INTEGRALS ALLIED TO AIRY'S INTEGRALS

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1. Introductory. Airy's integrals

$$\int_0^\infty \cos\,(\lambda^3 \pm x\lambda)\,d\lambda$$

can be expressed ([1], [2]) in terms of Bessel functions. In this paper integrals of the types

$$\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}\,(\lambda^n \pm x\lambda^l)\lambda^{k-1}\,d\lambda$$

are discussed. Various subsidiary formulae are given in § 2, some integrals of the type

$$\int_0^\infty \exp\,(-\lambda^n) f(\lambda^l)\lambda^{k-1}\,d\lambda$$

are evaluated in § 3, and from these the integrals of the Airy type are derived in § 4.

2. Formulae required in the proof. The first of these is the Gamma function formula

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{m}\right)\dots\,\Gamma\left(z+\frac{m-1}{m}\right)=(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{\frac{1}{2}-mz}\,\Gamma(mz). \quad \dots\dots\dots\dots\dots\dots(1)$$

The second is Ragab's formula ([2], p. 406, ex. 27, [3])

$$K_\mu(z)\,K_\nu(z)=\frac{1}{4z\sqrt{\pi}}\sum_{i,-i}\frac{1}{i}\,E\left(\frac{1+\mu+\nu}{2},\,\frac{1+\mu-\nu}{2},\,\frac{1-\mu+\nu}{2},\,\frac{1-\mu-\nu}{2}:\tfrac{1}{2}:e^{i\pi}z^2\right). \quad \dots\dots(2)$$

The following two formulae are also required.

If m is a positive integer and if $R(k)>0$, ([2], p. 406, ex. 30),

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}\,E(p\,;\;\alpha_r:q\,;\;\rho_s:z/\lambda^m)\,d\lambda=m^{k-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}\,E(p+m\,;\;\alpha_r:q\,;\;\rho_s:z/m^m), \quad \dots(3)$$

where $\alpha_{p+\nu+1}=(k+\nu)/m$, $\nu=0,\,1,\,2,\,\dots,\,m-1$.

If m is a positive integer and if $|\operatorname{amp} z|<\pi$, ([2], p. 407, ex. 32),

$$\frac{1}{2\pi i}\int e^{\zeta}\zeta^{-\rho}\,E(p\,;\;\alpha_r:q\,;\;\rho_s:\zeta^m z)\,d\zeta=m^{\frac{1}{2}-\rho}(2\pi)^{\frac{1}{2}m-\frac{1}{2}}\,E(p\,;\;\alpha_r:q+m\,;\;\rho_s:zm^m), \quad \dots\dots(4)$$

where the contour starts at $-\infty$ on the real axis, passes round the origin in the positive direction, and returns to $-\infty$, and $\rho_{q+\nu+1}=(\rho+\nu)/m$, $\nu=0,\,1,\,2,\,\dots,\,m-1$.

3. Some infinite integrals. Consider the integral

$$\int_0^\infty \exp\,(-\lambda^n+z\lambda^l)\lambda^{k-1}\,d\lambda,$$

where n and l are positive integers such that $l<n$, and $R(k)>0$. On expanding $\exp(z\lambda^l)$ in powers of z and putting $\lambda=\mu^{1/n}$, this becomes

$$\frac{1}{n}\sum_{r=0}^{\infty}\frac{z^r}{r!}\,\Gamma\left(\frac{k+rl}{n}\right),$$

and therefore

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$$\int_0^\infty \exp(-\lambda^n + z\lambda^l)\lambda^{k-1}d\lambda$$

$$=\frac{1}{n}\sum_{t=0}^{n-1}\Gamma\left(\frac{k+tl}{n}\right)\frac{z^t}{t!}\,F\left\{\begin{matrix}\frac{k+tl}{nl},\ \frac{k+tl+n}{nl},\ \ldots,\ \frac{k+tl+(l-1)n}{nl} : l^l\left(\frac{z}{n}\right)^n \\ \frac{t+1}{n},\ \frac{t+2}{n},\ \ldots*\ldots,\ \frac{t+n}{n}\end{matrix}\right\}, \qquad \ldots\ldots\ldots(5)$$

the asterisk indicating that the parameter n/n is omitted.

Now assume that n is odd, replace z by $-1/z$, and apply (1) with $(k+tl)/(nl)$ for z and l for m and also with $(t+1)/n$ for z and n for m; then the equation can be written

$$\int_0^\infty \exp(-\lambda^n)E(::z/\lambda^l)\lambda^{k-1}d\lambda$$

$$=n^{-3/2}l^{-\frac{1}{2}+k/n}(2\pi)^{\frac{1}{2}n-\frac{1}{2}l}\sum_{t=0}^{n-1}(-nl^{-l/n}z)^{-t}\,E\left\{\begin{matrix}\frac{k+tl}{nl},\ \frac{k+tl+n}{nl},\ \ldots,\ \frac{k+tl+(l-1)n}{nl} : l^{-l}(nz)^n \\ \frac{t+1}{n},\ \frac{t+2}{n},\ \ldots*\ldots,\ \frac{t+n}{n}\end{matrix}\right\}.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

On generalising, using (3) and (4), it is found that, if n and l are positive integers such that n is odd and $l<n$, and if $R(k)>0$,

$$\exp(-\lambda^n)\,E(p;\ \alpha_r : q;\ \rho_s : z/\lambda^l)\,\lambda^{k-1}\,d\lambda$$

$$=n^{\Sigma\alpha_r-\Sigma\rho_s-\frac{1}{2}p+\frac{1}{2}q-3/2}\,l^{-\frac{1}{2}+k/n}\,(2\pi)^{(\frac{1}{2}-\frac{1}{2}n)(p-q)+\frac{1}{2}n-\frac{1}{2}l}$$

$$\times\sum_{t=0}^{n-1}(-l^{-l/n}n^{q-p+1}z)^{-t}\,E\left\{\begin{matrix}\frac{k+tl}{nl},\ \ldots,\ \frac{k+tl+(l-1)n}{nl},\ \frac{\alpha_1+t}{n},\ \ldots,\ \frac{\alpha_p+t+n-1}{n} : l^{-l}(n^{q-p+1}z)^n \\ \frac{t+1}{n},\ \ldots*\ldots,\ \frac{t+n}{n},\ \frac{\rho_1+t}{n},\ \ldots,\ \frac{\rho_q+t+n-1}{n}\end{matrix}\right\}.$$

$$\ldots\ldots(7)$$

Note. If n is even the argument of the E-function should be multiplied by $e^{\pm i\pi}$.

For example, on applying formula (2) it is found that

$$\int_0^\infty \exp(-\lambda^n)\,K_\mu(z/\lambda^l)\,K_\nu(z/\lambda^l)\lambda^{k-1}\,d\lambda$$

$$=(2\sqrt{2z})^{-1}n^{-3/2}(2l)^{-\frac{1}{2}+(k+l)/n}(2\pi)^{1-n-l}\sum_{i,-i}\frac{1}{i}\sum_{t=0}^{n-1}\{(2l)^{-2l/n}n^{-2}z^2\}^{-t}$$

$$\times E\left\{\begin{matrix}\frac{k+l+2tl}{2nl},\ \ldots,\ \frac{k+l+2tl+(2l-1)n}{2nl},\ \frac{1+\mu+\nu+2t}{2n},\ \ldots,\ \frac{1-\mu-\nu+2t+2n-2}{2n} : \frac{e^{in\pi}(z/n)^{2n}}{(2l)^{2l}} \\ \frac{t+1}{n},\ \ldots*\ldots,\ \frac{t+n}{n},\ \frac{1+2t}{2n},\ \ldots,\ \frac{1+2t+2n-2}{2n}\end{matrix}\right\}, \qquad \ldots\ldots(8)$$

where n and l are positive integers such that n is odd and $n>2l$, and $R(k)>-l$.

4. Integrals of the Airy type. In formula (5) swing the line of integration through a positive angle $\pi/(2n)$, so that λ becomes $\eta e^{i\pi/(2n)}$, and let $z=\pm x e^{i(n-l)\pi/(2n)}$, where x is real and positive; then

$$\int_0^\infty \exp\,(-i\eta^n \pm ix\eta^l)\eta^{k-1}\,d\eta = \frac{1}{n}\,e^{-ik\pi/(2n)} \sum_{t=0}^{n-1} \Gamma\left(\frac{k+tl}{n}\right)\frac{(\pm x)^t}{t!}$$

$$\times\, e^{it(n-l)\pi/(2n)}\, F\left\{\begin{matrix}\dfrac{k+tl}{nl}, \dfrac{k+tl+n}{nl}, \ldots, \dfrac{k+tl+(l-1)n}{nl}\,;\ e^{i(n-l)\pi/2}l^l\left(\dfrac{\pm x}{n}\right)^n \\ \dfrac{t+1}{n}, \dfrac{t+2}{n}, \ldots * \ldots, \dfrac{t+n}{n}\end{matrix}\right\}, \quad \ldots\ldots(9)$$

where $n > R(k) > 0$ and $n > l$.

On putting $n=3$, $l=1$, $k=1$, and equating real parts, Airy's integrals are obtained.

The same method may be applied for other values of n and l.

For instance, if $n=5$, $l=3$, and if k is real and such that $0<k<5$, x real and positive,

$$\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}\,(\eta^5 \pm x\eta^3)\,\eta^{k-1}\,d\eta = \frac{1}{5}\sum_{t=0}^{4}\Gamma\left(\frac{k+3t}{5}\right)\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{k-2t}{10}\pi\right)\frac{(\mp x)^t}{t!}$$

$$\times\, F\left\{\begin{matrix}\dfrac{k+3t}{15}, \dfrac{k+3t+5}{15}, \dfrac{k+3t+10}{15}\,;\ \pm 27\left(\dfrac{x}{5}\right)^5 \\ \dfrac{t+1}{5}, \dfrac{t+2}{5}, \ldots * \ldots, \dfrac{t+5}{5}\end{matrix}\right\}. \quad \ldots\ldots\ldots(10)$$

Note. From these formulae numerous others can be deduced. For instance, if the cosines with arguments $\pm x$ on the left of (10) are added, integrals of the type

$$\int_0^\infty \cos\eta^5 \cos\,(x\eta^3)\,\eta^{k-1}\,d\eta$$

are obtained. The second cosine and the functions on the right can then be expressed as E-functions and generalised.

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