BULL. AUSTRAL. MATH. SOC. MOS 1650, 1694
VOL. 1 (1969), 353-355.

Subdirectly irreducible rings – some pathology

H. G. Moore

Every ring is isomorphic to a subdirect sum of subdirectly irreducible rings. Unfortunately, however, as is shown, the property of being subdirectly irreducible is not preserved under homomorphisms. An example is given of a finite non-commutative subdirectly irreducible ring R with heart (= the intersection of all non-zero ideals) H, such that R/H is isomorphic with GF(2) + GF(2). (GF(2) is the two element Galois Field.) Some additional properties of the ring R are listed and contrasts are made with results for commutative subdirectly irreducible rings; for example, the zero divisors of R do *not* form an ideal.

Because every ring is isomorphic to a subdirect sum of subdirectly irreducible rings, such rings are in a sense fundamental building blocks of ring theory. (Indeed subdirectly irreducible abstract algebras are fundamental in the theory of (universal) algebras, see [*6*].) Unfortunately, not as much is known about subdirectly irreducible rings as one would like. Some results have been given in the commutative case by McCoy [*4*], Divinsky [*2*], and others (see also [*3*]). Additional information concerning the *heart*, the intersection of all the non-zero ideals, of subdirectly irreducible rings in general is found in [*1*]. But as fundamental building blocks, such rings leave a great deal to be desired. It is not true, for example, that the homomorphic image of a subdirectly irreducible ring need also be subdirectly irreducible. An example of Divinsky [*2*] can be seen to demonstrate this for the commutative case.

We present here a finite non-commutative subdirectly irreducible ring

Received 23 June 1969.

which serves as a simple counter-example for this and other hopeful ring theoretic conjectures.

Let R be the linear associate algebra over GF(2) with basis 1, ξ, and η, such that $\xi^2 = \xi$, $\eta^2 = 0$, $\xi\eta = 0$, and $\eta\xi = \eta$.

It is fairly easy to see that R is subdirectly irreducible with heart $H = \{0,\eta\}$. The homomorphic image R/H of R is not, however, subdirectly irreducible, rather it is isomorphic with $GF(2) \oplus GF(2)$ via the mapping:

$$H \to (0,0) ,$$
$$1 + H \to (1,1) ,$$
$$\xi + H \to (1,0) ,$$
$$1 + \xi + H \to (0,1) .$$

While in a subdirectly irreducible commutative ring the zero divisors form an ideal, such is not the case in general, as this example shows.

McCoy [4] has classified subdirectly irreducible commutative rings as three types

(i) fields,

(ii) rings in which every element is a zero divisor, and

(iii) rings with both non-divisors of zero and nilpotent elements.

He proves that a commutative ring of type (iii) is subdirectly irreducible if, and only if, the set H^* of zero divisors is the annihilator of the heart H and the annihilator H^{**} of H^* is H. In the ring R of our example, however, $H^*H = HH^* = H$.

Our ring R is also much like the B-rings of [5] in that the elements of R satisfy the property "for each x in R there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$ or $x^{n(x)} = 0$." (In a B-ring an element is either nilpotent or else idempotent.) In this ring, for $x \notin H$, $x^3 = x$, while the heart H coincides with the Jacobson radical, J, of R, which is also the set of all the nilpotent elements of R.

Furthermore, R/J is a boolean ring, and the commutator ideal $C(R)$ of R is nil. (This last remark serves to demonstrate that R isn't all bad.)

References

[1] Nathan Divinsky, *Rings and radicals* (University of Toronto Press, Toronto, 1965).

[2] Nathan Divinsky, "Commutative subdirectly irreducible rings", *Proc. Amer. Math. Soc.* 8 (1957), 642-648.

[3] J. Lambek, *Lectures on rings and modules* (Blaisdell, Waltham, Massachusetts, 1966).

[4] Neal H. McCoy, "Subdirectly irreducible commutative rings", *Duke Math. J.* 12 (1945), 381-387.

[5] H.G. Moore and Adil Yaqub, "A generalization of boolean rings", to appear.

[6] Richard S. Pierce, *Introduction to the theory of abstract algebras* (Holt, Rinehart & Winston, New York, 1968).

Brigham Young University,
Provo, Utah.