# ON CONFORMALLY RECURRENT SPACES OF SECOND ORDER

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## Introduction

In a recent paper [1] Adati and Miyazawa studied conformally recurrent spaces, that is, Riemannian spaces defined by $C^h_{ijk,l} = \lambda_l C^h_{ijk}$ where $C^h_{ijk}$ is the conformal curvature tensor:

$$(1)\qquad \begin{aligned} C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}\,(R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j) \\ + \frac{R}{(n-1)(n-2)}\,(\delta^h_k g_{ij} - \delta^h_j g_{ik}), \end{aligned}$$

$\lambda_l$ is a non-zero vector and comma denotes covariant differentiation with respect to the metric tensor $g_{ij}$. The present paper is concerned with non-flat Riemannian spaces $V_n (n > 3)$ defined by

$$(2)\qquad C^h_{ijk,lm} = a_{lm} C^h_{ijk}$$

where $a_{lm}$ is a tensor not identically zero. We shall call a Riemannian space defined by (2) a conformally recurrent space of second order and shall denote an $n$-space of this kind by $C(^2K_n)$. A Riemannian space whose curvature tensor satisfies $R^h_{ijk,lm} = \bar{a}_{lm} R^h_{ijk}$ was called a Recurrent space of second order by A. Lichnerowicz [2]. Such an $n$-space shall be denoted by $^2K_n$. Evidently every $^2K_n$ is a $C(^2K_n)$ but the converse is not necessarily true. Sections 2 and 3 of this paper deal with Einstein and 2-Ricci-recurrent $C(^2K_n)$ respectively while section 4 deals with $C(^2K_n)$ admitting a parallel vector field. In the last section it will be shown that a Riemannian space satisfying $W^h_{ijk,lm} = a'_{lm} W^h_{ijk}$ where $W^h_{ijk}$ is Weyl's projective curvature tensor is a $C(^2K_n)$.

## 1. Tensor of recurrence in a $C(^2K_n)$

We have

$$(C_{hijk} C^{hijk})_{,lm} = 2C_{hijk\ lm} C^{hijk} + 2C^{hijk}{}_{,l} C_{hijk,m}$$

Therefore, in a $C(^2K_n)$



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$$(C_{hijk}C^{hijk})_{,lm} = 2a_{lm}C_{hijk}C^{hijk} + 2C^{hijk}{}_{,l}C_{hijk,m}$$

Hence

$$2(a_{lm}-a_{ml})C_{hijk}C^{hijk} = 0$$

So either

(i) $C_{hijk}C^{hijk} = 0$ or

(ii) $a_{lm} = a_{ml}$

If the space is of positive definite metric and not conformally flat, then (i) cannot hold and therefore $a_{lm}$ is symmetric.

Again, if in a $C(^2K_n)$, $R_{ij} = 0$, then from (1) and (2) it follows that $R^h_{ijk,lm} = a_{lm}R^h_{ijk}$, that is, the space is a $^2K_n$. It is already known that for a $^2K_n$ the tensor of recurrence is symmetric. Hence if for a $C(^2K_n)$, $R_{ij} = 0$, then its tensor of recurrence is symmetric. We can therefore state the following theorems:

THEOREM 1. *If a $C(^2K_n)$ with positive definite metric is not conformally flat, then its tensor of recurrence is symmetric.*

THEOREM 2. *If for a $C(^2K_n)$ the Ricci tensor is a zero tensor, then its tensor of recurrence is symmetric.*

## 2. Einstein $C(^2K_n)$

If a $C(^2K_n)$ is an Einstein space, defined by $R_{ij} = (R/n)g_{ij}$, then

$$R_{ij,lm} = 0 \tag{2.1}$$

Let us suppose that an Einstein $C(^2K_n)$ is a $^2K_n$. Then $R^h_{ijk,lm} = d_{lm}R^h_{ijk}$ for a non-zero tensor $d_{lm}$. Consequently $R_{ij,lm} = d_{lm}R_{ij}$. Therefore in virtue of (2.1) $R_{ij} = 0$, because $d_{lm} \neq 0$. Hence $R = 0$.

Again, if in an Einstein $C(^2K_n)$, $R = 0$ then $R_{ij} = 0$ and therefore the space is a $^2K_n$. In an Einstein $C(^2K_n)$ of zero scalar curvature

$$R^h_{ijk,lm} = a_{lm}R^h_{ijk} \tag{2.2}$$

Making use of (2.2) and the Bianchi identity we get

$$a_{lm}R^h_{ijk} + a_{jm}R^h_{ikl} + a_{km}R^h_{ilj} = 0 \tag{2.3}$$

Multiplying (2.3) by $a^l_t$ where $a^l_t = g^{lp}a_{pt}$ we have

$$a^l_t a_{lm}R^h_{ijk} + a^l_t a_{jm}R^h_{ikl} + a^l_t a_{km}R^h_{ilj} = 0 \tag{2.4}$$

$R_{ij} = 0$ implies $a_{hm}R^h_{ijk} = 0$ by contracting $h$ and $k$ in (2.3). Hence $a^t_m R^p_{kit} = 0$. Using this (2.4) reduces to $a^l_t a_{lm}R^h_{ijk} = 0$. Since the space is not flat, $a^l_t a_{lm} = 0$. Thus we have the following theorems:

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THEOREM 3. *A necessary and sufficient condition that an Einstein $C(^2K_n)$ may be a $^2K_n$ is that its scalar curvature is zero.*

THEOREM 4. *In an Einstein $C(^2K_n)$ of zero scalar curvature $a^l_i a_{lm} = 0$.*

We now consider an Einstein $C(^2K_n)$ of non-zero scalar curvature. From (2.1) as well as (1) and (2) we have

$$R_{hijk,lm} = a_{lm} T_{hijk} \tag{2.5}$$

where

$$T_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{hk}g_{ij} - g_{hj}g_{ik})$$

Using (2.5) Walker's Lemma 1 [3], namely

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0 \tag{2.6}$$

reduces to

$$b_{lm} T_{hijk} + b_{hi} T_{jklm} + b_{jk} T_{lmhi} = 0 \tag{2.7}$$

where

$$b_{lm} = a_{lm} - a_{ml}. \tag{2.8}$$

Since $T_{hijk} = T_{jkhi}$, by Walker's Lemma 2 [3] we have from (2.7) either $b_{lm} = 0$ or $T_{hijk} = 0$. Hence we have the following theorem:

THEOREM 5. *If a $C(^2K_n)$ is an Einstein space of non-zero scalar curvature, then either its tensor of recurrence is symmetric or it is a space of constant curvature.*

## 3. 2-Ricci-recurrent $C(^2K_n)$

In a previous paper [4] we called a non-flat Riemannian space a Ricci-recurrent space of second order, or briefly a 2-Ricci-recurrent space if its Ricci tensor satisfies

$$R_{ij,kl} = a^*_{kl} R_{ij} \tag{3.1}$$

and $R_{ij} \not\equiv 0$ for a non-zero tensor $a^*_{kl}$.

We put

$$\Pi_{ij} = \frac{1}{n-2} \left[ R_{ij} - \frac{R}{2(n-1)} g_{ij} \right] \tag{3.2}$$

and

$$D^h_{ijk} = \Pi^h_k g_{ij} - \Pi^h_j g_{ik} + \Pi_{ij}\delta^h_k - \Pi_{ik}\delta^h_j \tag{3.3}$$

where $\Pi^h_k = g^{ht}\Pi_{tk}$. Then

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(3.4) $$C^h_{ijk} = R^h_{ijk} - D^h_{ijk}$$

Moreover,

(3.5) $$\Pi = g^{ij}\Pi_{ij} = \frac{R}{2(n-1)}$$

and

$$D_{hijk} = -D_{ihjk} = -D_{hikj} = D_{jkhi}$$

where $D_{hijk} = g_{ht}D^t_{ijk}$. Let us suppose that $\Pi_{ij}$ is a non-zero tensor satisfying

(3.6) $$\Pi_{ij,kl} = a^*_{kl}\Pi_{ij}$$

where $a^*_{kl}$ is a non-zero tensor. Then $\Pi_{,kl} = a^*_{kl}\Pi$ or,

(3.7) $$R_{,kl} = a^*_{kl}R$$

From (3.2) we have

$$\Pi_{ij,kl} = \frac{1}{n-2}\left[R_{ij,kl} - \frac{R_{,kl}}{2(n-1)}g_{ij}\right].$$

Therefore

$$\frac{1}{n-2}R_{ij,kl} = a^*_{kl}\left[\Pi_{ij} + \frac{R}{2(n-1)(n-2)}g_{ij}\right] = a^*_{kl}\frac{1}{n-2}R_{ij}$$

Hence

(3.8) $$R_{ij,kl} = a^*_{kl}R_{ij}$$

Conversely, if (3.8) holds, then

$$\Pi_{ij,kl} = a^*_{kl}\Pi_{ij}$$

We can therefore state the following lemma:

LEMMA. *If in a Riemannian space, the tensor $\Pi_{ij}$, defined by* (3.2), *is a non-zero tensor, then the space is 2-Ricci-recurrent if and only if $\Pi_{ij,kl} = a^*_{kl}\Pi_{ij}$ for a non-zero tensor $a^*_{kl}$.*

We now suppose that in a $C(^2K_n)$, (3.6) holds. Differentiating (3.3) covariantly we have

$$D^h_{ijk,lm} = \Pi^h_{k,lm}g_{ij} - \Pi^h_{j,lm}g_{ik} + \Pi_{ij,lm}\delta^h_k - \Pi_{ik,lm}\delta^h_j$$

Using (3.6) we get

(3.9) $$D^h_{ijk,lm} = a^*_{lm}D^h_{ijk}.$$

From (3.4) we have

$$\begin{aligned} R^h_{ijk,lm} &= C^h_{ijk,lm} + D^h_{ijk,lm} \\ &= a_{lm}C^h_{ijk} + a^*_{lm}D^h_{ijk}. \end{aligned}$$

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Therefore

$$R_{hijk,lm} - R_{hijk,ml} = b_{lm} C_{hijk} + c'_{lm} D_{hijk} \tag{3.10}$$

where $b_{lm}$ is given by (2.8) and $c'_{lm} = a^*_{lm} - a^*_{ml}$. Now using (3.10) Walker's lemma (2.6) can be written as

$$\begin{aligned} b_{lm} C_{hijk} + b_{hi} C_{jklm} + b_{jk} C_{lmhi} + c'_{lm} D_{hijk} \\ + c'_{hi} D_{jklm} + c'_{jk} D_{lmhi} = 0. \end{aligned} \tag{3.11}$$

Let us suppose that $a_{lm}$ is symmetric. Then $b_{lm} = 0$. Hence (3.11) reduces to

$$c'_{lm} D_{hijk} + c'_{hi} D_{jklm} + c'_{jk} D_{lmhi} = 0. \tag{3.12}$$

Since

$$C_{ij} = 0, \quad \Pi_{ij} = \frac{1}{n-2}\left[D_{ij} - \frac{D}{2(n-1)} g_{ij}\right].$$

Hence $\Pi_{ij} \neq 0$ implies $D_{ijkl} \neq 0$. Also $D_{hijk} = D_{jkhi}$. Hence applying Walker's Lemma 2 to (3.12) we have $c'_{lm} = 0$. Hence $a^*_{lm}$ is symmetric. Next, we suppose that $a^*_{lm}$ is symmetric. Then $c'_{lm} = 0$ and it follows from (3.11) that

$$b_{lm} C_{hijk} + b_{hi} C_{jklm} + b_{jk} C_{lmhi} = 0. \tag{3.13}$$

Hence if $C_{hijk} \neq 0$, it follows from (3.13) that $b_{lm} = 0$ whence $a_{lm}$ is symmetric.

We can therefore state the following theorems:

THEOREM 6. *If a $C(^2K_n)$ which is not conformally flat is a 2-Ricci-recurrent space, then its tensor of recurrence is symmetric if and only if the tensor of 2-Ricci-recurrence is symmetric.*

THEOREM 7. *If a $C(^2K_n)$ is a 2-Ricci-recurrent space, then the tensor of 2-Ricci-recurrence is symmetric if the tensor of recurrence of $C(^2K_n)$ is so.*

## 4. $C(^2K_n)$ admitting a parallel vector field

Let us assume that there exists a parallel vector field $v^i$ in a $C(^2K_n)$. Then $v^i{}_{,l} = 0$. Therefore $v^i{}_{,lm} - v^i{}_{,ml} = 0$. Hence using the Ricci identity and the Bianchi identity we have

$$v^t R^h_{tlm} = 0, \qquad v^t R_{tl} = 0$$
$$v^t R^h_{tlm,n} = 0, \qquad v^t R_{tl,n} = 0.$$

Therefore,

$$v^t R^h_{ijk,t} = 0, \qquad v^t R_{ij,t} = 0, \qquad v^t R_{,t} = 0. \tag{4.1}$$

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From (2) we have $v^l C^h_{ijk,lm} = v^l a_{lm} C^h_{ijk}$ or,

$$v^l \Big[ R^h_{ijk,lm} - \frac{1}{n-2}(R^h_{k,lm} g_{ij} - R^h_{j,lm} g_{ik} + R_{ij,lm}\delta^h_k - R_{ik,lm}\delta^h_j) + \frac{1}{(n-1)(n-2)} R_{,lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \Big] = v^l a_{lm} C^h_{ijk}. \tag{4.2}$$

Using (4.1) the left hand side of (4.2) reduces to zero. Hence $v^l a_{lm} C^h_{ijk} = 0$.

Thus we have the following theorem:

THEOREM 8. *If a* $C(^2K_n)$ *admits a parallel vector field* $v^i$, *then either the space is conformally flat or* $v^l a_{lm} = 0$.

## 5. Projective recurrent spaces of second order

A. Riemannian space $V_n$ $(n \geqq 3)$ satisfying

$$W^h_{ijk,lm} = a'_{lm} W^h_{ijk} \tag{5.1}$$

for a non-zero tensor $a'_{lm}$ where $W^h_{ijk}$ is Weyl's projective curvature tensor

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik}) \tag{5.2}$$

shall be called a projective recurrent space of second order.

From (5.2) we have

$$W^h_{ijk,lm} = R^h_{ijk,lm} - \frac{1}{n-1}(\delta^h_k R_{ij,lm} - \delta^h_j R_{ik,lm}). \tag{5.3}$$

Substituting (5.2) and (5.3) in (5.1) we get

$$R^h_{ijk,lm} - \frac{1}{n-1}(\delta^h_k R_{ij,lm} - \delta^h_j R_{ik,lm}) = a'_{lm}\Big[R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik})\Big]. \tag{5.4}$$

Therefore,

$$R^h_{k,lm} = a'_{lm} R^h_k + \frac{1}{n}(R_{,lm} - a'_{lm} R)\delta^h_k \tag{5.5}$$

and

$$R_{ij,lm} = a'_{lm} R_{ij} + \frac{1}{n}(R_{,lm} - a'_{lm} R) g_{ij}. \tag{5.6}$$

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Again from (1) we have

$$(5.7)\quad \begin{aligned} C^h_{ijk,lm} = R^h_{ijk,lm} - \frac{1}{n-2}(R^h_{k,lm}g_{ij} - R^h_{j,lm}g_{ik} \\ + R_{ij,lm}\delta^h_k - R_{ik,lm}\delta^h_j) + \frac{1}{(n-1)(n-2)} R_{,lm}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \end{aligned}$$

Making use of (5.4), (5.5) and (5.6) we have from (5.7)

$$C^h_{ijk,lm} = a'_{lm} C^h_{ijk}.$$

Hence we have the following theorem:

THEOREM 9. *Every n-dimensional* $(n > 3)$ *projective recurrent space of second order is a* $C(^2K_n)$.

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