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A NOTE ABOUT THE DEFINITION OF *CW*-COMPLEXES

BY
MARCEL DÉRUAZ AND MARY O'KEEFE

In [1] Whitehead defined a *CW*-complex as a closure finite cell complex with the weak topology (i.e. a topology coherent with the family of closed cells, in Spanier's terminology). The purpose of this note is to show that these two conditions imposed on cell complexes can be replaced by a single one.

PROPOSITION. *A cell complex is a CW-complex iff its topology is coherent with the family of finite subcomplexes.*

The following consequence is immediate.

COROLLARY. *A cell complex is a CW-complex iff it is the direct limit of the family of finite subcomplexes.*

Proof. Let K be a *CW*-complex. Then K is closure finite and a subset $X \subset K$ is closed provided $X \cap L$ is closed in K for every finite subcomplex $L \subset K$, as remarked by Whitehead in [1]. Now if $X \cap L$ is closed in L for every finite subcomplex $L \subset K$, $X \cap L$ is closed in K and X is closed. Therefore the topology of K is coherent with the family of finite subcomplexes.

Conversely, let K be a cell complex such that a subset $X \subset K$ is closed (open) provided $X \cap L$ is closed (open) in L for every finite subcomplex $L \subset K$. According to [1], to show that K is a *CW*-complex, it is sufficient to check that K, and also every n-skeleton K^n, $n \geq 0$, all have a topology coherent with the family of closed cells. Let X be a subset of K and let $X \cap \bar{e}$ be closed in $\bar{e}$ for every cell $e \in K$. If L is any finite subcomplex of K, $X \cap L$ is the finite union of sets $X \cap \bar{e}$, $e \in L$, and $X \cap L$ is closed. By assumption it follows that X is closed and that the topology of K is coherent with the family of closed cells. For every $n \geq 0$, K^n is closed in K, since for every finite subcomplex $L \subset K$, $K^n \cap L$ is a finite subcomplex and thus is closed. Now let X be a subset of K^n and let $X \cap \bar{e}$ be closed in $\bar{e}$ for every cell $e \in K^n$. For every finite subcomplex $L \subset K$, $L \cap K^n$ is a finite subcomplex of K^n and thus a finite union of closed cells of K^n. Therefore $X \cap L = X \cap L \cap K^n$ is a finite union of sets $X \cap \bar{e}$, $e \in K^n$, and is closed in K^n. Hence $X \cap L$ is closed in K and in L; X is then closed by assumption and the topology of K^n is coherent with the family of closed cells.

REFERENCE

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UNIVERSITY OF OTTAWA,
OTTAWA, ONTARIO