a second larger and fairly interesting paper(7) of Poincaré on potential theory; I must confess, I would never be able to publish papers in the state Poincaré did.

August 20, 1894: Hermite has now studied everything I sent him of my book.(8) He writes about it full of delight to the translator: Je Crois voir la terre promise, etc.

May 30, 1896: I had a very nice letter from Picard about my book.

January 22, 1896: The papers of Jordan(9) are really quite interesting. But if already Kummer's computations were disagreeable to youth in Jordan's operations, his "pour fixer les idées" will truly nauseate you.

Comments: The revealing letter of February 14, 1885 refers to the six papers on quadratic forms published by H. Poincaré (1854–1912) between 1879 and 1882(10) in which the powerful French mathematician appropriated to himself a half century of German arithmetic developments, reinventing much of the theory of lattices, ideals, algebraic number theory, quadratic forms, and creating the beginnings of class field theory and the geometry of numbers, including some of the ideas Minkowski published a little later.(11)

However, Minkowski's fear of another anticipation by Poincaré was unfounded since the latter's work in arithmetics in subsequent years was directed towards conquering the reduction theory of forms of higher degree.(12) The problems only partially solved by Poincaré, as K. Mahler remarks, are still slumbering to be reawakened in the context of a constructive theory of algebraic numbers based on the reduction theory of higher norm forms. As we shall see in III, Minkowski worked for many years on the problem generalizing to higher algebraic number fields Lagrange's application of continued fraction algorithms to the arithmetic theory of real quadratic number fields,(13) apparently never fully to his own satisfaction and apparently not aware of the fact that his rival was trying his power on the hard problems of reduction theory.

In later letters Minkowski refers to intensive studies of the work of certain French mathematicians (like C. Hermite (1822–1901), J. Hadamard (1866–1963), C. Jordan (1838–1922)) which seem to indicate that he is after all much keener to learn than he is afraid of being anticipated. He obviously cherishes to be appreciated by his colleages "in Feindesland".

(7) s. Poincaré [2]. Minkowski's reviews of Poincaré [1], [2] are found in Jahrbuch d. Fort schritte d. Mathematik **22** (1899), 907–914 and 977–980.

(8) No copy of the letter seems to be published.

(9) s. C. Jordan (98)–(100) and (103) of chronological list of publications in Oeuvres, 4 vls. (Paris 1961–1964).

(10) s. Poincaré [3]–[8].

(11) s. Minkowski [5].

(12) s. Poincaré [9–11] and [12] p. 21.

(13) Joseph Louis Lagrange (1736–1813), Oeuvres VII, 56 & II, 606.

But some non-quoted remarks and allegations[14] seem to show that Minkowski never rid himself entirely of his awe before H. Poincaré's power of creative achievement and fear of anticipation even when the latter became seriously ill.

Yet it is equally true that Minkowski used to say at home that life is too short to waste on trivia,[15] for him its purpose was to behold the truth, to understand it well and to expound it perfectly.

II. On R. Dedekind and Kronecker

December 20, 1893: As I had planned, I have reproduced the comment you made in Cranz[16] in an appendix to my book,[17] in part I communicated it to Dedekind[18] on the occasion of receiving a copy of the new edition of his "Zahlentheone". It is a pity, I have not rushed more to finish my book. Dedekind certainly would have needed to refer to it at several places.

April 16, 1895: The latest paper of Dedekind[19] gives me the impression that you found a fellow sufferer in Hurwitz[20] in relation to Dedekind who really seems to have adopted Kronecker's[21] ways, to have had everything already.

March 31, 1896: I like your report[22] in its austere and yet complete form

(14) Minkowski [1], 155, [2] II, 401; note that the inspiring lecture of September 21, 1908 on "Raum und Zeit" ([2] II, 431–444) makes no mention of Poincaré's contribution in establishing the basic role of the group now named after him.

(15) Minkowski [2] I, XXIX 2nd paragraph. On Poincaré and Hilbert see also Reid, p. 22–24, 62–63, 99, 106, 120–121, 125, 133, 141, 185, 186.

In an obituary published in a Bonn newspaper by Professor E. Study the impact of Minkowski': personality was summarized as follows: A man of unpretentious behavior, he hardly ever came in public view. His inner eye was focused always on the things of the mind, never on the assertion of his personality. Yet he was not without ambition—the ambition of the kind that is becoming to the research scientist: only to present perfect work to the public. Thus he published relatively few papers, not influenced by the fashions of the day. Only a later period will be able to do full justice to his import. He did not have much influence as a teacher. It appears that he did not find it easy to adjust the exposition of his ideas to the average person's ability to understand. On the whole a man of quiet disposition, Minkowski liked to let things take care of themselves. But people near him learned to appreciate his harmonious nature, so amiable with all his critical talents and full of a happy sense of humor which sometimes could become sarcastic for good reasons. He offered the example of a self-disciplined person of easy-going manners to an epoch whose ideal seems to be the development of totally different character traits.

(16) Cranz, a summer resort at the Baltic Lake near Königsberg where Hilbert was professor at the University.

(17) There is no appendix to the "Geometrie der Zahlen." Probably Minkowski refers to the appendix to the first chapter (Anhang über lineare Ungleichungen, p. 39–45).

(18) s. Dirichlet-Dedekind. The preface was completed at Bad Harzburg, September 30, 1893 and contains no reference to H. Minkowski's break-through in the geometry of numbers though there is a footnote referring to it in Supplement XI (s. Dedekind [1] III, 142).

(19) s. Dedekind [3].

(20) s. Dedekind [2].

(21) s. Frobenius.

(22) s. Hilbert [2]. The "theorem on abelian fields" (§100) is Kronecker's theorem that finite normal extensions of the rational number field with abelian automorphism group always are contained in a suitable cyclotomic field extension of the rational number field. Its proof is given in §101–104.

extremely well, and it will certainly meet with great approval and push very much into the background the works of Dedekind and Kronecker.

July 27, 1896 (referring to the Zahlbericht): Your proof of the theorem on the abelian fields[22] is really extremely simple and beautiful. In order that the reader may enjoy it fully I would like to recommend you to finish off the necessary lemmas on abelian groups with hints of the proofs in advance of §100, at least you should give a reference to the reader where he would find a full exposition of the theory of abelian groups for this purpose.[23]

April 13, 1898: Your communications about your new reciprocity law[24] made me extremely excited, and I congratulate you for the far-reaching results which you have obtained. In the interest of the progress of science, I would like to ask you possibly not to keep back the most beautiful things for yourself for a long period, as it almost seems to be the case according to your pronouncements, but disclose everything you have achieved to others so that not again the state of affairs may develop as it did after the first publications of Kronecker.[21]

Comments: April 16, 1895: In his paper "Zur Theorie der Ideale" (Göttinger Nachrichten, 1894, Werke II, p. 272–277) Dedekind refers to the recently published paper of Hilbert (Grundzüge einer Theorie des Galoisschen Zahlkörpers, Göttinger Nachrichten 1894, p. 224–236) on the theory of decomposition and ramification of prime ideals of an algebraic number field in a finite extension. He points out firstly that his earlier publications contains those laws in outline, secondly that he had communicated the full theory to Frobenius in a letter of June 8, 1882 and thirdly that the topic is capable of much further generalization.

March 31, 1896: Hilbert did succeed in pushing back the influence of Dedekind and Kronecker in number theoretical research on his German contemporaries for a generation—at a price. As two of the editors of Dedekind's collected works (E. Noether and O. Ore) wrote at the occasion of Dedekind's 100th birthday:[25] "Es ist ein Zeichen, wie Dedekind seiner Zeit voraus war, dass seine Werke noch noch heute lebendig sind, ja dass sie vielleicht erst heute ganz lebendig sind." Dedekind's deep investigations on the foundations of ideal, ring module, and lattice theories make him the father of modern algebra.

In the sense in which C. Chevalley describes modern algebra as the language of modern mathematics[26] we can see in Dedekind more than in any other single man or woman the founder of the conceptual method of mathematical theorization man or woman the founder of the conceptual method of mathematical theorization in our century. The new generation of mathematicians under the leadership of E. Noether, H. Hasse, R. Brauer, A. A. Albert, O. Schreier, E. Artin, W. Krull,

[23] cp. Minkowski [1], 19–21.
[24] s. Hilbert [3].
[25] s. Dedekind [1] III, Nachwort der Herausgeber. Dedekind was born on October 6, 1831.
[26] s. Chevalley, Preface.

and B. L. van der Waerden after the First World War realized in full detail Dedekind's self confessed desire for conceptual clarity not only in the foundations of number theory, ring theory and algebra, but on a much broader front, in all mathematical disciplines.

It is not possible to blame this situation entirely on Hilbert himself who gave a full accounting of the available literature on algebraic number theory in his 'Zahlbericht' and who himself in his earlier work on the theory of invariants proved to be Dedekind's most powerful pupil in the abstract direction of algebraic research by giving a purely existential argument for the finite generation of the ring of invariants of any system of forms.(27) As well, Hilbert in his own work was fully aware of the strength and necessity of the constructive approach which he provided for invariant theory in another less well known paper.(28) Furthermore, Hilbert in a series of classical papers inaugurated and partially anticipated the research on class field theory which in the twenties and thirties of this century almost became synonymous with higher number theory(29) so that indeed a whole generation of scholars felt forced to think Hilbert's way.(30)

Now when the force of Hilbert's influence is about spent we realize more clearly the greater depth of the theories of Hilbert's teacher Kronecker in regard to his 'liebester Jugendtraum'. See for instance the comments made by Hassell on a letter of Kronecker to Dedekind in Werke V, p. 510–515; Minkowski was one of the editors of Kronecker's collected papers!. Consider also Dedekind's broad approach which led him to the following prophetic statement:

(From a letter to Frobenius, February 5, 1883 (Werke 2, p. 419)):

> I believe firmly, there will be a time when one will find very general laws permitting to derive immediately the prime ideals of a field from its discriminant and its other invariants (which maybe also ideals of related fields); yet we may be still far removed from this goal. In the last years, I have occupied myself very little with these questions to which I am very desirous to return since they seem to be the most interesting of all....

Dedekind knew the corresponding conjectures and results of classfield theory quite well(31) as did his contemporary H. Weber(32) but Dedekind outlines a research program which does full justice to the complexity of algebraic number fields as we meet them in nature while Hilbert picks one particular case, though perhaps the only one where to this day an elegant theory is developed.

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(27) s. Hilbert [4], [5].
(28) s. Hilbert [6].
(29) s. Hilbert [3], [7], [8].
(30) "You have made us all think only that which you would have us think" Reid, 214.
(31) s. Dedekind [4].
(32) s. Weber II §208–223, III §150–169.

III. On the Geometry of Numbers

November 6, 1889: I have made quite a bit of progress on the theory of positive definite quadratic forms indeed a lot turns out to be different for forms with a larger number of variables. Perhaps you or Hurwitz will be interested in the following theorem (that I can prove on half a page):

To every positive definite quadratic form of determinant D of n ($\geqq 2$) variables one can always give the argument such integral values that the form becomes $<nD^{1/n}$.

Hermite[33] has here for the coefficient n only $(4/3)^{1/2(n-1)}$, which is much larger in general.

December 22, 1890: I have generalized my proof for the theorem on the minimum of a positive quadratic form in an extraordinary way, I have realized that the special advantage of quadratic forms is illusory, in as much as other definite forms (though not necessarily rational forms) permit much further reaching consequences Thus I found the following result,[34] which could not be obtained by the use of quadratic forms: The discriminant of any algebraic number field which arises from an integral equation with $n-2\beta$ real and 2β complex roots is absolutely greater than

$$\left(\frac{\pi}{4}\right)^{2\beta}\frac{n^{2n}}{(n!)^2}$$

The letter of June 11, 1891 announces in detail four fundamental results of the new science of the geometry of numbers: asymptotic class number formula for positive definite quadratic forms with integral coefficients, Minkowski (Siegel-Hlawka) theorem,[35] estimates for densest lattice like sphere packing in n dimensions from above and below.[36]

August 30, 1892: With my book I have so far advanced that I shall turn to a publisher one of these days. I did not want to do it before everything was clear and definite. I have expounded all principles that I use, e.g. the auxiliary results from the theory of functions.

Another reference to the book is found in the letter of February 23, 1893 and again in the letter of June 2, 1893: Since I voluntarily contracted for awhile to paint doors and to varnish windows I must finish the job now. I hope it does not

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(33) s. Hermite [2].

(34) s. Minkowski [6], [7].

(35) Regarding the asymptotic class number formula, s. Minkowski [10]. Minkowski [8], [9] conjectured that any pointset of the n-dimensional space that is starred at the origin and of volume $\zeta(n)^n$ can be transformed by a volume preserving linear transformation so as to avoid all integral lattice points other than the origin. This conjecture was proved only for solid n-spheres by Minkowski [10], it was proved in general by E. Hlawka and put by C. L. Siegel into the context intended by Minkowski [9] (p. 279: Der Nachweis dieses Satzes erfordert eine arithmetische Theorie der kontinuierlichen Gruppe aus allen linearen Transformationen).

(36) s. Minkowski [10].

last much longer. The last item I put into my manuscript was a proof for the periodic-expansion of quadratic irrationalities as continued fractions. It is really strange, in how bad a state this old problem still was and how one based everything on a few formulae which had gotten in by change. Moreover I believe that every civilized mathematician must have a horror of the chapter on continued fractions in the textbooks. In my presentation there is hardly a trace left of computations and I believe I have suceeded to formulate more precisely the often uttered vague desire for generalization of the continued fraction theory.

February 8, 1894: There can be no argument about the educational value of (classical) geometry though there is more life in other areas of mathematics.

August 20, 1894: I am about ready with continued fractions for two real quantities. The now finished version of this investigation will be, I believe, quite instructive; it has much similarity with Jacobi's algorithm;[37] my algorithm is somewhat more sophisticated, which is compensated for by the possibility of deciding more plausible questions, to which the answer for the other algorithm remains in doubt.

February 10, 1896: The complete presentation of my investigations on continued fractions has needed almost 100 pages eventually. And yet it still lacked the only satisfactory end, a characteristic criterion for cubic irrationalities which was vaguely stirring in my mind. Without this completion, which I believed to be quite near, that chapter of my book would have been unsatisfactory at least for me, . . . as now recently again Weber urged me on to publish at least what was there already I resolved to do it . . . .

March 11, 1897: For my theorem that one can make n real linear forms of determinant $\pm D$ simultaneously absolutely less than $D^{1/n}$ by integral values of the variables Hurwitz has found a ridiculously simple proof after a recent conversation so that I must be very much ashamed not to find it myself. This proof takes care also of the boundary case regarding the equality sign in $\leqq D^{1/n}$, even though there are difficulties left in that latter case. The proof is as follows. It suffices to deal with the theorem for forms with rational integral coefficients: Let the forms be

$$y_h = a_{h1}x_1+a_{h2}x_2+\cdots+a_{hn}x_n \qquad (h = 1, 2, \ldots, n)$$

and abs $|a_{hk}|=D$. Then $Dx_k=A_{1k}y_1+A_{2k}y_2+\cdots+A_{nh}y_n=\xi_k$ when the A_{hk} are rational integers ($k=1, \ldots, n$). Now there are among the systems $\xi_1, \ldots, \xi_n$ modulo D all told D systems $\xi_1, \ldots, \xi_n$ that are not congruent as is shown by unimodular reduction of $|a_{hk}|$ to a quadratic system with non zero terms only in the diagonal. Suppose Δ is the greatest rational integer not larger than $D^{1/n}$ so that $(\Delta+1)^n>D$ and if $y_h=0, 1, 2, \ldots, \Delta (h=1, \ldots, n)$, then among those

[37] s. Leon Bernstein; The Jacobi-Perron Algorithm, its theory and application, Lecture notes 207, Springer 1971.

$(\Delta+1)^n$ systems there are necessarily some two, say $y_1', y_2', \ldots, y_n'$ and $y_1, y_2'', \ldots, y_n''$ for which $\xi_1, \ldots, \xi_n$ assume the same residue class systems modulo D. Then $y_1''-y_1'=b_1, \ldots, y_n''-y_n'=b_n$ are all $\leqq D^{1/n}$, but not all zero, and there are rational integers $x_1, \ldots, x_n$ for which the forms $y_1, \ldots, y_n$ will be equal to $b_1, \ldots, b_n$.

April 4, 1897: Hurwitz's proof for my theorem on linear forms really is essentially different from yours(38) though Dirichlet's idea finds its application in it too.

April 13, 1898: In recent times I have worked once again on generalizations of continued fraction algorithms. Thus each step requires long computations to which the results are in no comparable relation; in spite of everything I can't tear away myself from the subject. Certainly there are beautiful things hidden in the background.

July 20, 1898: For the computation of class numbers the bounds that the method of §42, p. 133 of my book gives are the best. But really the best methods to that end will be contained only in the second part of my book.(39) I myself do many new examples for my new algorithms, and I believe that much light, particularly on the theory of cubic extensions, will be shed due to the new computational tools.(40)

June 22, 1900: Since we parted from each other I worked with full vigor at my second volume. In its completed state I like some things quite well. I believe the computation with the new algorithms which I expound will find some friends, though perhaps not among the top mathematicians.

July 28, 1900: With my book I did not get as far as I wanted—I had hoped to present the final chapters to Hermite personally.(41) As Volterra indicated, probably Hermite will not be approachable at all.

Comments: Why does Minkowski see not much life in Geometry though he worked all his life on problems of convexity and on the geometry of numbers? Obviously he refers to euclidean geometry of the type taught in high schools of his period.

Minkowski had a remarkable geometric intuition which he brought to bear on problems of number theory among other things. His theories on pencils of convex bodies were instrumental in the creation of integral geometry(42) and the new trend in the geometry of numbers.(43) The arithmetization of geometry which Hilbert tried to establish(44) did not overly excite Minkowski as Minkowski's

(38) s. Hurwitz [3].

(39) s. Minkowski [14], [15].

(40) Minkowski's letter speaks about a good deal more than is contained in [11]–[13].

(41) The book did not get finished in the intended form, planned probably as the second volume of [14]. Some of the material went into [15] which if properly studied, provides one of the best introductions into algebraic number theory.

(42) s. Blaschke.

(43) s. R. L. Graham, Witenhausen, Zassenhaus.

(44) in the "Foundations of Geometry."

intuitive geometric ideas stirred up Hilbert's desire to get at the truth without geometric intuition.[45]

However Minkowski also saw what Hilbert did not see, that both geometric intuition and conceptual brilliance cannot stand in the 'user's court' if they are not underpinned by a broad algorithmic foundation. He tried in particular to establish algorithms for obtaining the units of an algebraic number field,[46] though he also very probably was convinced of the numerical inadequacy of his estimate for the class number of a field in terms of its discriminant. In other words he tried hard to establish the foundations of constructive algebraic number theory, a goal we begin to envisage anew now many years after Minkowski.[47] Note how little Minkowski is impressed by the pedagogical concerns of a text book writer!

IV. On Applied and Pure Mathematics

December 29, 1887 (from Bonn): My colleague v. Lilienthal is a very charming man; but I like to talk with him on anything else but mathematics. Quite soon he gets much too deep and raises questions on concepts and foundations when I want definite facts.

December 22, 1890: That I do not come to Königsberg at Christmas, you will have gathered already from the fact that I did not turn up, neither in your study nor at the certain Nachtigallensteigecke, nor anywhere else on the horizon. I do not know whether I must console you, nor even do I know whether it gives you comfort to hear from me that this time you would have had little joy from me as someone who is thoroughly infected by physics. Perhaps I would have to undergo a ten day quarantine before you would have readmitted me to our joint walks as mathematically pure and unapplied.

In order to find things in common with other mortals I have yielded to magics (meaning: to physics) entirely. I have practical laboratory sessions in the physics institute, at home I am studying Thompson, Helmholtz and consorts; and beginning with the end of next week I shall even work in a blue smock a few days a week in the institute for the construction of physical instruments, a type of practical activity more terrible than anything you can imagine.

April 16, 1895: ... I miss your company very much. I didn't quite make it yet to the apple tree, I believe, since your departure though I walked frequently with Garbe who always depicts the charms of the local professorial daughters.

July 1, 1895: The actual infinite is a term that I came across in a paper of Cantor, and I referred in my lecture[48] mostly to theorems of Cantor too; they found general interest though some people did not want to believe in them. The actual

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[45] s. Minkowski [15] Kap. I §6–9, s. also Hecke, 116–117.
[46] s. Minkowski [13].
[47] s. Zimmer.
[48] s. Minkowski [16].

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infinite in nature of which I mainly spoke—the title was worded somewhat excitingly so as to attract a few people inspite of the heat—was the position of points in space. Whether the attribute "in nature" was justified or not, I must leave undecided, even though I had constructed some amusing examples for Cantor's theorems in that regard. At this opportunity I have realized once again that Cantor is one of the most inspired living mathematicians.

February 11, 1899: Also I am still much occupied with the applications. From thermodynamics I went to chemistry. I always think one of these days to come to the rescue of (F.) Klein against his many attackers by way of demonstrating that the mathematicians really can do something for the practice, and even better things than to record the movements of a spinning top.

Comments: We observe that Minkowski is a 'naive' mathematician from the viewpoint of a contemporary having gone through all the strictures of the 'foundations of mathematics'. Nevertheless he develops subtly the consequences of Cantor's set theory in a physical interpretation leaving it open whether perhaps a discrete interpretation would be more appropriate. The perpetual discussion by word and example between Minkowski, F. Klein, and D. Hilbert on 'pure and applied mathematics' is still with us, even in about the same terms and so far with about the same results too.

F. Klein(49) in numerous books and articles tried to formalize the applications of mathematical thought as an independent discipline of teaching and research. Though this approach is most popular among us its dangers are a certain lack of depth and a lack of interest of the pure mathematicians. Hilbert, on the contrary, tried to purify the applications with his theory of gas equations(50) and general relativity.(51)

The dangers of this approach are its lack of applicability and the lack of interest it found with physicists in the precomputer age.

On the other hand, by the purely mathematical theory of integral equations, Hilbert and his pupils (like E. Schmidt and H. Bückner) perfected both the conceptual and the numerical aspects of Fredholm's theory so as to anticipate the need of a most useful tool for the modern quantum physicist.(52)

Minkowski, however, tries to listen to his colleagues, the physicists and chemists, hoping that what they tell him may strike a sympathetic chord in his own mind. In a manner of speaking he believes in team work. His hopes, as expressed in his letter of February 11, 1899, found their fulfilment in the last years of his life when he returned the stimulation he received from his friends, the physicists,

(49) s. e.g. Klein und Sommerfeld. For a critical appreciation of Klein's approach, s. Heun, in particular the summary at the end of the article (p. 117–120).

(50) s. Hilbert [9].

(51) s. Hilbert [10]–[11].

(52) s. C. Reid 171.

with interest in the form of classical treatises on special relativity theory[53] and electromagnetic theory.[54]

V. On Hilbert's Paris Address on "The Problems of Mathematics"[55]

December 30, 1899: Regarding your plans for a speech in Paris I cannot give any opinion yet, I shall think it over still and soon write about it to you.

January 5, 1900: I studied again Poincaré's Zürich address.[56] One can well subscribe to all his assertions in view of the mild way they are stated. Indeed he does full justice to pure mathematics. Therefore a speech exclusively in praise of pure mathematics does not fully convince me as the best thing to do. By the way only a few people will still remember what Poincaré said at that time. Since Poincaré was not there himself at the time and his draft was read, the impression was not nearly such as with Boltzmann at Munich for example.[57]

Most attractive would be the attempt of a preview on the future, in other words the determination of the problems, which future mathematicians should attack. Here your could achieve that people even after many years would still speak of your address. Yet to prophesy is of course a difficult thing to do. Perhaps you may be also afraid of revealing prematurely certain ideas which you have formed regarding the future treatment of problems. Topics of a more philosophical nature are perhaps better for a German public than for an international one. A review and a preview will be probably also given by a French mathematician who is likely to be the first to speak. You should try somehow to make sure about that. Since the audience are experts after all I think an address like Hurwitz's which also was liked very much at the time,[58] with definite facts would be more appropriate than a mere chat like Poincaré's.

I am really keen to know on which topic you will finally settle. Though it is much more the exposition than the topic itself which matters, nevertheless one could effect a doubling of the audience by an appropriate choice of a theme.

July 10, 1900: Of the more definite (?) problems of mathematical physics which are neither of a too special nor a too general character one should perhaps mention to find mechanical analogues of the mode of action of the forces in ether and furthermore the forces of chemical bonding . . . I believe that there are really many mathematical questions that are interesting and useful for physics in the areas touched by Boltzmann (Gastheorie, Bd I §1) Einleitung, S.4.5. Bd. II Schluss des §70. S. 206, S. 212 Mitte der Seite, §88–90.) Perhaps you also ask Nernst for his opinion. Also with the philosophers you will have to fight some tough battles.

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(53) s. Minkowski [18].
(54) s. Minkowski [17].
(55) s. Hilbert [12].
(56) s. Poincaré [13].
(57) s. Boltzmann.
(58) s. Hurwitz [2].

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At any rate, that is both Hurwitz's and my own opinion, you must shorten and cut the draft very substantially for oral presentation. For this will be better since some passages have to be spoken at a slower speed and you don't have to use all of your time. I shall read of course with great interest the galley proofs of your address.

July 17, 1900: I have received the first 3 proofs of your address. Since I don't know what is still to come I cannot yet give my definite opinion. At any rate it is extremely original to pose as a problem for the future that which mathematicians believed to possess already for the longest time, as the axioms of arithmetics. What will be the reaction of the laymen who will be represented among the audience in great numbers in any case?

July 28, 1900: I have finished reading your address with great pleasure. Since I waited for the end to get the correct picture my reading was somewhat delayed. I can only congratulate you on the address, it will certainly be the event of the congress and the success will be long lasting. In particular I believe that your power of attracting young mathematicians through this address which probably will be read by every mathematician without exception will still grow if this is possible at all. By elimination of small things the introduction has gained in perfection. Hurwitz and I had made ourselves an entirely wrong picture thinking that there should be called a halt to the 'Ignorabimus',(59) in particular since you had dealt with the variational calculus in such detail already.

Now you really have taken mathematics under your command and one will generally acknowledge you to be the director general. Now I am keen to know how you will proceed with the oral presentation. You can't say everything. Perhaps you read right on to the Ignorabimus, you give afterwards the 21 problems in the sectional session, referring to them in the address and still read the terminal passage. But, this procedure may not work in the end, if your address will be placed on the agenda of Friday rather than Monday.

Comment: The story of the preparation of the Paris address has been told very well already in C. Reid's book. There are undertones supplied by Hilbert's and Minkowski's relation to the French mathematicians as well as to German mathematicians like Dedekind and Frobenius.(60)

It is remarkable how well the design of the completed address fits the general pattern indicated by Minkowski.(61)

And yet Hilbert's address almost has gone its full course in the history of our

(59) s. Hilbert [12] 298 (2nd paragraph). Emil Dubois-Reymond (1818–1896), a German physiologist had stated in a famous address on the laws of nature, that we will never know the internal workings of nature: Ignorabimus. Hilbert answers: "In der Mathematik gibt es kein Ignorabimus."

(60) s. C. Reid 86, Biermann

(61) s. C. Reid 69–73.

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science so that even a large meeting of a predominantly historical bent relating to the Paris address can be held nowadays in the USA. But Minkowski's ideas are still as fresh as when they were first born and their full impact has hardly been explored. Thus we see in this country many young mathematicians of the most varied interests working on problems and areas of problems with which Minkowski wrestled already.

There are the number geometers all stout of heart and fearless. There are the functional analysts forgetful of Minkowski's more geometric ideas. There are the functional equationists taking up on a grand scale Minkowski-Hilbert's algorithmic ideas. There are combinatorialists, especially of the geometric type conjuring up the most sophisticated space divisions which would give Minkowski great joy to behold. There are applied mathematicians of all descriptions some working in teams, some working alone, some listening, some talking. There are number theorists of all denominations dealing with prime numbers and discriminants, so as to make them 'wiggle and waggle',(62) and we begin to out-dream Kronecker. There are the convex and also the non-convex geometers and there are the analysts sometimes painting doors and varnishing windows, sometimes engaging in sophisticated speculations of a topological nature which in principle were quite familiar to Minkowski as his recently rediscovered lecture notes show. And last but not least the algebraists who nowadays do a great deal more than Minkowski and Hilbert would dream of when they did not study Dedekind.

Here is a young generation full of vigor, apparently much less afraid of any lack of professional success than Minkowski and Hilbert were in their younger days, on the contrary we see them taking over gleefully one position after the other. May our science continue to prosper with them and after them!

Let me finish with a number of questions of continuing interest.(63)

1. Give algorithms that are best possible on the average for computing class number, unit group, discriminant and maximal order of any Dedekind order, and the Galois group of an algebraic number field.(64)

2. Classify algebraic number fields in terms of the behavior of its ramified prime numbers.(65)

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(62) In Fall 1893, Hilbert an Extraordinarius (associate professor) of the University of Königsberg, was offered by F. Althoff the position of an Ordinarius (full professor) at the same University vacated by Hilbert's teacher Lindemann. Hilbert, when asked by Althoff to name a successor for his former position, succeeded, in spite of some difficulties, in bringing Minkowski to Königsberg for Easter 1894. When Minkowski had accepted Althoff's offer he wrote to Hilbert on January 3, 1894 ([1], 59): Hearty thanks for all your efforts which lead to this happy end; let us hope for an agreeable and fruitful time together so that the prime-numbers and reciprocity laws may wiggle and waggle!

The two friends stayed jointly as colleagues at Königsberg until Easter 1895 when Hilbert moved to Göttingen.

(63) Growing out of the intimate acquaintance with Minkowski's work.

(64) s. Zimmer, Zassenhaus [1], [2].

(65) s. Zassenhaus [3].

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3. Study the optimal packing and covering inequalities in three and more dimensions.[66] Is it true that in case of the existence of a Lie group G for a packing or covering problem there are optimal distributions with an automorph contained as a discrete subgroup with compact factor space in G?

4. Apply higher arithmetics (reciprocity laws) effectively to the problem of prime factorization.[67]

5. Establish algebraic model theory as a theorem proving device.

6. Establish fully the relationship between the integral representations of a Dedekind order (e.g. the integral group ring of a finite group) and its representation theory modulo prime ideal powers.

7. If there are discrete space time universes, which mathematical features are in common to them to make them compatible with the macroscopic behavior of the observable world?

8. Will this country continue to encourage the growth and maturing of the apparently diverging research tendencies of its mathematicians or will it succeed in effectively manipulating them?

9. Do we work on a problem because it hounds us or because someone else holds us bound to it?

10. What is more important: to clarify a concept in our mind or to publish many papers?

11. Can we afford to continue proving the existence of mathematical objects which we cannot construct?

12. Does applied mathematics mean to profit from what we can do well, or to do routinely what once some people could do extremely well or to listen responsively to the questions of others?

Résumé

L'histoire des sciences mathématiques est beaucoup plus que l'étude de documents et la vérification de dates importantes.

Les documents et les dates sont nécessaires pour rappeler a ceux qui étudieront plus tard la profondeur véritable des mathématiques, quelles furent les étapes cruciales du passé.

Il est vrai cependant que chaque nouvelle génération de chercheurs va nécessairement réinterpreter l'essence et l'impact de découvertes passées à la lumière de leur propre perception. Ceci dans le but de regrouper le peu qui soit connu et d'ouvrir la voie à des nouvelles découvertes.

Prenons donc un moment de recul hors de nos sphères actuelles de préoccupation pour nous replonger dans notre passé commun et retrouver encore une fois une vision future de ce qui s'ouvre devant nous.

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[66] s. Graham, Witsenhausen and Zassenhaus.

[67] So far only elementary number theoretic and quadratic reciprocity law methods have been applied to the problem of effective prime factorization.

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