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# A SHORT PROOF OF EULER'S RELATION FOR CONVEX POLYTOPES

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ABSTRACT. The purpose of this paper is to present a short, self-contained proof of Euler's relation. The ingredients of this proof are (i) the principle of inclusion and exclusion of combinatorics and (ii) the Euler characteristic; a development of the Euler characteristic is included.

1. **Introduction.** This paper provides a short proof of Euler's relation for convex polytopes, which states that if $P$ is a nonempty convex polytope of dimension $d$ having $f_0$ vertices, $f_1$ edges, ..., and $f_{d-1}$ facets, then

$$f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d.$$

A brief history of Euler's relation prior to 1967 is provided by Grünbaum's book [5]. Euler knew the three-dimensional case. Schläfli studied the higher-dimensional cases in 1852, but his work wasn't published until 1902. Several other authors treated the higher-dimensional cases in 1880's. However, all proofs apparently assumed shellability of the boundary complex, so, as Grünbaum says, it seems that the "first real proof" was that of Poincaré in 1899, using homology. The first elementary proof was due to Hadwiger [6] (1955). Klee [7] gave another in 1963. Grünbaum's book contains an entirely geometrical proof.

Since the publication of Grünbaum's book there have been several developments relating to Euler's relation. Bruggesser and Mani [1] proved that the boundary complexes of all convex polytopes admit shellings, thereby validating the nineteenth century proofs. Tverberg [12] gave a simple geometrical proof of the relation based upon his triangulation method. Nef [10] gives a short and simple proof of Euler's relation.

For oriented matroid polytopes, Euler's relation follows from the topological representation theorem of Folkman and Lawrence [3] and Poincaré's topological argument. A direct proof can be obtained by using the result of Mandel [9] that oriented matroid polytopes are shellable. Cordovil, Las Vergnas, and Mandel [2] gave a different direct proof.

The proof, presented in the third section, is quite simple and although we present it for convex polytopes it applies almost verbatim for oriented matroid polytopes.

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The proof makes use of the "Euler characteristic," and resembles the proof of Nef [10]. We describe this function and verify its useful properties in the second section. There are other simple derivations of this function; for some of these, as well as for more on this function, see Hadwiger [6], Klee [7], Rota [11], and the survey article, McMullen and Schneider [8]. Groemer [4] describes a general abstract setting in which there can be defined an Euler characteristic, as does Zaslavsky [13].

2. **The Euler Characteristic.** By an *arrangement* of hyperplanes in $R^d$ we mean (following Grünbaum) a finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ of hyperplanes in $R^d$. Such an arrangement determines a decomposition of $R^d$ into "cells." For $x, y \in R^d$, write $x \sim y$ if $x$ and $y$ are in the same hyperplanes of $\mathcal{A}$ and on the same side of the hyperplanes that they are not in. This is an equivalence relation on $R^d$ and the equivalence classes are relatively open convex polyhedra—the *cells* of $\mathcal{A}$. An $\mathcal{A}$*-polyhedron* is a union of cells of $\mathcal{A}$. (Note that an $\mathcal{A}$-polyhedron need not be convex.)

Given an arrangement $\mathcal{A}$ and an $\mathcal{A}$-polyhedron $S = C_1 \cup C_2 \cup \cdots \cup C_m$, where the $C_i$'s are distinct cells, we define

$$\chi(\mathcal{A}, S) = \sum_{i=1}^{m} (-1)^{\dim(C_i)}.$$

The function satisfies the principle of inclusion and exclusion: If $S_1, \ldots, S_k$ are $\mathcal{A}$-polyhedra then

$$\chi(\mathcal{A}, S_1 \cup \cdots \cup S_k) = \\ \chi(\mathcal{A}, S_1) + \cdots + \chi(\mathcal{A}, S_k) - \chi(\mathcal{A}, S_1 \cap S_2) - \cdots + (-1)^{k-1} \chi(\mathcal{A}, S_1 \cap \cdots \cap S_k).$$

To say this differently, the function $\chi$ is, for fixed $\mathcal{A}$, a valuation on the Boolean lattice of $\mathcal{A}$-polyhedra.

LEMMA. *The value $\chi(\mathcal{A}, S)$ does not depend on $\mathcal{A}$.*

PROOF. That is to say, if $\mathcal{A}$ and $\mathcal{B}$ are arrangements and $S \subseteq R^d$ is both an $\mathcal{A}$-polyhedron and a $\mathcal{B}$-polyhedron, then $\chi(\mathcal{A}, S) = \chi(\mathcal{B}, S)$. To establish this we need only show that augmenting an arrangement by the addition of one hyperplane doesn't change the value of $\chi$; indeed, once this is shown, we have $\chi(\mathcal{A}, S) = \chi(\mathcal{A} \cup \mathcal{B}, S) = \chi(\mathcal{B}, S)$.

Let $\mathcal{A}$ be an arrangement, let $S$ be an $\mathcal{A}$-polyhedron, and let $H$ be a hyperplane. Let $S = C_1 \cup \cdots \cup C_m$, where the $C_i$'s are distinct cells of $\mathcal{A}$. Then $\chi(\mathcal{A}, S) = \sum_{i=1}^m \chi(\mathcal{A}, C_i)$ and $\chi(\mathcal{A} \cup \{H\}, S) = \sum_{i=1}^m \chi(\mathcal{A} \cup \{H\}, C_i)$ so we need only show that $\chi(\mathcal{A} \cup \{H\}, C) = \chi(\mathcal{A}, C)$ for each cell $C$ of $\mathcal{A}$; but either such a set $C$ is also a cell of $\mathcal{A} \cup \{H\}$, in which case $\chi(\mathcal{A} \cup \{H\}, C) = (-1)^{\dim(C)} = \chi(\mathcal{A}, C)$, or the hyperplane $H$ intersects but does not contain the relatively open convex set $C$, in which case the sets $C \setminus H^-$, $H \cap C$, and $C \setminus H^+$ are the cells of $\mathcal{A} \cup \{H\}$ which partition $C$, where $H^+$ and $H^-$ are the closed halfspaces determined by $H$, and we have

$$\begin{aligned} \chi(\mathcal{A} \cup \{H\}, C) &= (-1)^{\dim(C \setminus H^-)} + (-1)^{\dim(C \cap H)} + (-1)^{\dim(C \setminus H^+)} \\ &= (-1)^{\dim(C)} = \chi(\mathcal{A}, C). \end{aligned}$$

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We can now use the notation $\chi(S)$ instead of $\chi(\mathcal{A}, S)$, if we wish. This function $\chi$ is the "Euler characteristic." It is a valuation on the lattice of polyhedra in $R^d$.

As a consequence of the lemma we have $\chi(U) = (-1)^d$ for any open, nonempty, convex polyhedron $U$. To see this simply observe that $U$ is a cell in the arrangement of hyperplanes spanned by the facets of its closure. In particular, utilizing the empty arrangement, $\chi(R^d) = (-1)^d$.

3. **Proof of Euler's Relation.** Let $P \subseteq R^d$, $P \neq R^d$, be a closed, convex, polyhedral cone emanating from the origin and suppose that $P$ is full-dimensional, so that its interior is nonempty. Any convex polytope can be obtained as the intersection of such a cone with a hyperplane and it is clear that the number $f_i$ of $i$-dimensional faces of the cone is the same as the number of $(i-1)$-dimensional faces of the polytope. To verify Euler's relation we need only show that

$$f_0 - f_1 + f_2 - \cdots + (-1)^d f_d = 0$$

for such cones. Let $P = H_1^+ \cap \cdots \cap H_n^+$, where, for $1 \leq i \leq n$, $H_i^+$ is one of the closed halfspaces bounded by a hyperplane $H_i$ which contains the origin. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$. Then, utilizing the definition of $\chi(\mathcal{A}, -)$, we may write Euler's relation for $P$ as $\chi(\mathcal{A}, P) = 0$, since the faces of $P$ are precisely the cells of $\mathcal{A}$ which are contained in $P$.

Clearly $\chi(P) = \chi(R^d) - \chi(R^d \setminus P)$. We have seen that $\chi(R^d) = (-1)^d$ so we need only show that $\chi(R^d \setminus P) = (-1)^d$.

For $1 \leq i \leq n$ let $U_i = R^d \setminus H_i^+$, the open halfspace which is the complement of the closed halfspace appearing in the intersection for $P$. Since $R^d \setminus P = U_1 \cup \cdots \cup U_n$ we may use the principle of inclusion and exclusion, and we obtain

$$\chi(R^d \setminus P) = \chi(U_1) + \cdots + \chi(U_n) - \chi(U_1 \cap U_2) - \cdots + (-1)^n \chi(U_1 \cap \cdots \cap U_n).$$

The set $U_1 \cap \cdots \cap U_n$ is the interior of the reflection of $P$ through the origin. It is nonempty. Therefore all of the sets on the right in the equation are nonempty open convex polyhedra, so each of these values of $\chi$ is $(-1)^d$, and the right-hand side equals

$$(-1)^d \left( \binom{n}{1} - \binom{n}{2} + \cdots + (-1)^{n-1} \binom{n}{n} \right) = (-1)^d. \qquad \blacksquare$$

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