J. Austral. Math. Soc. (Series A) 54 (1993), 287–303

NON-WEAK COMPACTNESS OF THE INTEGRATION MAP FOR VECTOR MEASURES

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(Received 3 June 1991)

Communicated by P. G. Dodds

Abstract

Let m be a vector measure with values in a Banach space X. If $L^1(m)$ denotes the space of all m integrable functions then, with respect to the mean convergence topology, $L^1(m)$ is a Banach space. A natural operator associated with m is its integration map I_m which sends each f of $L^1(m)$ to the element $\int f\,dm$ (of X). Many properties of the (continuous) operator I_m are closely related to the nature of the space $L^1(m)$. In general, it is difficult to identify $L^1(m)$. We aim to exhibit non-trivial examples of measures m in (non-reflexive) spaces X for which $L^1(m)$ can be explicitly computed and such that I_m is not weakly compact. The examples include some well known operators from analysis (the Fourier transform on $L^1([-\pi, \pi])$, the Volterra operator on $L^1([0, 1])$, compact self-adjoint operators in a Hilbert space); such operators can be identified with integration maps I_m (or their restrictions) for suitable measures m.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 28B05, 47B05, 45P05.

Introduction

Given a Banach space X and an X-valued vector measure m, there is an associated Banach space $L^1(m)$ of m-integrable functions. From its definition (see Section 1) it is clear that properties of the space $L^1(m)$ are closely tied to those of X. So, even though $L^1(m)$ has certain properties in common with the classical L^1-spaces (such as completeness [5], separability [9] and

The first author acknowledges the support of the Australian Research Council and a University of New South Wales Research Grant.

order completeness [1, 5]) it is also to be expected that other Banach space properties are exhibited which are not typical of the classical situation (see Sections 1 and 2).

Our particular interest is in the integration map $I_m: L^1(m) \to X$ defined by $I_m f = \int f\, dm$, for every $f \in L^1(m)$. Just as for scalar measures it turns out that I_m is always bounded and linear. So, if X is reflexive, then I_m is necessarily weakly compact. If we admit non-reflexive spaces X it can happen that I_m is weakly compact for some measures m and not weakly compact for others. In this article we wish to concentrate on exhibiting non-trivial (and, hopefully, interesting) examples of non-weakly compact integration maps I_m. The case of maps I_m which are weakly compact (or even compact) will be taken up elsewhere.

The authors wish to thank Ben de Pagter for some useful discussions on this topic.

1. The Banach space $L^1(m)$

Let X be a Banach space. By a vector measure in X is meant a σ-additive map $m: \Sigma \to X$, where Σ is a σ-algebra of subsets of some non-empty set Ω. For each $x' \in X'$ (the continuous dual space of X), the complex measure $E \mapsto \langle m(E), x' \rangle$, $E \in \Sigma$, is denoted by $\langle m, x' \rangle$; its variation measure is denoted by $|\langle m, x' \rangle|$. The semivariation $|||m|||$, of m, is defined by

$$|||m|||(E) = \sup\{|\langle m, x' \rangle|(E);\, x' \in X',\ \|x'\| \le 1\}, \quad E \in \Sigma;$$

it satisfies the inequality [3; Proposition 1.11],

$$\text{(1)} \qquad \begin{aligned} \sup\{\|m(E)\|;\ E \subseteq F,\ E \in \Sigma\} &\le |||m|||(F) \\ &\le 4 \sup\{\|m(E)\|;\ E \subseteq F,\ E \in \Sigma\}, \end{aligned}$$

for every $F \in \Sigma$.

A Σ-measurable function $f: \Omega \to \mathbb{C}$ is called m-integrable if it is integrable for each complex measure $\langle m, x' \rangle$, $x' \in X'$, and if, for every $E \in \Sigma$, there exists an element of X, denoted by $\int_E f\, dm$, such that

$$\left\langle \int_E f\, dm, x' \right\rangle = \int_E f\, d\langle m, x' \rangle, \quad x' \in X'.$$

The Orlicz-Pettis lemma guarantees that the indefinite integral of f with respect to m, that is, the set function $fm: E \mapsto \int_E f\, dm$, $E \in \Sigma$, is again a vector measure in X. Identifying two m-integrable functions f and g if $|||m|||(\{\omega \in \Omega;\, f(\omega) \ne g(\omega)\}) = 0$, we obtain a linear space (of equivalence classes) which we denote by $L^1(m)$. With respect to the semivariation norm

$$(2)\quad \|f\|_{L^1(m)} = \|\|fm\|\|(\Omega) = \sup\left\{\int_\Omega |f|d|\langle m, x'\rangle|;\, x' \in X',\ \|x'\| \le 1\right\},$$

the space $L^1(m)$ is a Banach space.

Spaces of the kind $L^1(m)$ have been around for quite a while, although many of their Banach space properties have been investigated more recently. We include a brief summary of those properties which are related to this note.

The completeness of the spaces $L^1(m)$, even for more general locally convex spaces X, is well known; see [5, Chapter III], for example. If $f \in L^1(m)$ and g is a Σ-measurable function satisfying $|g| \le |f|$, then also $g \in L^1(m)$. Furthermore, the Monotone convergence and the Dominated convergence theorems are valid for m. In addition, every bounded, Σ-measurable function is m-integrable and the inclusion of $L^\infty(m)$ into $L^1(m)$ satisfies $\|f\|_{L^1(m)} \le \|\|m\|\|(\Omega) \cdot \|f\|_\infty$, for every $f \in L^\infty(m)$. These facts, all of which can be found in Chapter II of [5], for example, imply that $L^1(m)$ is a (complex) Banach lattice with order continuous norm; see also [1; Theorem 1].

If $\Sigma(m)$ denotes the set $\{\chi_E;\, E \in \Sigma\}$, considered as a part of $L^1(m)$, then the norm of $L^1(m)$ induces a metric d_m in $\Sigma(m)$ in the obvious way, that is,

$$d_m(\chi_E, \chi_F) = \|\chi_E - \chi_F\|_{L^1(m)}, \quad E, F \in \Sigma.$$

Analogous to the case for scalar measures, we say that the vector measure m is separable whenever $(\Sigma(m), d_m)$ is a separable metric space. Then, just as for the classical L^1-spaces, it turns out that m is a separable measure if, and only if, $L^1(m)$ is a separable Banach space, [9].

The spaces $L^1(m)$ are always weakly compactly generated and, if X does not contain an isomorphic copy of c_0, then neither does $L^1(m)$; see [1; §1]. We will see later that there are certain types of measures m and Banach spaces X such that $L^1(m)$ is actually isomorphic to X; this often allows one to show that $L^1(m)$ does not (or does!) have certain properties exhibited by the L^1-spaces of scalar measures m.

Finally, we make some remarks concerning the dual space $L^1(m)'$. The first comment is that it is not so well understood. For measures m with bounded variation the space $L^1(m)'$ was considered in [4] where, unfortunately, the main result is false. For the case of vector measures induced by spectral measures via evaluation, a concrete description of $L^1(m)'$ is given in [8]. One description of the individual elements of $L^1(m)'$ for an *arbitrary* vector measure m is presented in [6]. We give here a further description of a different (but related) kind; the method of proof is quite different to that given in [6].

Let $\nu = |\langle m, x'\rangle|$, where x' is some Rybakov functional for m, [5; p. 121]. Since m and ν have the same null sets we have $L^\infty(m) = L^\infty(\nu)$. Let $\mathscr{M}(\Sigma)$ denote the linear space of all Σ-measurable functions on Ω, in which case $L^1(m)$ is an ideal in $\mathscr{M}(\Sigma)$ in the sense of (complex) Riesz spaces, [11]. The order continuity of the norm in $L^1(m)$ guarantees that the order continuous linear functionals are actually continuous. Also, the natural inclusion of $L^1(m)$ into $L^1(\nu)$ is continuous. So it follows from (the complexified version of) the results in [11; §112] that $L^1(m)'$ can be identified with a certain space of Σ-measurable functions as follows.

PROPOSITION 1.1. *Let X be a Banach space, $m: \Sigma \to X$ be a vector measure and $\nu = |\langle m, x'\rangle|$, where x' is a Rybakov functional for m. Then $\xi \in L^1(m)'$ if, and only if, there exists a function $g_\xi \in \mathscr{M}(\Sigma)$ such that $\int_\Omega |f g_\xi|\, d\nu < \infty$, for every $f \in L^1(m)$, in which case the duality is given by*

$$\langle f, \xi\rangle = \int_\Omega f g_\xi \, d\nu, \quad f \in L^1(m).$$

REMARK 1. (a) If z' is another Rybakov functional, then $|\langle m, z'\rangle|$ and ν are mutually absolutely continuous and so the Radon-Nikodym theorem allows the description of $L^1(m)'$, which is dependent on the particular Rybakov functional chosen, to be formulated in terms of a different Rybakov functional.

(b) Since the linear map $x' \mapsto \xi_{x'}$, $x' \in X'$, is a contraction from X' into $L^1(m)'$, where $\xi_{x'}$ is the function $f \mapsto \int_\Omega f \, d\langle m, x'\rangle$, $f \in L^1(m)$, it follows that X' is always continuously imbedded in $L^1(m)'$. By considering scalar measures m (that is, $X = \mathbb{C}$) it is clear that this imbedding need not be onto.

2. The integration map

Let X be a Banach space, $m: \Sigma \to X$ be a vector measure and $I_m: L^1(m) \to X$ be the associated integration map defined by

$$I_m(f) = \int_\Omega f \, dm, \quad f \in L^1(m).$$

The continuity of the linear operator I_m is clear from (1) applied to the measure $fm: \Sigma \to X$, from (2) and from the observation that $I_m(f) = (fm)(\Omega)$.

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Since bounded subsets of reflexive Banach spaces are necessarily relatively weakly compact it is immediate that I_m is a weakly compact operator whenever the range space X is reflexive.

So, suppose that X is a non-reflexive space. The first observation is that it suffices to consider the closed subspace X_m, of X, generated by the range $m(\Sigma) = \{m(E); E \in \Sigma\}$, of m. For, since the Σ-simple functions are dense in $L^1(m)$ and I_m is continuous, it is clear that I_m assumes all of its values in X_m. Since $m(\Sigma)$ is always relatively weakly compact, [3; p. 14], it follows that X_m is a weakly compactly generated space. So, we may restrict our attention to non-reflexive, weakly compactly generated Banach spaces X. We remark that all separable spaces are weakly compactly generated.

Of course, we will be looking for "genuine examples" of integration maps $I_m: L^1(m) \to X$ in non-reflexive spaces X, meaning that X_m should not be reflexive. We now exhibit some non-trivial examples of such "genuine" maps. The first example comes from classical Fourier analysis, the second from the theory of integral operators and the final two from the theory of spectral operators.

EXAMPLE 1. (THE FOURIER TRANSFORM). Let λ denote Lebesgue measure in $\Omega = [-\pi, \pi]$, let $X = c_0(\mathbb{Z})$ where $\mathbb{Z}$ is the set of all integers, and let $\mathscr{F}: L^1(\lambda) \to X$ denote the Fourier transform. That is,

$$2\pi(\mathscr{F}\varphi)(n) = \int_\Omega \varphi(w)e^{-inw}\,d\lambda(w), \quad n \in \mathbb{Z},$$

for every $\varphi \in L^1(\lambda)$. Define $m: \Sigma \to X$ by $m(E) = \mathscr{F}(\chi_E)$, $E \in \Sigma$, where Σ is the σ-algebra of Borel subsets of Ω. The σ-additivity of m is clear from the continuity of $\mathscr{F}$, or, from the inequalities $2\pi\|m(E)\| \le \lambda(E)$, for all $E \in \Sigma$. It is routine to check that $L^1(m) = L^1(\lambda)$ as linear spaces and that

$$\|f\|_{L^1(m)} \le 4\int_\Omega |f|\,d\lambda \le 4\|f\|_{L^1(m)}, \quad f \in L^1(m).$$

Furthermore, the integration map I_m is just the map $\mathscr{F}$ (consider Σ-simple functions). Since the elements in X which have finite support are the image under I_m of the trigonometric polynomials it follows that $X_m = X$.

It remains to check that $I_m = \mathscr{F}$ is not weakly compact. The following result is a consequence of the Riemann-Lebesgue lemma and the identities

$$2\pi\langle e^{in(\cdot)}, \xi\rangle = (\mathscr{F}\xi)(-n), \quad n \in \mathbb{Z},$$

valid for each $\xi \in L^\infty(\lambda)$.

LEMMA 1.1. *The sequence* $\{e^{in(\cdot)}\}_{n=1}^\infty$ *is weakly convergent to zero in* $L^1(\lambda)$.

A simple calculation establishes the following

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LEMMA 1.2. $\|\mathscr{F}(e^{in(\cdot)})\| = 1$, *for every* $n \in \mathbb{Z}$.

So, suppose that $\mathscr{F}$ is weakly compact. Since $L^1(\lambda)$ has the Dunford-Pettis property $\mathscr{F}$ maps weakly convergent sequences to norm convergent sequences, [2; p. 113]. This contradicts Lemmas 1.1 and 1.2 above. Accordingly, $I_m = \mathscr{F}$ is not weakly compact.

EXAMPLE 2. (THE VOLTERRA INTEGRAL OPERATOR). Let $\Omega = [0, 1]$ and let Σ denote the σ-algebra of Borel subsets of Ω. Let λ denote Lebesgue measure on Σ and let $X = L^1(\lambda)$. Then the Volterra operator $V\colon L^1(\lambda) \to X$ is defined by

$$(Vf)(t) = \int_0^t f(s)\, d\lambda(s), \quad t \in \Omega,$$

for every $f \in L^1(\lambda)$. Since V is bounded (actually its operator norm $\|V\| \leq 1$) it follows that the set function $m\colon \Sigma \to X$ defined by $m(E) = V(\chi_E)$, $E \in \Sigma$, is σ-additive; it is also absolutely continuous with respect to λ (note that $\|m(E)\| \leq \lambda(E)$, $E \in \Sigma$). Even though X does not have the Radon-Nikodym property we have the following result; for the definition of the variation measure $|m|$, of m, we refer to [3, pp. 2–3].

LEMMA 2.1. *The function* $g\colon \Omega \to X$ *given by* $g(s) = \chi_{[s,1]}$, $s \in \Omega$, *is Bochner* λ*-integrable and satisfies*

$$m(E) = \int_E g\, d\lambda, \quad E \in \Sigma.$$

That is, g *is the Radon-Nikodym derivative of* m *with respect to* λ. *In particular, the variation* $|m|$ *satisfies*

$$|m|(E) = \int_E \|g(s)\|\, ds, \quad E \in \Sigma. \tag{3}$$

PROOF. Since $\|g(s) - g(w)\| = |s - w|$, for every $s, w \in \Omega$, it is clear that $g\colon \Omega \to X$ is continuous and hence, its range $g(\Omega)$ is a compact subset of X. In particular, g is strongly measurable and $\int_0^1 \|g(s)\|\, ds = \int_0^1 (1-s)\, ds < \infty$, so that g is Bochner λ-integrable. Let $E \in \Sigma$. An application of the Fubini theorem and the identities $\chi_{[0,t]}(s) = \chi_{[s,1]}(t)$, valid for all $s, t \in \Omega$, shows that

$$\langle m(E), \xi \rangle = \left\langle \int_E g\, d\lambda, \xi \right\rangle, \quad \xi \in X' = L^\infty(\lambda), \tag{4}$$

where $\int_E g\, d\lambda = \int_0^1 \chi_E(s) g(s)\, ds$. Accordingly, $\int_E g\, d\lambda = m(E)$.

The statement and formula concerning the finite variation $|m|$ of m now follows from [3; Chapter II, Theorem 2.4].

LEMMA 2.2. *The Volterra operator* $V: L^1(\lambda) \to X$ *is representable, that is, for every* $f \in L^1(\lambda)$, *the function* $fg: \Omega \to X$ *is Bochner* λ*-integrable and* $Vf = \int_\Omega fg\,d\lambda$. *Moreover,* V *is a compact operator.*

PROOF. The first statement follows from [3; Chapter III, Lemma 1.4]. The second statement follows from the compactness of $g(\Omega)$—see the proof of Lemma 2.1—and [3; Chapter III, Theorem 2.2].

LEMMA 2.3. *Let* $\varphi(s) = 1 - s$, *for every* $s \in \Omega$, *and let* $\mathbf{1}$ *denote the function constantly equal to* 1 *on* Ω, *interpreted as an element of* $X' = L^\infty(\lambda)$. *Then*

$$\langle m, \mathbf{1}\rangle(E) = \int_E \langle g, \mathbf{1}\rangle\,d\lambda = \int_E \varphi\,d\lambda = |m|(E), \quad E \in \Sigma.$$

PROOF. Direct computation.

LEMMA 2.4. *Let* $f: \Omega \to \mathbb{C}$ *be a* Σ*-measurable function. The following statements are equivalent.*

(i) $fg: \Omega \to X$ *is Bochner* λ*-integrable.*
(ii) $fg: \Omega \to X$ *is Pettis* λ*-integrable.*
(iii) f *is* $|m|$*-integrable.*
(iv) f *is* m*-integrable.*

PROOF. Clearly (i) implies (ii).

So, assume that (ii) holds, in which case $\langle fg, \mathbf{1}\rangle$ is λ-integrable. Lemma 2.3 implies that

$$\int_\Omega |\langle fg, \mathbf{1}\rangle|\,d\lambda = \int_\Omega |f| \cdot |\langle g, \mathbf{1}\rangle|\,d\lambda = \int_\Omega |f|\varphi\,d\lambda$$

from which (iii) follows via (3) and the identity

$$\|g(\cdot)\| = \varphi = \langle g, \mathbf{1}\rangle. \tag{5}$$

Clearly (iii) implies (iv); use Lemma 2.3 and (4).

Finally, assume that (iv) holds. Since g is continuous on the compact space Ω and f is Σ-measurable, it is clear that fg is weakly measurable and hence, by the separability of X, it is strongly measurable. As (iv) implies that $\langle m, \mathbf{1}\rangle$-integrability of f it follows that

$$\int_\Omega \|(fg)(s)\|\,ds = \int_\Omega |f| \cdot \|g(\cdot)\|\,d\lambda = \int_\Omega |f|\,d|m| = \int_\Omega |f|\,d|\langle m, \mathbf{1}\rangle|$$

is finite. This shows that fg is Bochner λ-integrable.

Lemma 2.5. *With the notation of Lemma 2.3 we have*

$$L^1(m) = L^1(\varphi\lambda) = L^1(|m|) \tag{6}$$

as equalities of vector spaces. In addition, $L^1(m)$ and $L^1(|m|) = L^1(\varphi\lambda)$ are isometric as Banach spaces. In particular, the inclusion $L^1(\lambda) \subseteq L^1(m)$ is proper. Moreover,

$$\int_\Omega f\,dm = \int_\Omega fg\,d\lambda, \quad f \in L^1(m). \tag{7}$$

Proof. The equalities of the linear spaces in (6) follow directly from Lemma 2.4. Let $J: L^1(m) \to L^1(|m|)$ be the identity map. It follows from (2) and Lemma 2.3 that

$$\|Jf\| = \|f\|_{L^1(|m|)} = \int_\Omega |f|\,d|\langle m, \mathbf{1}\rangle| \le |||fm|||(\Omega) = \|f\|_{L^1(m)},$$

for every $f \in L^1(m)$.

Suppose that $\xi \in X' = L^\infty(\lambda)$ and $\|\xi\|_\infty \le 1$. Then, for every $E \in \Sigma$, the inequality

$$|\langle g(s), \xi\rangle| = \left|\int_0^1 \chi_{[s,1]}(t)\xi(t)\,dt\right| \le \int_0^1 \chi_{[s,1]}(t)\,dt = \varphi(s),$$

for each $s \in \Omega$, implies that

$$|\langle m, \xi\rangle|(E) = \int_E |\langle g(s), \xi\rangle|\,ds \le \int_E \varphi\,d\lambda.$$

Accordingly, if $f \in L^1(m)$, then

$$\int_\Omega |f|\,d|\langle m, \xi\rangle| \le \int_\Omega |f|\varphi\,d\lambda = \|f\|_{L^1(|m|)} = \|Jf\|.$$

It follows from (2) that $|||fm|||(\Omega) \le \|Jf\|$. This establishes that J is an isometry.

The natural injection of $L^1(\lambda)$ into $L^1(m)$, which is a contraction since

$$\|f\|_{L^1(m)} = \|f\|_{L^1(|m|)} = \int_\Omega \varphi|f|\,d\lambda \le \int_\Omega |f|\,d\lambda = \|f\|_{L^1(\lambda)},$$

is clearly not surjective as $1/\varphi \in L^1(m)$ but $1/\varphi \notin L^1(\lambda)$.

The identity (7), which actually holds for all $E \in \Sigma$ and not just Ω, follows from Lemma 2.3 and the identities

$$\int_\Omega f\,d\langle m, \xi\rangle = \int_\Omega f\langle g, \xi\rangle\,d\lambda = \left\langle \int_\Omega fg\,d\lambda, \xi\right\rangle, \tag{8}$$

valid for any $f \in L^1(m)$ and $\xi \in X' = L^\infty(\lambda)$.

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LEMMA 2.6. *The integration map* $I_m: L^1(m) \to X$ *has the property that* $I_m f \geq 0$ *whenever* $f \in L^1(m)$ *satisfies* $f \geq 0$.

PROOF. We note that $h \in L^1(\lambda)$ is non-negative if, and only if, $\langle h, \xi \rangle = \int_\Omega \xi h \, d\lambda \geq 0$, for all $\xi \in L^\infty(\lambda)$ such that $\xi \geq 0$. So, choose such a $\xi \geq 0$. If $f \geq 0$ is an element of $L^1(m)$, then clearly $\langle I_m f, \xi \rangle = \int_\Omega f \langle g, \xi \rangle \, d\lambda \geq 0$ and so $I_m f \geq 0$.

We now have the main result.

PROPOSITION 2.7. *The integration map* $I_m: L^1(m) \to X$ *is not weakly compact.*

PROOF. For every $k \in \mathbb{N}$, let $f_k = (k/\varphi)\chi_{E(k)}$, where φ is defined in the statement of Lemma 2.3 and $E(k) = [1 - k^{-1}, 1)$. By Lemma 2.5 it follows that $\|f_k\|_{L^1(m)} = \|f_k\|_{L^1(\varphi\lambda)} = 1$, for every $k \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By Lemma 2.6

$$\int_{E(n)} |I_m f_k| \, d\lambda = \int_{E(n)} I_m f_k \, d\lambda = \langle I_m f_k, \chi_{E(n)} \rangle, \quad k \in \mathbb{N}. \tag{9}$$

Substituting $I_m f_k = \int_\Omega f_k g \, d\lambda$ (see (7)) and the definition of g into (9) gives

$$\int_{E(n)} |I_m f_k| \, d\lambda = \int_\Omega f_k(s) \lambda(E(n) \cap [s, 1]) \, ds \geq \int_{E(n)} \varphi(s) f_k(s) \, ds,$$

for every $k \in \mathbb{N}$. Accordingly,

$$\sup \left\{ \int_{E(n)} |I_m f_k| \, d\lambda ; k \in \mathbb{N} \right\} \geq \int_{E(n)} \varphi(s) f_n(s) \, ds = 1,$$

for every $n \in \mathbb{N}$, which shows that $\{I_m f_k\}_{k=1}^\infty$ is not a uniformly integrable subset of $X = L^1(\lambda)$, in the sense of Definition 10 in [3; p. 74]. Accordingly, the I_m-image of the unit ball of $L^1(m)$ is not a relatively weakly compact subset of $X = L^1(\lambda)$; see [3; Chapter III, Theorem 2.15].

REMARK 2. (a) Even though the measure m has very strong properties (it is of bounded variation and has relatively compact range) and the Banach space $L^1(m)$ is rather "nice" (it is a classical L^1-space by Lemma 2.5, which is not the case for all spaces of the type $L^1(m)$—see Example 3 below), the associated integration map $I_m: L^1(m) \to X$ is not even weakly compact.

(b) It may be worth recording that I_m is the (unique) extension of the Volterra operator $V: L^1(\lambda) \to X$ from the dense subspace $L^1(\lambda)$ of $L^1(m)$

to all of $L^1(m)$. This follows from the identities

$$\langle I_m f, \xi\rangle = \int_0^1 \xi(t)\left[\int_0^t f(s)\,ds\right] dt,$$

valid for every $f \in L^1(m)$ and $\xi \in L^1(m)' = L^\infty(m)$; see Lemma 2.5. That is, given $f \in L^1(m)$,

$$(I_m f)(t) = \int_0^t f(s)\,ds, \quad \text{for } \lambda\text{-a.e. } t \in \Omega.$$

These identities can be established from (8) and the Fubini theorem, which is applicable because

$$\int_0^1 \left[\int_0^1 |f(s)|\cdot|\xi(t)|\chi_{[s,1]}(t)\,ds\right] dt \le \|\xi\|_\infty \int_0^1 \left[\int_0^1 |f(s)|\varphi(s)\,ds\right] dt < \infty.$$

(c) Since the range of I_m contains all C^∞-functions with compact support in $(0, 1)$, a dense subspace of $X = L^1(\lambda)$, it follows that $X_m = X$.

(d) The lemma showing that $L^1(m) = L^1(|m|)$ played an important role. This is a special feature of this example and is not generally true for vector measures of bounded variation. Indeed, let X be any infinite dimensional Banach space. Let $\{x_n\}_{n=1}^\infty$ be an unconditionally summable sequence which is not absolutely summable, [2; p. 59]. If Σ denotes the σ-algebra of all subsets of $\Omega = \mathbb{N}$, then the set function $m: \Sigma \to X$ defined by

$$m(E) = \sum_{n\in E} (2^n\|x_n\|)^{-1} x_n, \quad E \in \Sigma,$$

is a vector measure of bounded variation. Then

$$|m|(E) = \sum_{n\in E} 2^{-n}, \quad E \in \Sigma.$$

The space $L^1(m)$ consists of those functions $f: \Omega \to \mathbb{C}$ such that $\{(2^n\|x_n\|)^{-1} f(n)x_n\}_{n=1}^\infty$ is unconditionally summable in X whereas $h \in L^1(|m|)$ if and only if $\{(2^n\|x_n\|)^{-1} h(n)x_n\}_{n=1}^\infty$ is absolutely summable. Hence, $f(n) = 2^n\|x_n\|$, $n \in \Omega$, is an example of an element in $L^1(m)\backslash L^1(|m|)$.

(e) Let $Y = C(\Omega)$, equipped with its supremum norm $\|\cdot\|_\infty$. Then the Volterra operator $\underset{\sim}{V}: L^1(\lambda) \to Y$ can formally be defined as for the case when X was the range space. That $\underset{\sim}{V}$ actually assumes its values in Y is clear from the inequality,

$$|(\underset{\sim}{V}f)(t) - (\underset{\sim}{V}f)(s)| \le \int_{[s,t]} |f(u)|\,du,$$

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valid for every $s, t \in \Omega$ with $s \leq t$, and the Lemma that $|f|\lambda$ is a σ-additive measure whenever $f \in L^1(\lambda)$. Then again $\underset{\sim}{m}: \Sigma \to Y$, defined by $\underset{\sim}{m}(E) = \underset{\sim}{V}(\chi_E)$, $E \in \Sigma$, is a Y-valued vector measure since

$$\|\underset{\sim}{m}(E)\|_\infty = \|\underset{\sim}{V}(\chi_E)\|_\infty \leq \|\chi_E\|_{L^1(\lambda)} = \lambda(E).$$

It turns out that $L^1(\lambda)$ is a linear subspace of $L^1(\underset{\sim}{m})$ and that the integration map $I_{\underset{\sim}{m}} : L^1(\underset{\sim}{m}) \to Y$ is the (unique) extension of $\underset{\sim}{V}$ from $L^1(\lambda)$ to all of $L^1(\underset{\sim}{m})$. It may be of interest to determine whether or not $I_{\underset{\sim}{m}}$ is weakly compact and to identify the Banach space $L^1(\underset{\sim}{m})$ more precisely. Of course, the Radon-Nikodym derivative g which is available when the range space is X is not available in Y. Rather than pursue this example we wish to concentrate on an example of a different nature.

EXAMPLE 3. (EVALUATIONS OF SPECTRAL MEASURES.) The space of all continuous linear operators of a Banach space X into itself is denoted by $L(X)$. If we wish to indicate that the uniform (resp. strong) operator topology is to be considered we will write $L_u(X)$, (resp. $L_s(X)$). Of course, $L_s(X)$ is no longer normable (if X is infinite dimensional); it is a quasicomplete, locally convex Hausdorff space.

A spectral measure in X is a σ-additive map $P: \Sigma \to L_s(X)$, where Σ is a σ-algebra of subsets of some non-empty set Ω, such that $P(\Omega) = I$ (the identity operator in X) and $P(E \cap F) = P(E)P(F)$, for every $E, F \in \Sigma$. The range $P(\Sigma)$ is always a uniformly bounded set in $L_u(X)$. The spectral measure P is called closed if $P(\Sigma)$ is a closed subset of $L_s(X)$. If X is separable, then every spectral measure in X is a closed measure. For each $x \in X$, we denote by $Px: \Sigma \to X$ the X-valued measure $E \mapsto P(E)x$, $E \in \Sigma$. Given $x \in X$, the closed subspace

$$P(\Sigma)[x] = \overline{\operatorname{span}}\{P(E)x;\ E \in \Sigma\},$$

of X (the bar denotes closure), is called the cyclic space generated by x with respect to P.

A Banach space X is called cyclic if there exist a closed spectral measure $P: \Sigma \to L_s(X)$ and $x \in X$ such that $X = P(\Sigma)[x]$. This agrees with the standard definition, as given in [10], for example.

LEMMA 3.1. (see [8]). *Let X be a cyclic Banach space. Let $P: \Sigma \to L_s(X)$ be any closed spectral measure and $x \in X$ any vector such that $P(\Sigma)[x] = X$. Then the integration map $I_{Px}: L^1(Px) \to X$ given by*

$$I_{Px}f = \int_\Omega f\,dPx, \quad f \in L^1(Px),$$

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is a Banach space isomorphism of $L^1(Px)$ *onto* X.

This result illustrates immediately how different Banach spaces of the type $L^1(m)$, for m a vector measure, can be from classical L^1-spaces. For instance, the spaces $L^p([0,1])$, $1 \leq p < \infty$, are cyclic Banach spaces. Accordingly, there exist vector measures m (of the form Px for suitable P and x) such that $L^1(m)$ is reflexive! Consequently, unlike the classical L^1-spaces $L^1(m)$ need not have the Dunford-Pettis property in general. Since c_0 is a cyclic Banach space it follows that spaces $L^1(m)$ need not be weakly sequentially complete.

Getting back to integration maps, we have the following result.

PROPOSITION 3.2. *Let* X *be a non-reflexive, cyclic Banach space. Then there always exist vector measures* $m: \Sigma \to X$ *with* $X_m = X$ *such that the integration map* $I_m: L^1(m) \to X$ *is not weakly compact.*

PROOF. Let $P: \Sigma \to L_s(X)$ be a closed spectral measure and $x \in X$ a vector such that $P(\Sigma)[x] = X$. Then $m = Px$ satisfies $X_m = X$. By Lemma 3.1 the integration map $I_m: L^1(m) \to X$ is a Banach space isomorphism. Since the set $\{I_m f;\, f \in L^1(m),\ \|f\|_{L^1(m)} \leq 1\}$ contains a multiple of the unit ball of X it cannot be relatively weakly compact (as X is non-reflexive).

This result makes it easy to produce specific examples of non-weakly compact integration maps of this type. Indeed, let $X = \ell^1$, $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbf{N}}$. Define a closed spectral measure $P: \Sigma \to L_s(X)$ by $P(E)x = y$, for every $x = \{x_n\}_{n=1}^{\infty} \in X$, where $y_n = \chi_E(n)x_n$, $n \in \mathbb{N}$, for every $E \in \Sigma$. Then $x = \{n^{-2}\}_{n=1}^{\infty}$ is a cyclic vector for X with respect to P. Accordingly, if $\{e_n\}_{n=1}^{\infty}$ are the standard unit basis vectors in ℓ^1, then the vector measure $m: \Sigma \to \ell^1$ given by

$$m(E) = P(E)x = \sum_{n \in E} n^{-2} e_n, \quad E \in \Sigma,$$

has the property that its integration map $I_m: L^1(m) \to X$ is not weakly compact. Similarly, if $X = c_0$ and $P: \Sigma \to L_s(X)$ and x are defined by using the "same formulae" as those used for ℓ^1, then P is also a closed spectral measure in c_0 with x a cyclic vector for P. Accordingly, the "same" m, now interpreted as c_0-valued, has a non-weakly compact integration map.

REMARK 3 (a). It may be worth noting that the vector measure m of Example 2 is *not* of the form Pf, for any spectral measure P in $X = L^1(\lambda)$

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and $f \in X$. To establish this claim we need the following result; the notation is as in Example 2.

LEMMA 3.3. *The integration map* $I_m: L^1(m) \to X = L^1(\lambda)$ *of Example* 2 *is injective, but not an isomorphism onto its range.*

PROOF. Let $f \in L^1(m)$ and suppose that $I_m f = 0$. By Remark 2(b) we have $\int_0^t f(s)\,ds = 0$, for λ-a.e. $t \in [0, 1]$ and hence, $f = 0$ in $L^1(\lambda)$. So, I_m is injective.

The only constant function in the range of I_m is zero. However, the closure of the range of I_m includes all constant functions. Indeed, let $c > 0$. Given $\varepsilon \in (0, c)$ define $f_\varepsilon(s) = 2\varepsilon^{-2}c^3 s\chi_{[0,\varepsilon/c]}(s)$, for every $s \in [0, 1]$. Since $\int_0^t f(s)\,ds = c^3\varepsilon^{-2}t^2$ if $0 \le t \le \varepsilon/c$ and equals c otherwise, it follows from Remark 2(b) that

$$\|c\mathbf{1} - I_m f_\varepsilon\|_{L^1(\lambda)} = \int_0^{\varepsilon/c} |c - c^3\varepsilon^{-2}t^2|\,dt \le \int_0^{\varepsilon/c} c\,dt = \varepsilon.$$

It follows that the closure of the range of I_m contains all (complex) constant functions, and so I_m is not an isomorphism onto its range.

That m is *not* an evaluation of some spectral measure at a point of X now follows from Lemma 3.3 and [6; Theorem 10], which states that the integration map $I_\nu: \Sigma \to Z$ (of a Banach space valued vector measure $\nu: \Sigma \to Z$) is an isomorphism onto a closed subspace of Z if and only if there exists a closed subspace Y of Z, a closed spectral measure $P: \Sigma \to L_s(Y)$ and a vector $y \in Y$ such that $\nu = J \circ Py$ (where J is the natural inclusion of Y into Z). Of course, one also needs to use the fact that the integration map $I_{Py}: L^1(Py) \to Y$ is an isomorphism of $L^1(Py)$ onto the closed linear span of $\{P(E)y;\ E \in \Sigma\}$, [8].

REMARK 3 (b). It is a well known fact from harmonic analysis that the (Fourier transform) map I_m of Example 1 is injective but not isomorphic onto its range. A similar argument as above shows that m (of Example 1) is not the evaluation of any spectral measure in X.

EXAMPLE 4. (SCALAR-TYPE SPECTRAL OPERATORS.) Let Z be a Banach space (infinite dimensional) and $T \in L(Z)$ be a scalar-type spectral operator. That is, if $\mathscr{B}$ denotes the σ-algebra of Borel subsets of $\mathbb{C}$, then there exists a (unique) spectral measure $P: \mathscr{B} \to L_s(Z)$, with support equal to the spectrum $\sigma(T)$ of T, such that $T = \int_{\sigma(T)} w\,dP(w)$. This integral is defined in the usual way for a locally convex space valued (in this case $L_s(Z)$) measure, [5]. The measure P is called the resolution of the identity of T. In addition, let us suppose that $\sigma(T)$ is a countable set with no limit points

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except, possibly, zero. Let $\Sigma = \mathscr{B} \cap \sigma(T)$. Define an operator-valued set function $m: \Sigma \to L(Z)$ by

(10) $$m(E) = \int_E w\, dP(w) = TP(E) = P(E)T, \quad E \in \Sigma.$$

Then m is σ-additive in $L_u(Z)$; see [7]. It is also shown in [7] that the range of m is a separable subset of $L_u(Z)$. Accordingly, if X denotes the closed subspace of $L_u(Z)$ generated by $m(\Sigma)$, then X is a separable Banach space.

Suppose $\sigma(T) = \{0\} \cup \{\lambda_n\}_{n=1}^{\infty}$, where $\{\lambda_n\}_{n=1}^{\infty}$ is the set of non-zero eigenvalues of T, and that $\lambda_n \to 0$ as $n \to \infty$. Let δ_n denote the Dirac point mass at λ_n (put $\lambda_0 = 0$) and let $P_n = P(\{\lambda_n\})$, $n \in \mathbb{N}$. If 0 belongs to the continuous spectrum of T let $P_0 = 0$. Otherwise 0 is an eigenvalue of T in which case P_0 will denote the associated eigenprojection. We note that $m(\{0\}) = 0$.

The space $L^1(m)$ consists of all functions $\varphi: \sigma(T) \to \mathbb{C}$ such that the series $\sum_{n=0}^{\infty} \varphi(\lambda_n) P_n T$ is unconditionally summable in X (or $L_u(Z)$), in which case

$$\int_E \varphi\, dm = \sum_{n=0}^{\infty} \delta_n(E) \varphi(\lambda_n) P_n T, \quad E \in \Sigma.$$

But, the identities $\delta_n(E) P_n = P(E) P_n$, $n \in \mathbb{N} \cup \{0\}$, and the continuity of $P(E)$ imply, for every $E \in \Sigma$, that

$$\sum_{n=0}^{\infty} \delta_n(E) \varphi(\lambda_n) P_n T = P(E) \sum_{n=0}^{\infty} \varphi(\lambda_n) P_n T$$
$$= P(E) \int_{\sigma(T)} \varphi\, dm = P(E) I_m(\varphi).$$

Accordingly, we have established that

(11) $$\int_E \varphi\, dm = P(E) I_m(\varphi) = I_m(\varphi) P(E), \quad E \in \Sigma.$$

LEMMA 4.1. *A function $\varphi: \sigma(T) \to \mathbb{C}$ belongs to $L^1(m)$, if and only if,*

(12) $$\lim_{n \to \infty} \lambda_n \varphi(\lambda_n) = 0.$$

PROOF. Suppose that $\varphi \in L^1(m)$, in which case $\sum_{n=0}^{\infty} \varphi(\lambda_n) P_n T$ is unconditionally summable in $L_u(Z)$. In particular, $\|\varphi(\lambda_n) P_n T\| \to 0$ as $n \to \infty$. Since $P_n P_j = 0$, if $n \neq j$, and $T = \sum_{j=1}^{\infty} \lambda_j P_j$ it is a routine calculation to check that $P_n T = \lambda_n P_n$, $n \in \mathbb{N} \cup \{0\}$. Accordingly, $\|\lambda_n \varphi(\lambda_n) P_n\| = \|\varphi(\lambda_n) P_n T\| \to 0$ as $n \to \infty$. Since $\|P_n\| \geq 1$, for every $n \in \mathbb{N}$, property (12) follows.

Conversely, suppose that (12) holds for a function $\varphi\colon \sigma(T) \to \mathbb{C}$. Then $\psi\colon \sigma(T) \to \mathbb{C}$ defined by

$$\psi(\lambda_n) = \lambda_n \varphi(\lambda_n), \quad n = 0, 1, 2, \ldots, \tag{13}$$

is P-essentially bounded and so is P-integrable in $L_s(Z)$. In particular, the series $\sum_{n=1}^{\infty} \lambda_n \varphi(\lambda_n) P_n$ is unconditionally summable in $L_s(Z)$. To show that $\varphi \in L^1(m)$ it suffices to show that the series $\sum_{n=0}^{\infty} \varphi(\lambda_n) m(\{\lambda_n\})$ is unconditionally summable in X. So, let $\{n(k)\}_{k=1}^{\infty}$ be an increasing sequence of non-negative integers. If $N > M$, then

$$\begin{aligned}\left\| \sum_{k=M}^{N} \varphi(\lambda_{n(k)}) m(\{\lambda_{n(k)}\}) \right\| &= \left\| \sum_{k=M}^{N} \lambda_{n(k)} \varphi(\lambda_{n(k)}) P_{n(k)} \right\| \\ &\le 4\alpha \max_{M \le k \le N} |\lambda_{n(k)} \varphi(\lambda_{n(k)})|,\end{aligned}$$

where $\alpha = \max\{\|P(E)\|;\ E \subseteq \{\lambda_{n(k)}\}_{k=M}^{N}\}$. We have used the fact from the theory of spectral operators that $\|\int_\Omega g\,dQ\| \le 4\beta\|g\|_Q$, with $\beta = \sup\{\|Q(F)\|; F \in \Lambda\}$, whenever $Q\colon \Lambda \to L_s(Z)$ is a spectral measure on a σ-algebra Λ, g is a Q-essentially bounded function and $\|\cdot\|_Q$ is the Q-essentially supremum norm. Since $\alpha \le K = \sup\{\|P(E)\|; E \in \Sigma\}$ we conclude that

$$\left\| \sum_{k=M}^{N} \varphi(\lambda_{n(k)}) m(\{\lambda_{n(k)}\}) \right\| \le 4K \max_{M \le k \le N} |\lambda_{n(k)} \varphi(\lambda_{n(k)})|, \tag{14}$$

with K independent of the sequence $\{n(k)\}_{k=1}^{\infty}$. Since $\lambda_n \varphi(\lambda_n) \to 0$ as $n \to \infty$ and each term $\varphi(\lambda_{n(k)}) m(\{\lambda_{n(k)}\})$, $k = 1, 2, \ldots$, belongs to X, it follows from (14) that the series $\sum_{k=1}^{\infty} \varphi(\lambda_{n(k)}) m(\{\lambda_{n(k)}\})$ is summable in X.

Let us equip $L^1(m)$ with the equivalent norm (see (1) and (2))

$$|\varphi| = \sup\left\{ \left\| \int_E \varphi\, dm \right\|;\ E \in \Sigma \right\}, \quad \varphi \in L^1(m).$$

LEMMA 4.2. *The integration map $I_m\colon L^1(m) \to X$ is a Banach space isomorphism of $L^1(m)$ onto X.*

PROOF. Let K be as in the proof of Lemma 4.1. It follows from (11) that

$$|\varphi| \le K\|I_m \varphi\|, \quad \varphi \in L^1(m).$$

Since $E = \sigma(T) \in \Sigma$ and $P(\sigma(T)) = I$ it is clear from the definition of $|\cdot|$ that $\|I_m \varphi\| \le |\varphi|$, for every $\varphi \in L^1(m)$. Accordingly, we have

$$|\varphi| \le K\|I_m \varphi\| \le K|\varphi|, \quad \varphi \in L^1(m), \tag{15}$$

which shows that I_m is a bicontinuous isomorphism of $L^1(m)$ onto its range in X.

To see that this range is all of X, let $S \in X$. Then $S = \lim_{k\to\infty} S_k$ (in $L_u(Z)$) where each S_k is in the linear span of $m(\Sigma)$. Accordingly, there exist Σ-simple functions f_k, $k \in \mathbb{N}$, such that $S_k = \int_{\sigma(T)} f_k\, dm = I_m f_k$. It follows from (15) and the completeness of $L^1(m)$ that there is $f \in L^1(m)$ such that $f_k \to f$ in $L^1(m)$. Then $I_m f = S$.

Proposition 4.3. *Whenever $\sigma(T)$ is infinite, the integration map $I_m\colon L^1(m) \to X$ is not compact.*

Proof. If I_m were compact it would follow from Lemma 4.2 that X is finite dimensional.

Proposition 4.4. *Suppose that $\sigma(T)$ is a countably infinite set with zero as only limit point. Then both $L^1(m)$ and X are isomorphic to c_0. In particular, the integration map $I_m\colon L^1(m) \to X$ is not weakly compact.*

Proof. Let F be the vector space

$$\left\{\varphi\colon \sigma(T) \to \mathbb{C};\ \lim_{n\to\infty} \lambda_n \varphi(\lambda_n) = 0\right\},$$

equipped with the seminorm $q(\varphi) = \sup\{|\lambda\varphi(\lambda)|;\ \lambda \in \sigma(T)\}$. Since $m(\{0\}) = 0$ it follows that $L^1(m)$ coincides with the quotient space $Y = F/q^{-1}(0)$; see Lemma 4.1. The quotient space is then a Banach space with norm

$$|||\varphi||| = \sup\{|\lambda\varphi(\lambda)|;\quad \lambda \in \sigma(T)\backslash\{0\}\},\quad \varphi \in Y;$$

the equivalence class of $\varphi \in F$, which is again denoted by φ, can be interpreted as a function on $\sigma(T)\backslash\{0\}$.

Given $\varphi \in Y$, let ψ be defined by (13). Then, with K as in the proof of Lemma 4.2, we have that

$$|\varphi| \le K\|I_m\varphi\| = K\left\|\int_{\sigma(T)} \psi\, dP\right\|.$$

But, from the theory of spectral operators it is known that $\|\int_{\sigma(T)} \psi\, dP\| \le 4K\|\psi\|_P$, where $\|\cdot\|_P$ is the P-essential supremum norm. Since $\{0\}$ is the only possible P-null set and $\psi(0) = 0$ we conclude that

$$\|\psi\|_P = \sup\{|\psi(\lambda_n)|;\ n \ge 1\} = |||\varphi|||.$$

Accordingly,

$$|\varphi| \le 4K^2|||\varphi|||,\quad \varphi \in Y,$$

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which shows that the identity map from Y onto $L^1(m)$ is continuous. Since it is also injective, the open mapping theorem implies that $L^1(m)$ and Y are isomorphic Banach spaces. Since Y is isomorphic to c_0 the proof is complete.

REMARK 4. If we choose T to be *compact*, in addition to being a scalar-type spectral operator, then it is clear from (10) and the definition of X that X consists entirely of compact operators.

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