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REFINEMENT CONDITIONS ON OPERATIONS IN SAMPLE SPACES

ROBERT J. WEAVER

1. Introduction. The recent study of operational statistics (see [**2**; **4**; **5**; **10**; **11**; **12**; **13**; **14**; **15**; **16**]) describes a generalized sample space which represents the set of all possible outcomes of a collection of coherently related operations (experiments). This approach to probability generalizes the classical notion of a sample space due to A. N. Kolmogorov [**8**], and it gives the concept somewhat wider applicability. For instance in [**4**] and [**14**], D. J. Foulis and C. H. Randall set out the start of a program wherein a generalized sample space (hereafter called a GSS) and its affiliated partially ordered set of generalized propositions could be a framework within which a genuinely operational interpretation of the so called "logic" of quantum mechanical systems may be found. In particular, in the GSS, as in quantum mechanics, there need not exist any single operation which simultaneously "refines" all other operations available in the system (as does the "grand canonical operation" of Kolmogorov).

In Theorem 20 of [**14**] the notion of refinement among operations in a GSS is given and is shown to imply a number of other conditions, each of which has a fairly appealing intuitive interpretation. Furthermore, in the classical case (of Kolmogorov) these conditions are mutually equivalent and coincide with the usual meaning of refinement among operations. It is our purpose in this paper to explore and set forth precisely those properties of operations which will make these "refinement-like" notions mutually equivalent. The local conditions which we find become conditions on the entire GSS when required of all operations, and in this global sense, the properties have particularly interesting "visible" consequences as will be seen in Theorems 6 and 12. Before proceeding with our discussion of the conditions, we introduce some definitions.

The basic structure on which a GSS is constructed is the orthogonality space. An *orthogonality space*, $(X, \perp)$, consists of a nonempty set X equipped with a symmetric binary relation $\perp$ (called *orthogonality*) which satisfies the condition that if $x, y \in X$ and $x \perp y$, then $x \neq y$ (anti-reflexivity). Now borrowing some more notation from Hilbert space (the nonzero vectors of which furnish us with an important example) we define, for $W \subset X$, $W^\perp = \{x \in X | x \perp w$ for all w in $W\}$, $W^{\perp\perp} = (W^\perp)^\perp$, and so forth. If $x \in X$, then we write simply $x^\perp$ for $\{x\}^\perp$. For $M \subset X$, let $M^u = \bigcup\{m^{\perp\perp} | m \in M\}$. Also, if $W \subset X$, then $W \subset W^u \subset W^{\perp\perp}$; $M^\perp \subset W^\perp$ whenever $W \subset M$; and $W^{\perp\perp\perp} = W^\perp$. A set $V \subset X$ is said to be *orthogonal* provided that distinct elements of V are pairwise

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orthogonal. Let

$$\mathscr{O}(X, \perp) = \{V \subset X | V \text{ is orthogonal}\}.$$

With this background we are ready for the following definition due to Foulis and Randall.

1. *Definition. A* generalized sample space (GSS) *consists of a triple* $(X, \perp, \mathscr{A})$ *where* $(X, \perp)$ *is an orthogonality space and* $\mathscr{A} \subset \mathscr{O}(X, \perp)$ *satisfies the following three conditions:*

(i) $E \in \mathscr{A} \Rightarrow E^{\perp} = \emptyset$,

(ii) $\bigcup \mathscr{A} = X$,

(iii) *if* $A \subset E \in \mathscr{A}$, $B \subset F \in \mathscr{A}$, *and* $A \subset B^{\perp}$, *then there exists* $G \in \mathscr{A}$ *such that* $A \cup B \subset G$. (*This is called the* coherency condition.)

We call $\mathscr{A}$ the *manual of admissible operations*, and X is interpreted as the *outcome set* for the entire experimental procedure being depicted. As is customary, the actual physical operations for the procedure are identified with their outcome sets, the elements (sets) in the manual $\mathscr{A}$. These operations are orthogonal sets which (by (i)) are maximal as such with respect to set inclusion. It is easy to see that if X is a finite set, all maximal orthogonal sets are operations by the coherency condition. If $D \subset X$ and there exists $E \in \mathscr{A}$ such that $D \subset E$, then D is called an *admissible event*. The set of all events is denoted $\mathscr{O}(X, \perp, \mathscr{A})$. It is clear that $\mathscr{O}(X, \perp, \mathscr{A}) \subset \mathscr{O}(X, \perp)$. If $x, y \in X$ and $x \perp y$, then $x \neq y$, and (by (ii) and (iii)), there exists an operation $E \in \mathscr{A}$ such that $x, y \in E$. We interpret this orthogonality then to mean that x "operationally rejects" y. If $D \in \mathscr{O}(X, \perp, \mathscr{A})$, then $D^{\perp}$ is the set of all outcomes which *reject* the event D, whereas D is said to be *confirmed* by the outcomes in $D^{\perp\perp}$. An ordered pair of the form $(D^{\perp\perp}, D^{\perp})$ is called an *operational proposition* for the sample space, and the collection of operational propositions, ordered by set inclusion of the confirming sets, is called the *logic* of the space. The logic, then, is a partially ordered set (see [**1**]) where the order relation corresponds to implication of propositions. Let $x \in X$ and $E \in \mathscr{A}$. Then we say that E *tests* $\{x\}$ if $E \subset x^{\perp\perp} \cup x^{\perp}$. Thus E tests a singleton event $\{x\}$ if every outcome in E either confirms or refutes the outcome x. More generally, E *tests* a proposition $(D^{\perp\perp}, D^{\perp})$ if $E \subset D^{\perp\perp} \cup D^{\perp}$. Hence any outcome in E must either confirm or reject the event D. A detailed and interesting discussion of the physical interpretation of the GSS may be found in [**4**; **5**; **10**; **11**; **12**; **13**; **14**].

An *orthogonality diagram* may represent a finite orthogonality space $(X, \perp)$ as a graph whose vertices represent the elements of X while the orthogonality of two elements is depicted by an edge connecting the corresponding vertices. Note that such a graph is undirected since $\perp$ is symmetric, and it has no loops (edges which begin and end at the same vertex) since $\perp$ is antireflexive. Figure 1 shows three examples of orthogonality diagrams. Occasionally such diagrams give a partial, though useful, representation of infinite spaces as well.

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The sample space in Figure 1 (i) will be called a *hook* while that in (ii) will be a *house*.

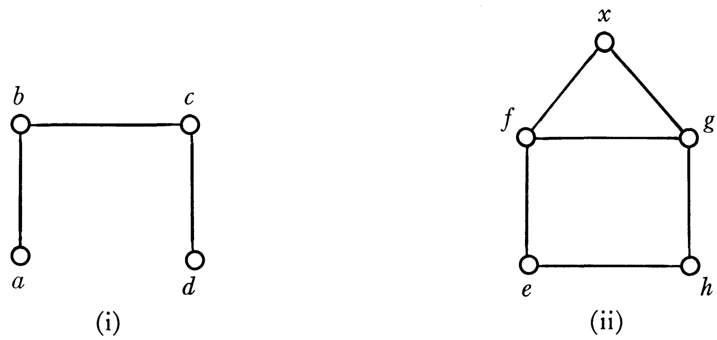

FIGURE 1

Finally we shall need to consider subsets of the outcome set considered as a sample space. Suppose $(X, \perp, \mathscr{A})$ is a GSS, $Y \subset X$, $\#$ is $\perp$ restricted to Y, and $\mathscr{B}$ is the collection of all maximal $\#$-orthogonal subsets of Y. Then $(Y, \#, \mathscr{B})$ is a sample space in its own right called the *completely coherent sample space induced on the subset* Y, hereafter called simply a *subset space*.

2. Refinement and coarsening of operations. Frequently we are interested in two operations E and F which are related in such a way that all of the information obtainable with the execution of E can be gotten also by carrying out F. More precisely, we say that operation F is a *refinement* of E (or E is a *coarsening* of F), denoted $E \prec F$, if and only if for each e in E, there exists a set $F_e \subset F$ such that $e^{\perp\perp} = F_e^{\perp\perp}$. Thus every outcome in E is confirmed precisely when some event contained in F is confirmed.

As an example, consider the operation E of throwing a die, observing the number of dots on its upper face, and recording whether that number is even or odd: $E = \{\text{even, odd}\}$. Let F be carried out in the same way except that the outcome recorded is the actual number of dots observed: $F = \{1, 2, 3, 4, 5, 6\}$. Now let us add to the manual operations of observing the throw of the die and recording the appropriate symbols as follows: $G = \{1, 2, 3, \text{even}\}$, $H = \{2, 4, 6, \text{odd}\}$. This is done so that we may reasonably claim that, for instance, the appearance of outcome "3" rejects the outcome "even". Now arising from manual $\mathscr{A} = \{E, F, G, H\}$ we have a sample space whose orthogonality diagram is shown in Figure 2. Certainly $E \prec G$, $G \prec F$, $E \prec H$, $H \prec F$, and $E \prec F$.

In [14, Theorem 20], mentioned earlier, Foulis and Randall list seven conditions on pairs of operations, conditions either equivalent to or weaker than the relation $E \prec F$. In particular, of the seven, the following implications are shown to occur: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Rightarrow$ (vii), and (i) is the

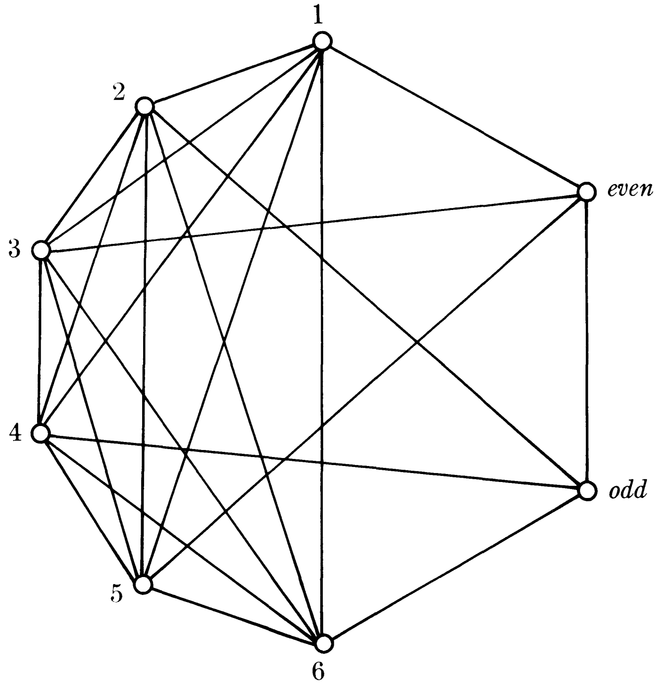

FIGURE 2

same as our coarsening condition. We shall choose one condition from each equivalence class, say (i), (v), and (vii), calling them henceforth respectively, (a), (b), and (c). We express these in the following using slightly different notations from their counterparts in [**14**].

2. *Refinement conditions. Let* $(X, \perp, \mathscr{A})$ *be a* GSS *and suppose* $E, F \in \mathscr{A}$. *Consider these conditions:*

(a) $E \prec F$.

(b) $e \in E \Rightarrow F$ *tests* $\{e\}$.

(c) $e \in E \Rightarrow$ *there exists* $f \in F$ *such that* $f \in e^{\perp\perp}$ (*i.e., each element of* E *is confirmed by some element of* F).

Then $(a) \Rightarrow (b) \Rightarrow (c)$ and, as we shall see, the reverse implications do not in general hold. In the example given above, however, all of these conditions are equivalent for all pairs of operations. Furthermore, by [**14**] we know that (b) is equivalent to our condition

$(b)'$ $f \in F \Rightarrow$ *there exists e in* E *such that* $f \in e^{\perp\perp}$.

This says that every element of F confirms some element of E. Note in Figure 1 (i) (the "hook") that $E = \{a, b\}$ and $F = \{b, c\}$ satisfy (b) but not (a). Furthermore in the "house" (Figure 1 (ii)), $E = \{e, f\}$, $F = \{x, f, g\}$ obey (c) and not (b). We shall pay special attention to these examples in what follows.

3. Weak Dacey sample spaces. J. C. Dacey [**2**] has described those GSS's whose logics are orthomodular posets (see [**1**; **3**; **6**]). A special significance in this situation is that it identifies in the logic of a *GSS* an important property almost always taken to hold in the "logic" of quantum mechanics (see [**7**; **9**]). Such *GSS*'s have come to be known as *Dacey sample spaces*, or just *D-spaces* [**4**]. In particular $(X, \perp, \mathscr{A})$ is a D-space if and only if $a, b \in X$, $E \in \mathscr{A}$, and $E \subset a^\perp \cup b^\perp$ together imply that $a \perp b$. We will show that a condition slightly weaker than this "D" property will be necessary and sufficient to insure that 2(a) is equivalent to 2(b).

3. *Definition. Let* $(X, \perp, \mathscr{A})$ *be a* GSS.

(*a*) *An operation* $F \in \mathscr{A}$ *is said to be a* weak operation *if whenever* $a, b \in X$ *where* $F \subset a^\perp \cup b^\perp$ *and* F *tests either* a *or* b, *we have* $a \perp b$.

(*b*) $(X, \perp, \mathscr{A})$ *is a* weak Dacey sample space (WDSS) *if every operation* $F \in \mathscr{A}$ *is weak.*

4. Theorem. *Let* $F \in \mathscr{A}$ *where* $(X, \perp, \mathscr{A})$ *is a* GSS. *Then a necessary and sufficient condition that* 2(a) *and* 2(b) *be equivalent for* F *and each* E *in* $\mathscr{A}$ *is that* F *is a weak operation.*

Proof. Suppose $F \in \mathscr{A}$ and 2(a),(b) are mutually equivalent for F and each E in $\mathscr{A}$. We must show that F is a weak operation. Suppose $F \subset a^\perp \cup b^\perp$ and that F tests $\{a\}$. Let

$$D = (F \cap a^\perp) \cup \{a\} \in \mathscr{O}(X, \perp, \mathscr{A}).$$

(Here we have used the coherency condition.) Then there exists E in $\mathscr{A}$ such that $D \subset E$, and with respect to E and F it can easily be seen that 2(b)′ holds since F tests $\{a\}$. Hence 2(a) holds by hypothesis. Therefore there exists $F_a \subset F$ such that $F_a^{\perp\perp} = a^{\perp\perp}$. But since F tests $\{a\}$, we have $F_a = F \cap a^{\perp\perp}$. Now F was chosen so that $F \subset a^\perp \cup b^\perp$, so $F \cap a^{\perp\perp} \subset b^\perp$. Thus $b \in b^{\perp\perp} \subset (F \cap a^{\perp\perp})^\perp = a^\perp$; that is, $a \perp b$. Hence F is a weak operation.

Conversely, suppose F is a weak operation and $E \in \mathscr{A}$ where E and F satisfy 2(b). Suppose $e \in E$, and let $F_e = F \cap e^{\perp\perp}$. Then $F_e \subset e^{\perp\perp}$, so $F_e^{\perp\perp} \subset e^{\perp\perp}$. We must show that the reverse containment holds. If $F \subset e^{\perp\perp}$, then $F_e = F$ and we are done. Thus we may suppose that an element $f \in F \backslash e^{\perp\perp}$ exists. Now if $F_e^{\perp\perp} \neq e^{\perp\perp}$, then $e^\perp \subsetneq F_e^\perp$ and we may choose $a \in F_e^\perp \backslash e^\perp$. Then by 2(b)′ there exists $e_1 \in E$ such that $f \in e_1^{\perp\perp} \subset e^\perp$, so that F clearly tests $\{e\}$. Also $F = F_e \cup (F \backslash e^{\perp\perp}) \subset a^\perp \cup e^\perp$. Since F is a weak operation, necessarily $a \in e^\perp$, a contradiction to the choice of a. Hence 2(b) $\Rightarrow$ 2(a) for E, F.

5. Corollary. $(X, \perp, \mathscr{G})$ *is a* WDSS *if and only if for all* $E, F \in \mathscr{A}$, 2(a) *and* 2(b) *are equivalent.*

If a GSS together with all of its subset spaces is a D-space, then the space is called a *hereditary Dacey space* (*HD-space*). Dacey has shown that a GSS is an *HD*-space if and only if it contains no subset space which is in the form of a hook (Figure 1 (i)). Furthermore [**14**, Theorem 20] asserts that if $(X, \perp, \mathscr{A})$ is a D-space, then 2(b) is equivalent to 2(a). Our next theorem shows us that the

"*HD*" property is exactly what we need to give us a "hereditary" property for WDSS's.

6. THEOREM. $(X, \perp, \mathscr{A})$, *together with all of its subset spaces is a* WDSS *if and only if* $(X, \perp, \mathscr{A})$ *is an HD-space.*

Proof. If X and all of its subset spaces are WDSS's, then no subset space of X may be a hook (since a hook is not a WDSS). Conversely, if X contains no subset space in the form of a hook, then $(X, \perp, \mathscr{A})$ is an HD-space, so that X and all of its subset spaces are D-spaces, hence WDSS's.

The orthogonality diagram in Figure 3 shows a GSS which is a WDSS but has a subset space $\{a, c, d, e\}$ which is not.

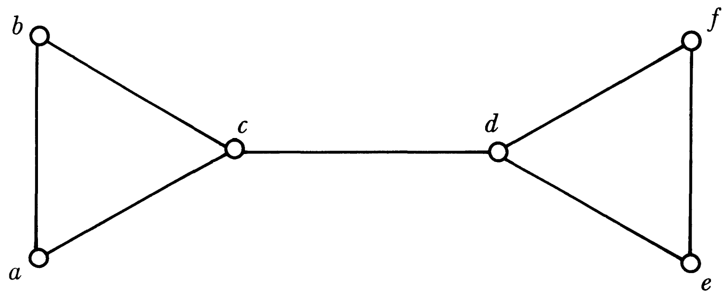

FIGURE 3

4. Operation inclusive sample spaces. We turn our attention to the equivalence of 2(b) and 2(c).

7. *Definition. Let* $(X, \perp, \mathscr{A})$ *be a* GSS.

(*a*) *If* $E \in \mathscr{A}$ *and* $D \in \mathscr{O}(X, \perp, \mathscr{A})$, *then we say that D* partially refines E *provided for each e in E,* $D \cap e^{\perp\perp} \neq \emptyset$. (That is, every outcome in E is confirmed by some outcome in the event D.)

(*b*) $E \in \mathscr{A}$ *is* inclusive *if for every event* $D \in \mathscr{O}(X, \perp, \mathscr{A})$ *where D partially refines E, we have* $D \subset E^u$.

(*c*) $(X, \perp, \mathscr{A})$ *is an* operation inclusive sample space (OISS) *if every operation* $E \in \mathscr{A}$ *is inclusive.*

(*d*) $(X, \perp, \mathscr{A})$ *is a* hereditary OISS (HOISS) *provided* $(X, \perp, \mathscr{A})$ *together with each of its subset spaces is an* IOSS.

If $E \in \mathscr{A}$ is not inclusive, then there exists an event D which is a partial refinement of E and an element x in D where $x \notin E^u$. In this case we say that E is *not inclusive by x*, and we observe that for all e in E, $x^\perp \cap e^{\perp\perp} \neq \emptyset$.

8. THEOREM. *Let* $E \in \mathscr{A}$ *where* $(X, \perp, \mathscr{A})$ *is a* GSS. *Then a necessary and sufficient condition that* 2(b) *and* 2(c) *be equivalent for E and each* $F \in \mathscr{A}$ *is that E is inclusive.*

Proof. Suppose $E \in \mathscr{A}$ and 2(b) and (c) are mutually equivalent for E and each $F \in \mathscr{A}$. Now let $D \in \mathscr{O}(X, \perp, \mathscr{A})$ where D partially refines E. We must show that $D \subset E^u$ in order to show E inclusive. For each $e \in E$, set

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$D_e = D \cap e^{\perp\perp} \neq \emptyset$. If $D \not\subseteq E^u$, then there exists $d \in D$ such that $d \notin E^u$. Let

$$D_1 = \{d\} \cup [\bigcup\{D_e | e \in E\}].$$

Then $D_1 \in \mathscr{O}(X, \perp, \mathscr{A})$ so by definition we may extend D_1 to an operation $F \in \mathscr{A}$ where $D_1 \subset F$. Now E, F satisfy 2(c) and so, by hypothesis, 2(b) also. But 2(b)$'$, which also holds here, says that $F \subset E^u$. Thus we have $d \in D_1 \subset F \subset E^u$, but d was originally chosen outside of E^u, a contradiction. Thus E is an inclusive operation.

Conversely, suppose E is inclusive, and let $F \in \mathscr{A}$ where E, F satisfy 2(c). Then for all $e \in E$, $F \cap e^{\perp\perp} \neq \emptyset$ so that F is a partial refinement of E. (F being an operation is certainly an event.) Hence by hypothesis, $F \subset E^u$, and so each f in F confirms some element of E; i.e., 2(b)$'$ holds, which is equivalent to 2(b).

9. Corollary. *$(X, \perp, \mathscr{A})$ is an* OISS *if and only if for all E, $F \in \mathscr{A}$,* 2(b) *and* 2(c) *are equivalent.*

In analogy with our discussion of weak D-spaces, the next objective is to find a condition on an OISS $(X, \perp, \mathscr{A})$ so that each of its subset spaces will also be an OISS.

10. Lemma. *Suppose $(X, \perp, \mathscr{A})$ contains no subset space in the form of a house (Figure* 1 (ii)) *and that $E \in \mathscr{A}$ is not inclusive by x. Then either $E \cap x^\perp = \emptyset$ or else there exists a unique point g in E such that $E \backslash \{g\} \subset x^\perp$.*

Proof. Suppose $x \cap E \neq \emptyset$ and for each g in E, $E\backslash\{g\} \not\subset x^\perp$. Choose $e \in E \cap x^\perp$. Now $E \not\subset x^\perp$, so we may choose a particular $g \in E$ so that $g \notin x^\perp$. Then $e \neq g$ so that $e \perp g$. Hence $g^{\perp\perp} \subset e^\perp$. Now $x^\perp \cap g^{\perp\perp} \neq \emptyset$ since E is not inclusive by x. Thus choosing $z \in x^\perp \cap g^{\perp\perp}$, we have $z \in e^\perp$ and $z \notin g^\perp$. Let $h \in E\backslash\{g\}$ where $h \notin x^\perp$ (by assumption such an h exists). Then $\{h, e, g\} \in \mathscr{O}(X, \perp, \mathscr{A})$ and $z \in g^{\perp\perp} \subset h^\perp$. Now if we let $b \in h^{\perp\perp} \cap x^\perp \neq \emptyset$, then $b \in \{x, e, g, z\}^\perp$ and $b \notin h^\perp$. Thus we have constructed the orthogonality diagram in Figure 4 (where it is known that only those orthogonalities which are shown by connecting edges exist among the given points). But here the subset space $\{x, z, h, g, b\}$ is in the form of a house, a contradiction. Hence, such

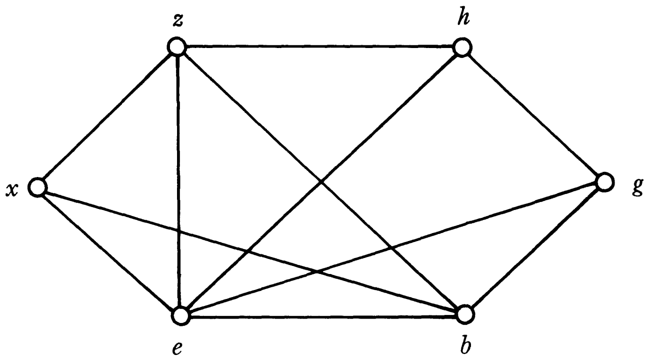

Figure 4

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a g as mentioned in the statement of the lemma must exist. Clearly no more than one such element can have this property since otherwise we would have that $x \in E^{\perp} = \emptyset$, a contradiction. Hence the conclusion holds.

11. THEOREM. *If a* GSS $(x, \perp, \mathscr{A})$ *contains no subset space in the form of a house, then* $(X, \perp, \mathscr{A})$ *is an* OISS.

Proof. Assuming that $(X, \perp, \mathscr{A})$ is not an OISS yet contains no house, there is an $E \in \mathscr{A}$ which is not inclusive by some element $x \in X$. By (10) there are two cases, each of which leads rather easily to the construction of a forbidden house as a subset space. This contradiction completes the proof.

Note that a house is not itself an OISS since, in Figure 1 (ii), $\{e, f\}$, $\{e, h\}$, and $\{h, g\}$ are each operations which are not inclusive by x (and the event $\{x, f, g\}$).

12. THEOREM. $(X, \perp, \mathscr{A})$ *is an* HOISS *if and only if* $(X, \perp, \mathscr{A})$ *contains no subset space in the form of a house.*

Proof. If $(X, \perp, \mathscr{A})$ is an HOISS, yet contains a house, then that house is not an OISS, a contradiction to definition 7(d). On the other hand, suppose $(X, \perp, \mathscr{A})$ contains no house and is not an HOISS. Then there exists a subset space $(Y, \#, \mathscr{B})$ of $(X, \perp, \mathscr{A})$ which is not an OISS. Thus, by (11), $(Y, \#, \mathscr{B})$ contains a subset space $(H, b, \mathscr{G})$ which is a house. But then $(H, b, \mathscr{G})$ is a subset space of $(X, \perp, \mathscr{A})$ also, a contradiction.

We conclude with an example of an OISS which is clearly not an HOISS. Its orthogonality diagram is given in Figure 5. This example has other interesting properties which have been noticed by M. F. Janowitz and reported in [**14**].

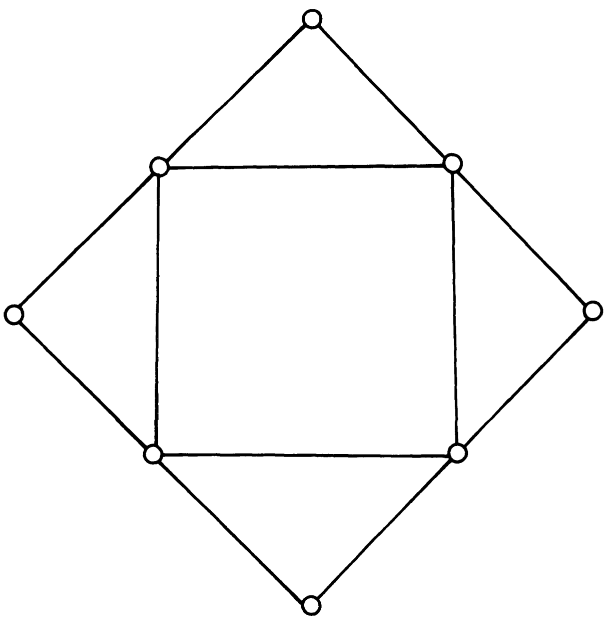

FIGURE 5

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Mount Holyoke College,
South Hadley, Massachusetts