# PROGRAMMES IN PAIRED SPACES

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**1. Introduction.** The basic problem of linear programming is to minimize (or maximize) a linear function of a finite number of variables constrained by a finite number of linear inequalities. From a mathematical point of view the subject may be regarded as being divided into two areas. One is primarily analytical and deals with certain questions of duality and consistency. The other is algorithmic and is concerned with computational questions and methods.

The analytical aspects of the problem were first discussed in **(12)**. Later papers which expand on this work appear in **(6)** and **(23)**. The principal computational technique used in linear programming was first discussed in **(9)**. The results of some of the later research in this area are reported in **(13)**.

This paper is concerned with the duality and existence theorems of linear programming under more general conditions. The problem is discussed from the standpoint of paired linear spaces rather than that of locally convex spaces since the theory is slightly easier to develop from this point of view. It also has the advantage of containing certain aspects of previous work in this area **(5**; **10**; **15)** as special cases. In preparing this work the author has benefited greatly from the work in **(5)** and **(10)**. The work in **(15)** appeared after the main part of the paper was completed. It should be mentioned that several theorems in this paper can be obtained from those in **(5)** and **(10)** if one modifies the topologies considered there in an appropriate manner.

The main part of the paper starts with § 2 which contains a few lemmas which will be required later in the paper. In § 3 a programme and its dual are defined and several relations between these programmes are established. Section 4 is devoted to the form a programme can take in locally convex spaces. Section 5 contains several examples intended to illustrate some of the inherent complexities of the subject and to indicate how the theory can be applied in certain kinds of non-linear programming problems, continuous games, and Kantorovitch's generalization of the transportation problem.

**2. Preliminary lemmas.** It is assumed that the reader is somewhat familiar with the theory of linear topological spaces as developed in **(3**; **4)**. The only

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departures from the terminology of these references is that "net" is used instead of "filter", the word "neighbourhood" means "open neighbourhood" and "equilibrée," "tonneau," and "espace tonnelé" are translated into "circled," "disk," and "disk space," respectively. The real numbers are denoted by $R$ and the non-negative real numbers by $R_0$. Bilinear functionals are denoted: $((\ ,\ ))$. If the linear spaces $X$ and $Y$ are in duality, the weak topology on $X$ is denoted as $w(X, Y)$ and the Mackey topology on $X$ is denoted as $s(X, Y)$. If $C$ is a cone in $X$, the set $C^+$ is the set $\{y; y \in Y \text{ and } ((x, y)) \geqslant 0 \text{ for all } x \in C\}$ and the set $C^{++}$ is the set $\{x; x \in X \text{ and } ((x, y)) \geqslant 0 \text{ for all } y \in C^+\}$.

LEMMA 1. *Let $C_1$ and $C_2$ be convex cones in a linear topological space $(X, \mathfrak{T})$ and suppose that $C_2$ has a non-null $\mathfrak{T}$-interior $C_2^0$. If $(C_1 + C_2)^0$ denotes the $\mathfrak{T}$-interior of $C_1 + C_2$ then $(C_1 + C_2)^0 \subset C_1 + C_2^0$.*

*Proof.* Since $C_2^0$ is not empty, $(C_1 + C_2)^0$ is not empty. If $x_0 \in (C_1 + C_2)^0$ and $x \in C_2^0$ then there exists a $\delta > 0$ such that $x_0 - \delta x \in (C_1 + C_2)^0$ and we may assume that $\delta < 1$. Suppose $x_0 - \delta x = x_1 + x_2$ where $x_1 \in C_1$ and $x_2 \in C_2$. Then $x_0 = x_1 + (x_2 + \delta x)$ and consequently $x_2 + \delta x \in C_2^0$ (**3**, p. 51, Proposition 5). Therefore $x_0 \in C_1 + C_2^0$.

LEMMA 2. *Let $X$ and $Y$ be linear spaces paired under $((\ ,\ ))$ and $C$ be a convex cone in $X$. Then $C^+$ is $\mathfrak{T}_1$-closed and $C^{++}$ coincides with the $\mathfrak{T}_2$-closure of $C$. $\mathfrak{T}_1$ $(\mathfrak{T}_2)$ is any topology on $Y(X)$ which is compatible with the duality between $X$ and $Y$.*

*Proof.* By (**4**, p. 67, Proposition 4) and the convexity of $C^+$ and $C^{++}$ it is sufficient to prove these statements for $\mathfrak{T}_1 = w(Y, X)$ and $\mathfrak{T}_2 = w(X, Y)$. It is immediate that $C^+$ is $w(Y, X)$-closed and $C^{++}$ is $w(X, Y)$-closed. Since $C \subset C^{++}$ the $w(X, Y)$-closure of $C$ is contained in $C^{++}$. Suppose $x_0$ is not an element of the $w(X, Y)$-closure of $C$. Since $w(X, Y)$ is locally convex, there exists a $y \in Y$ which satisfies the condition that $((x_0, y)) < 0$ and $((x, y)) \geqslant 0$ for all $x$ in the $w(X, Y)$-closure of $C$ (**3**, p. 73, Proposition 4; **4**, pp. 50 and 69). Consequently, there exists a $y \in C^+$ which satisfies $((x_0, y)) < 0$. Thus $x_0$ is not an element of $C^{++}$.

LEMMA 3. *Let $X$ and $Y$ be linear spaces paired under $((\ ,\ ))_1$, $Z$ and $W$ be linear spaces paired under $((\ ,\ ))_2$, and suppose that $T$ is a linear transformation from $X$ into $Z$. In order that $T$ be $w(X, Y) - w(Z, W)$ continuous it is necessary and sufficient that there exists a dual map $T^*$. If $T^*$ exists, it is unique and $w(X, Y) - w(Z, W)$ continuous. Furthermore, if $T^*$ exists $T$ is $s(Z, W)$ continuous.*

*Proof.* The first statement is proved in (**4**, p. 100, Proposition 1). The first part of the second statement follows from the fact that the bilinear functionals separate points. The second part is proved in (**4**, p. 101). The last statement is proved in (**17**, § 30.2).

LEMMA 4. *Let $X$ and $Y$ be vector spaces paired under $((\,,\,))_1$ and $Z$ and $W$ be vector spaces paired under $((\,,\,))_2$. Suppose that $P$ is a convex cone in $X$ which is $w(X, Y)$-closed, $Q$ is a convex cone in $Z$ which is $w(Z, W)$-closed, and $T$ is a linear transformation of $X$ into $Z$ which is $w(X, Y) - w(Z, W)$ continuous. Then $(P \cap T^{-1}(Q))^+$ coincides with the $w(Y, X)$-closure of $T^*(Q^+) + P^+$ and hence with the closure of $T^*(Q^+) + P^+$ for any topology on $Y$ which is compatible with the duality between $X$ and $Y$.*

*Proof.*

$$(T^*(Q^+) + P^+)^+ = \{x;\ ((x, T^*w + y))_1 \geqslant 0 \text{ for all } w \in Q^+,\ y \in P^+\}$$
$$= \{x;\ ((Tx, w))_2 + ((x, y))_1 \geqslant 0 \text{ for all } w \in Q^+, y \in P^+\}.$$

It is clear that the vector spaces $Z \times X$ and $W \times Y$ are paired under the bilinear functional $((\,,\,))$ defined by

$$(( (z, x), (w, y) )) = ((z, w))_2 + ((x, y))_1 \quad \text{for all} \quad x \in X, y \in Y, z \in Z, \text{ and } w \in W.$$

Moreover, $(T^*(Q^+) + P^+)^+ = \{x;\ (( (Tx, x), (w, y) )) \geqslant 0 \text{ for } (w, y) \in Q^+ \times P^+\}$. Applying Lemma 2, we obtain that

$$(T^*(Q^+) + P^+)^+ = \{x;\ (Tx, x) \in Q \times P\}$$
$$= \{x;\ Tx \in Q \text{ and } x \in P\} = T^{-1}(Q) \cap P.$$

Consequently, $(T^{-1}(Q) \cap P)^+ = (T^*(Q^+) + P^+)^{++}$. The proof is completed by referring to Lemma 2 and (**4**, p. 67, Proposition 4).

LEMMA 5. *In addition to the conditions of Lemma 4, assume that $Q$ has a non-empty $s(Z, W)$-interior, $Q^0$, and that $P \cap T^{-1}(Q^0)$ is not empty. Then $(P \cap T^{-1}(Q))^+ = T^*(Q^+) + P^+$. In other words, $T^*(Q^+) + P^+$ is $w(Y, X)$-closed and hence closed for any topology on $Y$ which is compatible with the duality between $X$ and $Y$.*

*Proof.* The proof is not difficult if the null element of $Z$ belongs to $Q^0$. When this condition does not obtain, the ensuing argument applies. Lemma 4 established that $T^*(Q^+) + P^+ \subset (P \cap T^{-1}(Q))^+$. The converse inclusion may be proved as follows. Suppose $y \in (P \cap T^{-1}(Q))^+$. If $y = 0$ then $y \in T^*(Q^+) + P^+$. If $y \neq 0$ then $x \in P$ and $Tx \in Q$ implies $((x, y))_1 \geqslant 0$. By assumption there exists $x_0 \in P$ with $Tx_0 \in Q^0$. It is easy to show that $((x_0, y))_1 > 0$. If $y \notin T^*(Q^+) + P^+$ then $r \in R$, $w \in Q^+$, and $ry - T^*w \in P^+$ implies $r \leqslant 0$. Equivalently, if $B(r, w) = ry - T^*w$ then $(r, w) \in R \times Q^+$ and $B(r, w) \in P$ implies $(( (-1, 0), (r, w) )) = -r + ((0, w))_2 \geqslant 0$. Thus, $(-1, 0)$ is an element of the $s(R \times Z, R \times W)$-closure of $B^*(P) + (\{0\} \times Q)$, where $\{0\}$ is the cone consisting of the real number zero and $B^*$ is the dual map of $B$, that is, $B^*x = ( ((x, y))_1, -Tx )$ for all $x \in X$. Consequently there exist nets $\{x_\alpha\} \subset P$ and $\{z_\alpha\} \subset Q$ such that $\{-Tx_\alpha + z_\alpha\}$ $s(Z, W)$-converges to 0 and $\{((x_\alpha, y))_1\}$ converges to $-1$.

If $x_\alpha' = 2((x_0, y))_1 x_\alpha + x_0$ and $z_\alpha' = 2((x_0, y))_1 z_\alpha$ then $\{Tx_\alpha' - z_\alpha'\}$ $s(Z, W)$-converges to $Tx_0$ and $\{((x_\alpha', y))_1\}$ converges to $-((x_0, y))_1$. Since $Tx_0 \in Q^0$

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there exists an $\bar{\alpha}$ such that $Tx_\alpha' - z_\alpha' \in Q^0$ for $\alpha > \bar{\alpha}$. The same reasoning as in Lemma 1 insures that $Tx_\alpha' \in Q^0$ for $\alpha > \bar{\alpha}$. In addition, $x_\alpha' \in P$ so $((x_\alpha', y))_1 > 0$ for $\alpha > \bar{\alpha}$. This implies that the greatest lower bound of the net $\{((x_\alpha', y))_1\}$ is non-negative. This is a contradiction since the net $\{x_\alpha'\}$ was constructed so that the net $\{((x_\alpha', y))_1\}$ converged to $-((x_0, y))_1 < 0$. Thus $y \in T^*(Q^+) + P^+$ whenever $y \in (P \cap T^{-1}(Q))^+$. The proof is completed by referring to (**4**, p. 67, Proposition 4).

LEMMA 6. *In addition to the conditions of Lemma* 4, *assume that* $y_0 \in Y$, *and* $z_0 \in Z$. *Then a necessary and sufficient condition for* $x \in P$ *and* $Tx - z_0 \in Q$ *to imply that* $((x, y_0))_1 \geqslant k$ *is that* $x \in P$, $r \in R_0$, *and* $Tx - rz_0 \in Q$ *implies* $((x, y_0))_1 \geqslant rk$.

*Proof*. The sufficiency of the condition is obvious. The necessity of the condition is straightforward except for the case when $r = 0$. In this case it must be shown that if $x \in P$ and $Tx - z_0 \in Q$ implies that $((x, y_0))_1 \geqslant k$, then $x \in P$ and $Tx \in Q$ implies that $((x, y_0))_1 \geqslant 0$. To prove this, assume that the latter condition is false. Then there exists an $x_1 \in P$ with $Tx_1 \in Q$ and $((x_1, y_0))_1 = h < 0$. Then for every $n \geqslant 1$, $nx_1 \in P$, $T(nx_1) \in Q$, and $((nx_1, y_0))_1 = nh < 0$. If $x \in P$ and $Tx - z_0 \in Q$ then $x + nx_1$ also satisfies the same condition for $n \geqslant 1$. However, it is clear that $((x + nx_1, y_0))_1 - k$ cannot remain non-negative for all $n \geqslant 1$. Consequently the condition is also necessary.

**3. Programmes in paired spaces.** Let $X$ and $Y$ be linear spaces paired under $((\,,))_1$ and Z and W be linear spaces paired under $((\,,))_2$. A *programme* for these paired spaces is a quintuple $(A, P, Q, y_0, z_0)$. In this quintuple it is assumed that $A$ is a linear transformation from $X$ into $Z$ which is $w(X, Y) - w(Z, W)$ continuous, $P$ is a convex cone in $X$ which is $w(X, Y)$-closed, $Q$ is a convex cone in $Z$ which is $w(Z, W)$-closed, $y_0$ is an element of $Y$, and $z_0$ is an element of $Z$. The programme is said to be *consistent* if and only if there exists $x \in P$ such that $Ax - z_0 \in Q$. Such an $x$ is called *feasible*. If the programme is consistent, its *value* is defined as $\underline{M} = \inf((x, y_0))_1$ (the infimum taken over feasible $x$). A feasible $x$ is called *extremal* if and only if $((x, y_0))_1 = M$. The programme is said to be *convergent* if it is consistent, has a finite value $M$, and there is an extremal element. The programme is called *subconsistent* if and only if $z_0$ is an element of the $w(Z, W)$-closure of the set $A(P) - Q$. A net $\{x_\alpha\} \subset P$ is called *feasible* if the programme is subconsistent and there exists a net $\{z_\alpha\} \subset Q$ such that $\{Ax_\alpha - z_\alpha\}$ $w(Z, W)$-converges to $z_0$. The net $\{z_\alpha\}$ is called an *associated* net for $\{x_\alpha\}$. If the programme is subconsistent its *subvalue* is defined as $\underline{m} = \inf \text{lim-inf}\{((x_\alpha, y_0))_1\}$ (the infimum taken over feasible nets $\{x_\alpha\}$). A feasible net $\{x_\alpha\}$ is called *extremal* if and only if $\lim \{((x_\alpha, y_0))\} = m$.

Associated with the programme $(A, P, Q, y_0, z_0)$ is the programme $(A^*, Q^+, -P^+, -z_0, y_0)$ for $W$ and $Z$ paired under ${}_2((\,,))$ and $Y$ and $X$ paired under

$_1((\ ,\ ))$. The bilinear functionals $_2((\ ,\ ))$ and $_1((\ ,\ ))$ are defined by $_2((w, z)) = ((z, w))_2$ for all $w \in W$ and $z \in Z$, and $_1((,y\ x)) = ((x, y))_1$ for all $y \in Y$ and $x \in X$. The programme $(A^*, Q^+, -P^+, -z_0, y_0)$ is said to be the *dual* of the programme $(A, P, Q, y_0, z_0)$. It is also called the *dual programme*. Its value is denoted by $\underline{M}'$ and its subvalue is denoted by $\underline{m}'$. It is easy to see that the dual programme is well defined, consistent if and only if there exists $w \in Q^+$ such that $y_0 - A^*w \in P^+$, and subconsistent if and only if $y_0$ is an element of the $w(Y, X)$-closure of the set $A^*(Q^+) + P^+$. If the dual programme is consistent, then $M' = -\sup ((z_0, w))_2$ (the supremum taken over feasible $w$). If the dual programme is subconsistent, then $m' = -\sup$ lim-sup $\{((z_0, w_\alpha))_2\}$ (the supremum taken over feasible nets $\{w_\alpha\}$). A feasible $w$ is extremal if and only if $((z_0, w))_2 = -M'$ and a feasible net $\{w_\alpha\}$ is extremal if and only if $\lim \{((z_0, w_\alpha))\} = -m'$.

Several elementary relations are given in the first theorem.

THEOREM 1.

(a) *If a programme is consistent, it is subconsistent and $M \geqslant m$.*

(b) *If the dual programme is consistent, it is subconsistent and $M' \geqslant m'$.*

(c) *If a programme is subconsistent and has a finite subvalue, there always exist extremal nets.*

(d) *If a programme and its dual are consistent then $M$ and $M'$ are finite and $M \geqslant -M'$.*

(e) *If $\bar{x}$ is feasible for the programme and $\bar{w}$ is feasible for the dual programme then $((\bar{x}, y_0 - A^*\bar{w}))_1 \geqslant 0$ and $((A\bar{x} - z_0, \bar{w}))_2 \geqslant 0$. If $M = -M'$ then $\bar{x}$ and $\bar{w}$ are extremal if and only if $((\bar{x}, y_0 - A^*\bar{w}))_1 = 0$ and $((A\bar{x} - z_0, \bar{w}))_2 = 0$.*

(f) *If $\bar{x}$ is feasible for the programme, $\bar{w}$ is feasible for the dual programme, and $((\bar{x}, y_0))_1 = ((z_0, \bar{w}))_2$ then $\bar{x}$ and $\bar{w}$ are extremal.*

(g) *If the dual programme is consistent and has a finite value $M'$ then an $\bar{x}$ which is feasible for the programme and satisfies $((\bar{x}, y_0))_1 = -M'$ is extremal.*

(h) *If the programme is consistent and has a finite value $M$ then a $\bar{w}$ which is feasible for the dual programme and satisfies $((z_0, \bar{w}))_2 = -M$ is extremal.*

(i) *The dual of the dual programme is equivalent to the programme itself.*

THEOREM 2. *The programme $(A, P, Q, y_0, z_0)$ is consistent and has a finite value $M$ if and only if the dual programme $(A^*, Q^+, -P^+, -z_0, y_0)$ is subconsistent and has the finite number $-M$ as its subvalue.*

*Proof.* Let $T$ be the mapping: $X \times R \rightarrow Z$ defined by $T(x, r) = Ax - rz_0$ and let $X \times R$ and $Y \times R$ be paired under the bilinear functional $((\ ,\ ))$ defined by $(( (x, r), (y, s) )) = ((x, y))_1 + rs$. By appealing to Lemma 6 and the definition of the value of a programme, we easily obtain that the programme $(A, P, Q, y_0, z_0)$ is consistent and has a finite value $M$ if and only if $(x, r) \in P \times R_0$ and $T(x, r) \in Q$ implies $(( (x, r), (y_0, -M) )) \geqslant 0$, and for every $\epsilon > 0$ there exists an $x \in X$ with $T(x, 1) \in Q$ and $(( (x, 1), (y_0, -(M + \epsilon)) )) < 0$. By Lemma 2 these latter conditions are satisfied if and only if

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$(y_0, -M)$ is an element in the dual of the cone $(P \times R_0) \cap T^{-1}(Q)$, and $(y_0, -(M + \epsilon))$ is not. By appealing to Lemma 4, we see that this is possible if and only if there exists a feasible net $\{w_\alpha'\}$ and a net $\{r_\alpha'\} \subset R_0$ with the property that the net $\{((z_0, w_\alpha'))_2 - r_\alpha'\}$ converges to $M$ and for every feasible net $\{w_\alpha\}$ and net $\{r_\alpha\} \subset R_0$, the net $\{((z_0, w_\alpha))_2 - r_\alpha\}$ does not converge to $M + \epsilon$. This is clearly equivalent to the statement that the dual programme $(A, Q^+, -P^+, -z_0, y_0)$ is subconsistent and has a finite subvalue $m'$ which is equal to $-M$.

COROLLARY 2.1. *The dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *is consistent and has a finite value* $M'$ *if and only if the programme* $(A, P, Q, y_0, z_0)$ *is subconsistent and has the finite number* $-M'$ *as its subvalue.*

*Proof.* Theorems 2 and 1(i).

COROLLARY 2.2. *If* $K$ *is a finite real number then the programme* $(A, P, Q, y_0, z_0)$ *has* $K$ *as its value and subvalue if and only if the dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *has* $-K$ *as its value and subvalue.*

*Proof.* Theorem 2 and Corollary 2.1.

THEOREM 3. *Let the programme* $(A, P, Q, y_0, z_0)$ *be consistent and have a finite value* $M$. *The dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *is convergent and has* $-M$ *as its value, if the set* $G = \{(A^*w + y, r - ((z_0, w))_2); y \in P^+, w \in Q^+, \text{and } r \in R_0\}$ *is* $w(Y \times R, X \times R)$*-closed. In particular,* $G$ *is* $w(Y \times R, X \times R)$*-closed if* $Q$ *has a non-empty* $s(Z, W)$*-interior* $Q^0$ *and there exists an* $x \in P$ *such that* $Ax - z_0 \in Q^0$.

*Proof.* Define $T$ as in the proof of Theorem 2. Then $G = T^*(Q^+) + (P^+ \times R_0)$. The closure assumption on $G$ and Lemma 4 assure that $(y_0, -M) \in G$ and for every $\epsilon > 0$, $(y_0, -(M + \epsilon))$ is not an element of $G$. Hence there exist $y' \in P^+, w' \in Q^+$ and $r' \in R_0$ such that $A^*w' + y' = y_0$ and $((z_0, w'))_2 - r' = M$. Moreover, $y \in P^+, w \in Q^+$ and $A^*w + y = y_0$ implies that $((z_0, w))_2 < M + \epsilon$ whenever $\epsilon > 0$. It is clear that $r' = 0$ and that $w'$ is extremal. The last statement in the theorem follows from Lemma 5.

COROLLARY 3.1. *Let the dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *be consistent and have a finite value* $M'$. *The programme* $(A, P, Q, y_0, z_0)$ *is convergent and has* $-M'$ *as its value if the set* $H = \{(Ax - z, r + ((x, y_0))_1); x \in P, z \in Q, \text{and } r \in R_0\}$ *is* $w(Z \times R, W \times R)$*-closed. In particular* $H$ *is* $w(Z \times R, W \times R)$*-closed if* $P^+$ *has a non-empty* $s(Y, X)$*-interior* $(P^+)^0$ *and there exists a* $w \in Q^+$ *such that* $y_0 - A^*w \in (P^+)^0$.

*Proof.* Theorems 3 and 1(i).

THEOREM 4. *Let the programme* $(A, P, Q, y_0, z_0)$ *be consistent and have a finite value* $M$. *Suppose the set* $U = \{(Ax - z, ((x, y_0))_1); x \in P \text{ and } z \in Q\}$ *has a non-empty* $s(Z \times R, W \times R)$*-interior* $U^0$ *and there exists an* $r_0 \in R$

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*such that* $(z_0, r_0) \in U^0$. *Then the dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *is consistent and has* $-M$ *as its value.*

*Proof.* Let $E = \{r; (z_0, r) \in U\}$ and $F = \{r; (z_0, r) \in \bar{U}\}$, where $\bar{U}$ is the $w(Z \times R, W \times R)$-closure of $U$. Clearly, $M$ is the greatest lower bound of $E$, and either $F$ has no lower bound, or $m$ is the least element of $F$. Assume that $F$ has no lower bound. Then $(z_0, r) \in \bar{U}$ for all $r \leqslant r_0$. By definition of $M$ we have $M \leqslant r_0$ so that $2M - r_0 - 2 < r_0$. Therefore $(z_0, 2M - r_0 - 2) \in \bar{U}$. By (**3**, p. 51, Proposition 15), $\frac{1}{2}(z_0, r_0) + \frac{1}{2}(z_0, 2M - r_0 - 2) \in U^\circ$. Thus $(z_0, M - 1) \in U^0$. This contradiction of the minimality of $M$ shows that $F$ has a lower bound $m$. Clearly $m \leqslant M$. Applying the same proposition once more, we see that $t(z_0, r_0) + (1 - t)\ (z_0, m) \in U^0$ for all $0 < t < 1$. Consequently $m = M$. Corollary 2.2 insures that the dual programme is consistent and has a finite value $-M$.

COROLLARY 4.1. *Let the dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *be consistent and have a finite value* $M'$. *Suppose the set* $V = \{(A^*w + y, ((z_0, w))_2); w \in Q^+$ *and* $y \in P^+\}$ *has a non-empty* $s(Y \times R, X \times R)$-*interior* $V^0$ *and there exists an* $r_0 \in R$ *such that* $(y_0, r_0) \in V^0$. *Then the programme* $(A, P, Q, y_0, z_0)$ *is consistent and has* $-M$ *as its value.*

*Proof.* Theorems 4 and 1(i).

The next few theorems concerning the consistency of programmes follow easily from Lemma 6 and the preceding theorems.

THEOREM 5. *The programme* $(A, P, Q, y_0, z_0)$ *is consistent if and only if the programme* $(A^*, Q^+, -P^+, -z_0, 0)$ *has zero as its subvalue.*

COROLLARY 5.1. *The dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *is consistent if and only if the programme* $(A, P, Q, y_0, 0)$ *has zero as its subvalue.*

THEOREM 6. *The programme* $(A, P, Q, y_0, z_0)$ *is subconsistent if and only if the programme* $(A^*, Q^+, -P^+, -z_0, 0)$ *has zero as its value. If* $A(P) - Q$ *is* $w(Z, W)$-*closed, the programme* $(A, P, Q, y_0, z_0)$ *is consistent.*

COROLLARY 6.1. *The dual programme* $(A^*, Q^+, -P^+, -z_0, y_0)$ *is subconsistent if and only if the programme* $(A, P, Q, y_0, 0)$ *has zero as its value. If* $A^*(Q^+) + P^+$ *is* $w(Y, X)$-*closed, the dual programme is consistent.*

THEOREM 7. *Let* $Q$ *have a non-null* $s(Z, W)$-*interior* $Q^0$. *Then there exists an* $x \in P$ *with* $Ax - z_0 \in Q^0$ *if and only if the programme* $(A^*, Q^+, -P^+, -z_0, 0)$ *has zero as its value and the zero vector is the only extremal element.*

*Proof.* Necessity: If there exists an $x \in P$ with $Ax - z_0 \in Q^0$ then $w \in Q^+$ and $w \neq 0$ implies that $((Ax - z_0, w))_2 > 0$. That is, $((x, -A^*w))_1 < -((z_0, w))_2$. Consequently, $w \in Q^+$, $w \neq 0$, and $-A^* w \in P^+$ implies $-((z_0, w))_2 > 0$. Sufficiency: If $Q^0$ is non-empty then the $s(Z, W)$-interior of $A(P) - Q$ is non-empty. Thus if $w \in Q^+$, $w \neq 0$, and $-A^*w \in P^+$ implies $((-z_0, w))_2 > 0$, then

$z_0$ is in the $s(Z, W)$-interior of $A(P) - Q$. Because $Q^0$ is non-empty, Lemma 3 insures that $z_0 \in A(P) - Q^0$. In other words, there exists an $x \in P$ with $Ax - z_0 \in Q^0$.

COROLLARY 7.1. *Let $P^+$ have a non-null $s(Y, X)$-interior $(P^+)^0$. Then there exists a $w \in Q^+$ with $y_0 - A^* w \in (P^+)^0$ if and only if the programme $(A, P, Q, y_0, 0)$ has zero as its value and the zero vector is the only extremal element.*

THEOREM 8. *Let $\{z_\alpha\}$ be a net in $Z$ which $w(Z, W)$-converges to $z_0$ and satisfies the condition that $\alpha_2 > \alpha_1$ implies*

$$z_{\alpha_1} - z_{\alpha_2} \in Q.$$

*If the programme $(A, P, Q, y_0, z_{\alpha'})$ is consistent then the programme $(A, P, Q, y_0, z_\alpha)$ is consistent for all $\alpha > \alpha'$. If the programme $(A, P, Q, y_0, z_{\alpha'})$ is consistent, has a finite value $M_{\alpha'}$, and the programme $(A, P, Q, y_0, z_0)$ has a finite value $M_0$ then for $\alpha > \alpha'$ the programme $(A, P, Q, y_0, z_\alpha)$ has a finite value $M_\alpha$ which satisfies $M_{\alpha'} \geqslant M_\alpha \geqslant M_0$. If $\overline{\phantom{x}}$ denotes the $w(X, Y)$-* or *$w(Z,W)$-closure and*

$$\overline{A^{-1}(U_\alpha\{z_\alpha + Q\})} \supset A^{-1}(\overline{U_\alpha\{z_\alpha + Q\}})$$

*then* $\lim M_\alpha = M_0$.

*Proof.* If $K_\alpha = \{x;\, x \in P \text{ and } Ax - z_\alpha \in Q\}$ then

$$K_0 \supset K_{\alpha_2} \supset K_{\alpha_1}$$

whenever $\alpha_2 > \alpha_1$. Thus if $K_{\alpha'}$ is non-empty then $K_\alpha$ is non-empty for $\alpha > \alpha'$. If the programme $(A, P, Q, y_0, z_0)$ has a finite value $M_0$ then $M_0$ is a lower bound for $M_\alpha$. To prove that $\lim M_\alpha = M_0$ when the condition of the last statement is satisfied it is sufficient to show that $\overline{U_\alpha K_\alpha} = K_0$. By definition of $U_\alpha K_\alpha$ it is seen that

$$\overline{U_\alpha K_\alpha} = P \cap \overline{A^{-1}(U_\alpha\{z_\alpha + Q\})}$$

and that

$$\overline{U_\alpha\{z_\alpha + Q\}} = z_0 + Q.$$

Thus

$$K_0 = P \cap A^{-1}(\overline{U_\alpha\{z_\alpha + Q\}}).$$

Since $A$ is $w(X, Y) - w(Z, W)$ continuous,

$$\overline{A^{-1}(U_\alpha\{z_\alpha + Q\})} \subset A^{-1}(\overline{U_\alpha\{z_\alpha + Q\}}).$$

By assumption the reverse inclusion is satisfied. Consequently $\overline{U_\alpha K_\alpha} = K_0$.

COROLLARY 8.1 *Let $\{y_\alpha\}$ be a net in $Y$ which $w(Y, X)$-converges to $y_0$ and satisfies the condition that $\alpha_2 > \alpha_1$ implies*

$$y_{\alpha_2} - y_{\alpha_1} \in P^+.$$

*If the dual programme $(A^*, Q^+, -P^+, -z_0, y_{\alpha'})$ is consistent then the programme $(A^*, Q^+, -P^+, -z_0, y_\alpha)$ is consistent for $\alpha > \alpha'$. If the programme $(A^*, Q^+,$*

*$-P^+$, $-z_0$, $y_{\alpha'}$) is consistent, has a finite value $M_{\alpha'}'$, and the programme ($A^*$, $Q^+$, $-P^+$, $-z_0$, $y_0$) has a finite value $M_0'$ then for $\alpha > \alpha'$ the programme ($A^*$, $Q^+$, $-P^+$, $-z_0$, $y_\alpha$) has a finite value $M_\alpha'$ which satisfies $M_{\alpha'}' \geqslant M_\alpha' \geqslant M_0'$. If $\overline{\phantom{x}}$ denotes the $w(W, Z)$- or $w(Y, X)$-closure and*

$$\overline{A^{-1}(U_\alpha y_\alpha - P^+)} \supset A^{-1}\,(\overline{U_\alpha y_\alpha - P^+})$$

*then* $\lim M_\alpha = M_0$.

*Proof.* Theorems 8 and 1 (h).

**4. Programmes in locally convex separated spaces.** In this section a locally convex separated linear topological space will be abbreviated to lcs-space. We shall depart slightly from the notation of the previous sections and denote the dual of an lcs-space $(E, \mathfrak{T})$ by $E^*$. If $(E, \mathfrak{T})$ is an lcs-space it is clear that $E$ and $E^*$ are paired under the bilinear functional $((\,,\,))$ defined as $((e, e^*)) = e^*(e)$ and that $E^*$ and $E$ are paired under the bilinear functional $((\,,\,))_1$ defined as $((e^*, e))_1 = e^*(e)$. Consequently the theorems established in the preceding section apply in the case of lcs-spaces.

It is also helpful to remember that if $(U, \mathfrak{T}_1)$ and $(V, \mathfrak{T}_2)$ are lcs-spaces then one can form a programme $(A, P, Q, y_0, z_0)$ for the paired spaces $X$ and $Y$ and the paired spaces $Z$ and $W$ in four ways:

(1) $X = U, \quad Y = U^*, \quad Z = V, \quad W = V^*;$

(2) $X = U, \quad Y = U^*, \quad Z = V^*, \quad W = V;$

(3) $X = U^*, \quad Y = U, \quad Z = V, \quad W = V^*;$ and

(4) $X = U^*, \quad Y = U, \quad Z = V^*, \quad W = V.$

It is assumed here and in the remainder of this section that an lcs-space and its dual are paired in the manner given in the previous paragraph.

If $(U, \mathfrak{T}_1)$ and $(V, \mathfrak{T}_2)$ are disk spaces then $s(U, U^*) = \mathfrak{T}_1$, and $s(V, V^*) = \mathfrak{T}_2$. Consequently, Theorem 3 and its Corollary insure the following. Case (1): if $Q$ has a non-empty $\mathfrak{T}_2$-interior $Q^0$ and there exists a $u \in P$ with $Au - z_0 \in Q^0$ then the dual programme is convergent and has value $-M$ if the programme has $M$ as its value. Case (3): if $Q$ has a non-empty $\mathfrak{T}_2$-interior $Q^0$ and there exists a $u^* \in P$ with $Au^* - z_0 \in Q^0$ then the dual programme is convergent and has value $-M$ if the programme has $M$ as its value. If $P^+$ has a non-empty $\mathfrak{T}_1$-interior $(P^+)^0$ and there exists a $v^* \in Q^+$ with $y_0 - A^*v^* \in (P^+)^0$ then the programme is convergent and has value $-M'$ if the dual programme has $M'$ as its value. Case (4): if $P^+$ has a non-empty $\mathfrak{T}_1$-interior $(P^+)^0$ and there exists a $v \in Q^+$ with $y_0 - A^*v \in (P^+)^0$ then the programme is convergent and has value $-M'$ if the dual programme has $M'$ as its value.

**5. Examples.** 5.1. *The necessity of subvalues.* If $U$ and $V$ are finite dimensional Euclidean spaces and $P$ and $Q$ are the positive orthants (or, more generally,

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polyhedral cones) then one easily obtains from Theorem 3 the duality theorem of linear programming. Namely, a programme is consistent and has the finite number $M$ as its value if and only if the dual programme is consistent and has the finite number $-M$ as its value. Moreover, extremal vectors exist for both programmes. If $P$ and $Q$ are not polyhedral the result concerning the values is still true if both programmes are consistent. However, in this case it is quite easy to see that there needn't exist extremal vectors for consistent programmes which have finite values, nor need there exist extremal vectors for consistent dual programmes which have finite values. The following example demonstrates that in a Hilbert space setting both the programme and the dual programme may be consistent, and have finite values $M$ and $M'$, respectively, which satisfy $M + M' > 0$.

Let $L_2$ denote the set of Lebesgue measurable functions on the unit interval which are square integrable and consider the problem of determining $M = \inf \int_0^1 x\, g(x)dx + 2r$ subject to (1) $g$ an element of $L_2$ which is non-negative almost everywhere; (2) $r \in R_0$; and (3) $\int_t^1 g(x)dx + r \geqslant 1$ almost everywhere ($0 \leqslant t \leqslant 1$). It is easy to see that this problem is equivalent to determining the value of the programme $(A, P, Q, y_0, z_0)$ for the spaces $L_2 \times R$ and $L_2 \times R$ paired under $((\,,\,))_1$ defined by

$$(((g_1, r_1), (g_2, r_2)))_1 = \int_0^1 g_1(x)g_2(x)dx + r_1 r_2$$

and the spaces $L_2$ and $L_2$ paired under $((\,,\,))_2$ defined by

$$((g_1, g_2))_2 = \int_0^1 g_1(x)g_2(x)dx.$$

$A$ is the linear transformation from $L_2 \times R \to L_2$ defined by

$$A(g, r)(t) = \int_t^1 g(x)dx + r \ (0 \leqslant t \leqslant 1), \qquad P = P_1 \times R_0$$

where $P_1$ consists of those elements of $L_2$ which are non-negative almost everywhere, $Q = P_1$, $y_0 = (h, 2)$ where $h$ is the element of $L_2$ defined by $h(t) = t$ ($0 \leqslant t \leqslant 1$), and $z_0$ is the element of $L_2$ defined by $z_0(t) = 1 (0 \leqslant t \leqslant 1)$.

The dual programme is the programme $(A^*, Q^+, -P^+, -z_0, y_0)$ for the spaces $L_2$ and $L_2$ and the spaces $L_2 \times R$ and $L_2 \times R$. $A^*$ is the linear transformation from $L_2 \to L_2 \times R$ defined by

$$A^*g(t) = \left(\int_0^t g(x)dx, \int_0^1 g(x)dx\right) (0 \leqslant t \leqslant 1),\ Q^+ = P_1,$$

$$\text{and } -P^+ = (-P_1) \times (-R_0).$$

The problem of determining the value of the dual programme can be written: determine $M' = -\sup \int_0^1 f(x)dx$ subject to (1) $f$ an element of $L_2$ which is non-negative almost everywhere, (2) $\int_0^t f(x)dx \leqslant t$ almost everywhere ($0 \leqslant t \leqslant 1$), and (3) $\int_0^1 f(x)dx \leqslant 2$.

It is not difficult to show that (1) $M = 2$ and $M' = -1$; (2) the pair $(\bar{g}, 1)$ where $\bar{g}(t) = 0$ $(0 \leqslant t \leqslant 1)$ is extremal for the programme; (3) the net $\{\bar{g}_n, \bar{r}_n\}$ where $\bar{g}_n$ is defined by $\bar{g}_n(t) = nt^{n-1}$ $(0 \leqslant t \leqslant 1)$ and $\bar{r}_n = 0$ is extremal for the programme and has the net $\{\bar{h}_n\}$ where $\bar{h}_n$ is defined by $\bar{h}_n(t) = 0$ $(0 \leqslant t \leqslant 1)$ as an associated net; (4) $\bar{f}$ defined by $\bar{f}(t) = 1$ $(0 \leqslant t \leqslant 1)$ is extremal for the dual programme; and (5) the net $\{\bar{f}_n\}$ where $\bar{f}_n$ is defined by $\bar{f}_n(t) = 1 + nt^{n-1}$ $(0 \leqslant t \leqslant 1)$ is extremal for the dual programme and has the net $\{\bar{k}_n\}$ where $\bar{k}_n(t) = 0$ $(0 \leqslant t \leqslant 1)$ as an associated net.

5.2. *Unattained infima and unbounded extremal nets.* In 5.1 it was stated that extremal vectors always exist for consistent linear programming problems which have finite values. This example shows that a programme may be consistent and have a finite value which is not attained. Moreover, all extremal nets are unbounded in a sense to be defined.

Let $C$ denote the continuous real-valued functions on the unit interval and let $BV$ be the set of all real-valued functions $g$ on the unit interval which have bounded variation and satisfy the normalizing conditions: $g(0) = 0$ and $g(t + 0) = g(t)$ $(0 < t < 1)$. Then the spaces BV and $C$ are paired under the bilinear functional $((\ ,\ ))$ defined by $((g, f)) = \int_0^1 f(t)dg(t)$. Let $P$ be those elements of $BV$ which are non-decreasing and define the linear transformation from $BV \to R$ by $A(g) = \int_0^1 t\, dg(t)$. Let $y_0$ be the element of $C$ defined by $y_0(t) = t^2 (0 \leqslant t \leqslant 1)$, $z_0$ be the unit element of $R$ (that is, the real number one), and denote the cone consisting of the real number zero by $\{0\}$.

The programme $(A, P, \{0\}, y_0, z_0)$ for the paired spaces $BV$ and $C$ and the space $R$ paired with itself is consistent and has zero as its value and subvalue. There exist no extremal elements, the sequence $\{\bar{g}_n\}$ where $\bar{g}_n$ is defined by $\bar{g}_n(t) = 0$ $(0 \leqslant t < 1/n)$ and $\bar{g}_n(t) = n(1/n \leqslant t \leqslant 1)$ is an extremal sequence and every extremal sequence is unbounded in the sense that the total variation of the elements of the sequence increase without limit. Specifically, the value of the programme is the $\inf \int_0^1 t^2 dg(t)$ subject to $g$ a non-decreasing normalized function of bounded variation which satisfies the condition: $\int_0^1 t\, dg(t) = 1$.

If $g$ is a feasible vector it is clear that $\int_0^1 t^2 dg(t) \geqslant 0$. Consequently, the value of the programme is non-negative. Moreover, by Schwarz's inequality:

$$\int_0^1 t(t+1)dg(t) \leqslant \int_0^1 t^2 dg(t) \left[\int_0^1 t^2 dg(t) + 2\int_0^1 t\, dg(t) + \int_0^1 dg(t)\right].$$

Thus

$$\int_0^1 t^2 dg(t) \left[\int_0^1 t^2 dg(t) + 1 + \int_0^1 dg(t)\right] \geqslant 1$$

and consequently $\int_0^1 t^2 dg(t) > 0$. The elements of the sequence $\{\bar{g}_n\}$ defined above are non-decreasing, normalized functions of bounded variation which satisfy the conditions: $\int_0^1 t\ dg(t) = 1$ and $\int_0^1 t^2 dg_n(t) = 1/n$. Thus the value of the programme is zero and it is not attained. Suppose $\{g_n\}$ is any feasible sequence. Then $\int_0^1 t\,dg_n(t)$ converges to 1 and $\int_0^1 t^2 dg_n(t) \geqslant 0$. Hence the subvalue of the programme is also zero. If $\{g_n\}$ is extremal then $\int_0^1 t\,dg_n(t)$ converges to 1 and $\int_0^1 t^2 dg_n(t)$ converges to 0. By Schwarz's inequality we deduce that

$$\int_0^1 t^2 dg_n(t)\left[\int_0^1 t^2 dg_n(t) + 2\int_0^1 t\,dg_n(t) + \int_0^1 dg_n(t) - 1\right] \geqslant \int_0^1 t\,dg_n(t).$$

As $n$ increases without limit the right side approaches 1 so the limit-infima of the left side must be greater than or equal to 1. Since $\int_0^1 t^2 dg_n(t)$ converges to zero and $\int_0^1 t\,dg_n(t)$ converges to 1 it is clear that $\int_0^1 dg_n(t)$ must increase without limit. Since $g_n$ is non-decreasing $\int_0^1 dg_n(t)$ is the total variation of $g_n$ on the unit interval. This establishes that the total variation of the elements of every extremal sequence increases without limit.

It is relatively easy to modify this example to obtain a programme which is not consistent but is subconsistent and has a finite subvalue. This implies that there exists a programme (namely, the dual programme) which is consistent and has a finite value, but whose subvalue does not exist. A problem similar to this example arises in statistics **(7, 16, 19)** and may be handled in the same way.

5.3. *Discontinuity of programmes.* Theorem 8 pertains to a kind of continuity property of programmes. This example shows that it is possible to have a sequence of programmes $(A, P, Q, y_0, z_n)$ $(n = 1, 2, \ldots,)$ each of which is convergent and has unity as its value. Moreover, $\{z_n\}$ converges to 0 in a decreasing manner, (that is, $z_n - z_{n+1} \in Q$) and the limit programme $(A, P, Q, y_0, 0)$ has zero as its value.

Let $C$ and $BV$ be defined as in the previous example and let $P$ be those elements of $C$ which are non-negative. For $n \geqslant 2$, let $z_n$ be the element of $C$ defined by $z_n(t) = t(0 \leqslant t \leqslant 1/n)$ and $z_n(t) = (1-t)/(n-1)(1/n \leqslant t \leqslant 1)$. Let $A$ be the linear transformation from $R$ into $C$ defined by $(Ar)(t) = rt(0 \leqslant t \leqslant 1)$. It is clear that for $n \geqslant 2$ the programme $(A, P, R, 1, z_n)$ for the paired spaces $C$ and $BV$ and the space $R$ paired with itself is consistent and has unity as its value. The sequence $z_n$ converges monotonically to 0 and the programme $(A, P, R, 1, 0)$ for the same spaces is consistent and has zero as its value.

5.4. *Haar's extension of the Minkowski-Farkas lemma.* The Minkowski-Farkas lemma may be stated as follows: "Let $A$ be an $n \times m$ matrix and let $A^*$ be its transpose. Let $P_n$ denote the positive orthant of Euclidean $n$-space $E_n$ and let $u^* = (u^*_1, \ldots, u^*_m)$ be an element of Euclidean $m$-space. If $Au \in P_n$ implies

$$\sum_{i=1}^{m} u_i u_i^* \geqslant 0$$

then there exists a $v \in P_n$ such that $A^*v = u^*$."

This lemma follows easily from Lemma 11 when one recognizes that the

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image of a polyhedral cone under a continuous linear transformation is itself a polyhedral cone and is therefore closed.

It does not seem to be well known that the Minkowski-Farkas lemma was extended about thirty-five years ago by A. Haar **(14)** to cover a more general case. In particular the following theorem was established: "*Let $g_0, g_1, \ldots, g_n$ be a finite collection of linearly independent functions of bounded variation on the unit interval. Define the linear transformation A mapping C, the continuous real-valued functions on the unit interval, into $E_n$ by $Af = (\int_0^1 f(t)dg_1(t), \ldots, \int_0^1 f(t)dg_n(t))$. If $Af \in P_n$ implies $\int_0^1 f(t)dg_0(t) \geqslant 0$ then there exists a vector $(u_1, \ldots, u_n) \in P_n$ such that $\sum_{i=1}^n u_i g_i = g_0$ in the sense that $\int_0^1 f(t)d(\sum u_i g_i)(t) = \int_0^1 f(t)dg_0(t)$ for all $f \in C$.*" The proof given above for the Minkowski-Farkas lemma applies in this case if we assume that the functions $g_0, g_1, \ldots, g_n$ are normalized as in 5.2.

It is now evident that the following theorem is true and contains both the Minkowski-Farkas lemma and Haar's extension as special cases. Let $(U, \mathfrak{T})$ be an lcs-space and let $u^*_0$ be an element of the dual space $(U, \mathfrak{T})^*$. Let $A$ be a continuous linear transformation from $U$ into $E_n$ and let $A^*$ denote the dual map. Let $P_n$ denote the positive orthant of $E_n$. If $Au \in P_n$ implies $u^*_0(u) \geqslant 0$ then there exists a $u^* \in (U, \mathfrak{T})^*$ such that $A^*u^* = u^*_0$.

Duffin has suggested a similar theorem which is proved using a slightly different argument. Let $P$ be a closed convex cone in an lcs-space $(U, \mathfrak{T})$ and let $u^*_1, \ldots, u^*_n$ be $n$ elements in the dual space $(U, \mathfrak{T})^*$. If for every $u \in P$ there exists at least one $u^*_i (i = 1, \ldots, n)$ such that $u^*_i(u) \geqslant 0$ then there exist $n$ non-negative real numbers $a_1, \ldots, a_n$ not all zero such that $\sum_{i-1}^n a_i u^*_i \in P^+$.

The proof is accomplished by introducing a continuous linear transformation $T$ which maps $U \times R$ into $U \times E_n$. Suppose $A(u, r) = (r - u^*_1(u), \ldots, r - u^*_n(u))$. The assumptions imply that if the transformation $T$ is defined as $T(u, r) = (u, A(u, r))$ then $T(u, r) \in P \times P_n$ implies $(( (u, r), (0, 1) )) \geqslant 0$. The bilinear functional $((\,,\,))$ is defined on the Cartesian product $(U \times R) \times (U^* \times R)$ as $(( (u, r), (u^* s) )) = u^*(u) + rs$. Lemma 11 insures that $(0, 1)$ belongs to the weak-star closure of $T^*(P^+ \times P_n)$. Therefore, there exist sequences $\{u^*_{(k)}\} \subset P^+$ and $\{a^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)})\} \subset P_n$ such that the sequence $\{u^*_{(k)} - \sum_{i=1}^n a_i^{(k)} u^*_i\}$ converges to the zero element of $(U, \mathfrak{T})^*$ in the weak-star topology and $\{\sum_{i=1}^n a_i^{(k)}\}$ converges to unity. The non-negativity of the $a_i^{(k)}$ and the convergence of $\{\sum_{i=1}^n a_i^{(k)}\}$ implies that the sequence $\{a_i^{(k)}\}$ converges for each $i$, to say $\bar{a}_i$. Thus $\{\sum_{i=1}^n a_i^{(k)} u^*_i\}$ converges in the weak-star topology to $\sum_{i=1}^n \bar{a}_i u^*_i$ and consequently so does $\{u^*_{(k)}\}$. Therefore $\sum_{i=1}^n \bar{a}_i u^*_i \in P^+$. Clearly $\bar{a}_i \geqslant 0$ and $\sum_{i=1}^n \bar{a}_i = 1$.

5.5. *A non-linear problem.* Consider the problem of determining the maximum value of $ar + bt$ subject to $r$ and $t$ real numbers which satisfy $(r^2 + b^2)^{\frac{1}{2}} \leqslant 1$. It is easy to see that this maximum exists and is equal to $(a^2 + b^2)^{\frac{1}{2}}$. The extremal points are $\bar{r} = a/(a^2 + b^2)^{\frac{1}{2}}$, $\bar{t} = b/(a^2 + b^2)^{\frac{1}{2}}$. It is not obvious that this problem can be expressed as the problem of determining the value of a programme. A proof of this fact follows.

Let $E_2$ be Euclidean 2-space. Then a point $(r, t) \in E_2$ satisfies the condition $(r^2 + t^2)^{\frac{1}{2}} \leqslant 1$ if and only if $|ru + tv| \leqslant 1$ for all $(u, v) \in E_2$ which satisfy the condition $(u^2 + v^2)^{\frac{1}{2}} \leqslant 1$. Let $S$ denote the unit circle, $C(S)$ denote the set of all real-valued functions which are continuous on $S$, $Q$ be the non-negative elements of $C(S)$, and denote $C(S) \times C(S)$ by $C_2(S)$. Let $M(S)$ denote the set of all regular countably additive real-valued set functions defined on the Borel sets in $S$ and denote $M(S) \times M(S)$ by $M_2(S)$. Then $C_2(S)$ and $M_2(S)$ are paired under the bilinear functional $((\,,\,))$ defined by $(( (f, g), (h, k) )) = \int_S f(x)h(ds) + \int_S g(s)k(ds)$ (**11**, p. 265, Theorem 3). Let $A$ be the linear transformation from $E_2$ into $C_2(S)$ defined by $[A(r, t)]\,(u, v) = (ru + tv, -ru - tv)$. Let $\delta$ be the element of $C(S)$ defined as $\delta(u, v) = 1$ $((u, v) \in S)$.

The negative of the value of the programme $(-A, E_2, -Q, (-a, -b), (-\delta, -\delta))$ for the space $E_2$ paired with itself and the paired spaces $C_2(S)$ and $M_2(S)$ is equal to the maximum value of $ar + bt$ subject to $r$ and $t$ real numbers which satisfy $(r^2 + t^2)^{\frac{1}{2}} \leqslant 1$. It is interesting to note that the problem of determining the value of the dual programme is equivalent to determining the minimum variation of those elements $g$ of $M(S)$ which satisfy the conditions $\int_S ug(d(u, v)) = a$ and $\int_S vg(d(u, v)) = b$. The solution to this problem is the measure $\bar{g}$ on $S$ defined as $\bar{g}(F) = (a^2 + b^2)^{\frac{1}{2}}$ if $F$ is a Borel subset of $S$ containing the point $(a/(a^2 + b^2)^{\frac{1}{2}}, b/(a^2 + b^2)^{\frac{1}{2}})$ and $\bar{g}(F) = 0$ if the Borel subset $F$ of $S$ does not contain the point $(a/(a^2 + b^2)^{\frac{1}{2}}, b/(a^2 + b^2)^{\frac{1}{2}})$. The variation of $\bar{g}$ is $(a^2 + b^2)^{\frac{1}{2}}$.

It is clear that the same method can be applied to convert similar types of non-linear programming problems in Euclidean and other Banach spaces into programmes in paired spaces.

5.6. *Continuous games.* It is well known that one can use the duality theory of linear programming to prove that every finite game has a value (**12**, p. 326). Continuous games on the unit square have been considered from the same point of view (**24**, § 10) although a complete proof of Ville's theorem based on this approach has not appeared in the literature. It will now be shown that this can be done using the duality theory of § 3. This application of programming theory was suggested to the author by Duffin.

*Ville's theorem. Let $K$ be a continuous function on the unit square. Let $G$ denote the set of all real-valued functions $g$ on the unit interval which are non-decreasing and satisfy $g(0) = 0$ and $g(1) = 1$. Let*

$$V(g, h) = \int_0^1 \int_0^1 K(r, t)dg(r)dh(t),$$

$$V_1 = \min_{h \in G} \max_{g \in G} V(g, h) \quad \text{and} \quad V_2 = \max_{g \in G} \min_{h \in G} V(g, h).$$

*Then $V_1 = V_2$.*

*Proof.* We may assume without loss of generality that $K$ is strictly positive. It is clear that $V_1$ and $V_2$ exist and satisfy the relation $V_2 \leqslant V_1$. It is not difficult to show that $V_1 \leqslant V_2$ if there exist $g_0, h_0 \in G$ and a real number $v$ such that $\int_0^1 K(r, t)dg_0(r) \geqslant v(0 \leqslant t \leqslant 1)$, and $\int_0^1 K(r, t)dh_0(t) \leqslant v(0 \leqslant r \leqslant 1)$. It will now be shown that $g_0$, $h_0$, and $v$ can be constructed from the extremal vectors of a certain programme.

Let $C$, $BV$, and $P$ be defined as in 5.2 and let $Q$ be the non-negative elements of $C$. Define the linear transformation $A$ from $BV$ into $C$ by $(Ag)(t) = \int_0^1 K(r, t)dg(r)$ $(0 \leqslant t \leqslant 1)$. Let $y_0$ be the element of $C$ defined by $y_0(t) = 1$ $(0 \leqslant t \leqslant 1)$. The value of the programme $(A, P, Q, y_0, y_0)$ for the paired spaces $BV$ and $C$ and the paired spaces $C$ and $BV$ is equal to the $\inf \int_0^1 1\, dg(t)$ subject to the conditions that $g \in P$ and $\int_0^1 K(r, t)dg(r) \geqslant 1 (0 \leqslant t \leqslant 1)$. The topology $s(C, BV)$ is the topology induced by the norm on $C$ defined as $||f|| = \max_{0 \leqslant t \leqslant 1} |f(t)|$. This follows from the fact that $C$ together with this norm is a Banach space. It is clear that $Q$ has a non-null $s(C, BV)$-interior. In fact any element of $C$ which is strictly positive is in the $s(C, BV)$-interior of $Q$.

Since $K$ is strictly positive there exists a $g \in P$ such that $\int_0^1 K(r, t)dg(r)$ $1(0 \leqslant t \leqslant 1)$, that is, $Ag - y_0$ belongs to the $s(C, BV)$-interior of $Q$. Theorem 3 insures that the programme $(A, P, Q, y_0, y_0)$ has a finite value $M$ and that the dual programme has $-M$ as its value. Moreover, both programme have extremal elements, say $g_1$ and $h_1$. Since $g_1 \in P$ and $\int_0^1 K(r, t)dg_1(r) \geqslant 1$ it is clear that $M > 0$. The functions $g_0 = (1/M)g_1$, $h_0 = (1/M)h_1$ and the real number $v = 1/M$ satisfy the conditions in the first paragraph of the proof.

It is interesting to note that Theorem 3 would not be applicable if the theorem were stated using the $w(C, BV)$ topology. For in this topology the set $Q$ does not have an interior **(11**, p. 265, Corollary 4**)**.

5.7. *A generalization of the transportation problem.* The transportation problem may be expressed as determining the minimum value of

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} c_{ij}x_{ij}$$

subject to the conditions that the $x_{ij}$ are non-negative,

$$\sum_{j=1}^{m_2} x_{ij} = a_i(i = 1, \ldots, m_1), \quad \textit{and} \quad \sum_{i=1}^{m_1} x_{ij} = b_j \qquad (j = 1, \ldots, m_2).$$

It is assumed that $c_{ij}$, $a_i$, and $b_j$ are real numbers and that $a_i$ and $b_j$ satisfy

$$\sum_{i=1}^{m_1} a_i = \sum_{j=1}^{m_2} b_j, \quad a_i \geqslant 0, \quad \textit{and} \quad b_j \geqslant 0.$$

A continuous analogue of this problem has been treated in the literature by Kantorovitch **(18)**. The example presented below is similar but more general.

Let $(F_1, \mathfrak{T}_1)$ and $(F_2, \mathfrak{T}_2)$ be compact Hausdorff spaces and let $(F_3, \mathfrak{T}_3)$ be their topological product. Let $B_i$ denote the smallest $\sigma$-field of subsets of $F_i$ which contain the closed subsets of $(F_i, \mathfrak{T}_i)$, $H_i$ denote the set of all regular countably additive (finite) real-valued set functions on $B_i$, $C_i$ denote the set of all continuous real-valued functions on $(F_i, \mathfrak{T}_i)$, $Q_i$ denote the non-negative elements of $C_i$, $P_i$ denote the elements of $H_i$ which are non-negative on $B_i$,

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and $O_i$ denote the cone consisting of the zero element of $H_i$. For $g \in H_3$ define the set functions $g_1$ on $B_1$ and $g_2$ on $B_2$ by $g_1(G_1) = g(G_1 \times F_2)$ and $g_2(G_2) = g(F_1 \times G_2)$. Let $A$ be the linear transformation from $H_3$ into $H_1 \times H_2$ defined by $Ag = (g_1, g_2)$. Let $p_1$ be an element of $P_1$, $p_2$ be an element of $P_2$ which satisfies $p_2(F_2) = p_1(F_1)$, and $c \in C_3$. Then $H_i$ and $C_i$ are paired spaces (**11**, p. 265, Theorem 3) and $P_i^+ = Q_i$.

The programme $(A, P_3, O_1 \times O_2, c, (p_1, p_2))$ for the paired spaces $H_3$ and $C_3$ and the paired spaces $H_1 \times H_2$ and $C_1 \times C_2$ and its dual are convergent and the values satisfy $M + M' = 0$. Specifically,

$$M = \inf \int_{E_3} c(r, t) g(d(r, t))$$

subject to $g \in P_3$, $g_1 = p_1$ and $g_2 = p_2$; and

$$M' = -\sup\left[\int_{E_1} f_1 dh_1 + \int_{E_2} f_2 dh_2\right]$$

subject to $f_i \in C_i$ and $f_1(r) + f_2(t) \leqslant c(r, t)((r, t) \in E_1 \times E_2)$. The formulation of the dual programme uses the fact that the dual map $A^*$ of $A$ is defined by $A^*(f_1, f_2)(r, t) = f_1(r) + f_2(t)$. The statements about the values and the convergent nature of the programme and its dual follow from Corollary 3.1.

That this result implies the familiar theorem about the transportation problem is seen by letting $F_1 = \{1, \ldots, m_1\}$ $F_2 = \{1, \ldots, m_2\}$, $\mathfrak{T}_1$ be the set of all subsets of $F_1$ and the null set, and $\mathfrak{T}_2$ be the set of all subsets of $F_2$ and the null set.

Kantorovitch did not consider the dual programme as such, although he did recognize that the variables of the dual programme were of importance in proving whether or not a feasible $g$ was extremal. In particular he proved a special case of the following theorem.

If $(F_1, \mathfrak{T}_1) = (F_2, \mathfrak{T}_2)$, $c \in Q_3$, and $c(r, r) = 0$ for all $r \in F_2$, then a $g$ which is feasible for the programme is extremal if and only if there exists an $h \in C_2$ which satisfies $|h(r) - h(t)| \leqslant c(r, t)$ for all $r, t \in F_2$ and $h(r) - h(t) = c(r, t)$ whenever $g(N_r \times N_t) > 0$ for all neighbourhoods $N_r$ of $r$ and $N_t$ of $t$ Kantorovitch presented an analytical proof which is valid when $(F_2, \mathfrak{T}_2)$ is a compact metric space. The proof presented below is based on duality theory and is valid under the conditions specified.

*Proof.* Let $g_0$ be an extremal element for the programme. It is not difficult to show that

$$M = \min \int_{E_3} c(s)(g + k)(ds) \quad (s = (r, t))$$

subject to $g, k \in P_3$, $g_1 - k_1 = p_1$ and $g_2 - k_2 = p_2$. Moreover, $(g_0, 0)$ attains the minimum. If the dual of the programme corresponding to this problem is constructed it is seen that $M$ is also equal to the

$$\max\left[\int_{E_2} f_1 dp_1 + \int_{E_2} f_2 dp_2\right]$$

subject to $f_1, f_2 \in C_2$ and $|f_1(r) + f_2(t)| \leqslant c(r, t)$ for all $r, t \in F_2$. Since $c(r, r) = 0$ for all $r \in F_2$ it is clear that if the maximum is attained for $f^*_1$ and $f^*_2$, then $f^*_1(r) + f^*_2(r) = 0$ or $f^*_2(r) = -f^*_1(r)$ for all $r \in F_2$. Theorem 1($e$) shows that

$$\int_{E_2} \{c(r, t) - [f_1^*(r) - f_1^*(t)]\} g_0(ds) = 0 \quad (s = (r, t)).$$

Thus $f^*_1(r) - f^*_1(t) = c(r, t)$ except on sets of $g_0$-measure zero. This shows that the condition is necessary. The proof that the condition is sufficient is straightforward.

For other problems which can be expressed as programmes in paired spaces the reader may refer to (**1**, chapters 4, 5, 6, 7, 11, and 12; **2**, Chapters 6 and 7; **8**; **10**; **15**; **21**, pp. 105-126; **22**, chapter 2; **24**).

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