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ON THE ESSENTIAL SPECTRA OF QUASISIMILAR OPERATORS

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1. Introduction. B. Sz.-Nagy and C. Foiaş gave the first examples of quasisimilar operators with different spectra [**25**]. Indeed, quasisimilar operators can even have different spectral radius [**19**] (see also [**15**]). Nevertheless, T. B. Hoover has shown in [**19**] that if T and S are quasisimilar, then $\sigma(T) \cap \sigma(S) \neq \emptyset$, where $\sigma(R)$ denotes the spectrum of the (bounded linear) operator R. In [**12**], the author improved Hoover's result by showing that each component of $\sigma(S)$ intersects $\sigma(T)$, and viceversa. This, in turn, was further improved by L. A. Fialkow in [**7**]. (Actually, Fialkow's results were obtained independently of [**12**].)

L. A. Fialkow [**7**] and L. R. Williams [**27**] independently proved that $\sigma_e(T) \cap \sigma_e(S) \neq \emptyset$, where $\sigma_e(R)$ denotes the essential spectrum of R. Several authors have raised the following natural question:

Is it also true that each component of $\sigma_e(S)$ intersects $\sigma_e(T)$, and viceversa? [**7**] [**21**] [**26**].

The main purpose of the present article is to solve this riddle.

Let $\mathcal{L}(\mathcal{X})$ denote the algebra of all operators acting on the complex, infinite dimensional Banach space $\mathcal{X}$, and let $\mathcal{K}(\mathcal{X})$ denote the ideal of all compact operators (so that $\sigma_e(R)$ is the spectrum of the canonical projection $R \in \mathcal{L}(\mathcal{X})$ in the quotient Calkin algebra $\mathcal{L}(\mathcal{X})/\mathcal{K}(\mathcal{X})$). Recall that $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are *quasisimilar* if there exist injective operators $X : \mathcal{X} \longrightarrow \mathcal{Y}$ and $Y : \mathcal{Y} \longrightarrow \mathcal{X}$ with dense range such that

$$XT = SX \quad \text{and} \quad TY = YS$$

(An injective operator with dense range between two Banach spaces is called a *quasiaffinity*.)

THEOREM 1. *If $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are quasisimilar, then each component of $\sigma_e(S)$ intersects $\sigma_e(T)$, and viceversa.*

The proof of this theorem is the content of Section 3. In Section 4 we shall discuss some consequences of this result for the case when $\mathcal{X} = \mathcal{Y} = \mathcal{H}$ is a separable Hilbert space, along with several related examples. Among other results, the article includes the answers to two questions of L. A. Fialkow [**10**] about the structure of the quasisimilarity orbits of a quasinilpotent Hilbert space operator and of an operator with disconnected spectrum. A simple example shows that quasisimilarity can modify the structure of the invariant subspace

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lattice so deeply that it does not even preserve reflexivity. (An operator $T \in \mathcal{L}(\mathcal{X})$ is called *reflexive* if every A in $\mathcal{L}(\mathcal{X})$ that leaves invariant every subspace of T is necessarily a weak limit of polynomials in T.)

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2. Preliminary results. The notion of quasisimilarity was introduced by B. Sz.-Nagy and C. Foiaş, in connection with their Harmonic Analysis of contractions in Hilbert space [**24**] [**25**] [**26**]. Quasisimilarity preserves the existence of hyperinvariant subspaces [**19**], although it does not preserve the structure of the lattice of hyperinvariant subspaces [**13**].

If $XT = SX$ and $TY = YS(X, Y$ quasiaffinities), then it is an easy exercise to check that

$$X[q^{-1}(T)p(T)] = [q^{-1}(S)p(S)X] \text{ and } [q^{-1}(T)p(T)]Y = Y[q^{-1}(S)p(S)]$$

for all pairs of polynomials (p, q), provided q has no roots in $\sigma(T) \cup \sigma(S)$. Indeed, $Xf(T) = f(S)X$ and $f(T)Y = Yf(S)$ for each function f analytic on some neighborhood of $\sigma(T) \cup \sigma(S)$.

For each complex λ,

$$X\ker(\lambda - T) \subset \ker(\lambda - S) \quad \text{and} \quad Y\ker(\lambda - S) \subset \ker(\lambda - T)$$

and

$$\text{nul}(\lambda - T) = \text{nul}(\lambda - S) \quad \text{and} \quad \text{nul}(\lambda - T)^* = \text{nul}(\lambda - S)^*,$$

where $\text{nul}\, A = \dim \ker A$. Thus, if either $\ker(\lambda - T)$ or $\ker(\lambda - S)$ is finite dimensional, then

$$X\ker(\lambda - T) = \ker(\lambda - S) \quad \text{and} \quad Y\ker(\lambda - S) = \ker(\lambda - T).$$

Recall that $A \in \mathcal{L}(\mathcal{X})$ is a *semi-Fredholm operator* if $\text{ran}\, A := A\mathcal{X}$ is a *subspace* (i.e.; a closed linear manifold) and at least one of the cardinal numbers $\text{nul}\, A$ and $\text{nul}\, A^*$ is finite. In this case, the *index* of A is defined by

$$\text{ind}\, A = \text{nul}\, A - \text{nul}\, A^*.$$

It is completely apparent that quasisimilar semi-Fredholm operators have the same index.

A semi-Fredholm operator A is *Fredholm* if ind A is finite (equivalently, both nul A and nul A^* are finite). The well-known Atkinson's theorem says that

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : \lambda - T \text{ is not Fredholm}\}.$$

The nonempty compact set

$$\sigma_{lre}(T) = \{\lambda \in \mathbf{C} : \lambda - T \text{ is not semi-Fredholm}\}$$

is the *Wolf spectrum* of T. Its complement, $\rho_{s-F}(T) = \mathbf{C}\backslash\sigma_{lre}(T)$ is the semi-Fredholm domain of T. The reader is referred to **[11]** **[20]** for the properties of the semi-Fredholm operators.

Another important (and easy to prove) property of quasisimilarity is that it preserves multiplicity. More precisely, $T \in \mathcal{L}(\mathcal{X})$ is *n-multicyclic* if there exist $e_1, e_2, \ldots, e_n \in \mathcal{X}$ such that

$$\mathcal{X} = \vee\{T^k e_j : \quad j = 1, 2, \ldots, n\}_{k=0}^{\infty}$$

(where $\vee$ denotes "the closed linear span of"), but no set of $n-1$ vectors has the above property. If T and S are quasisimilar and T is n-multicyclic, then so is S.

Suppose $T \in \mathcal{L}(\mathcal{X}), S \in \mathcal{L}(\mathcal{Y})$, and $X : \mathcal{X} \longrightarrow \mathcal{Y}$ and $Y : \mathcal{Y} \longrightarrow \mathcal{X}$ are quasiaffinities such that $XT = SX$ and $TY = YS$, and σ is a component of $\sigma_e(S)$ such that $\sigma \cap \sigma_e(T) = \emptyset$. Since $\sigma_e(S)$ is compact, σ is the intersection of the clopen subsets of $\sigma_e(S)$ including it. Therefore, there exist a clopen subset σ' of $\sigma_e(S)$ and a connected open set $\Omega \subset \mathbf{C}$ such that

$$\sigma \subset \sigma' \subset \Omega \subset \mathbf{C}\backslash\sigma_e(T).$$

Furthermore, by replacing (if necessary) Ω by a smaller set, and by using the properties of the semi-Fredholm operators, we can directly assume that Ω is bounded, its boundary, $\partial\Omega$, consists of finitely many pairwise disjoint rectifiable Jordan curves, $\Omega^- \cap \sigma_e(T) = \emptyset$, and $\lambda - T$ and $\lambda - S$ are Fredholm operators such that nul$(\lambda - T)$ and nul$(\lambda - S)$ are constant for all λ in $\partial\Omega$.

Clearly, this observation reduces the proof of Theorem 1 to that of the following.

Claim. Let $T \in \mathcal{L}(\mathcal{X})$, $S \in \mathcal{L}(\mathcal{Y})$, and let $X : \mathcal{X} \longrightarrow \mathcal{Y}$ and $Y : \mathcal{Y} \longrightarrow \mathcal{X}$ be quasiaffinities such that $XT = SX$ and $TY = YS$. If Ω is a bounded, connected, open set, $\partial\Omega$ is the union of finitely many pairwise disjoint rectifiable Jordan curves,

$$\partial\Omega \cap [\sigma_e(T) \cup \sigma_e(S)] = \emptyset,$$

and $\operatorname{nul}(\lambda - T) = \operatorname{nul}(\lambda - S)$ is constant for all $\lambda \in \partial\Omega$, then

$$\Omega \cap \sigma_e(T) = \emptyset \Rightarrow \Omega \cap \sigma_e(S) = \emptyset.$$

The main ingredient for the proof is the author's result about the Fredholm structure of a multicyclic operator [**17**] (see also [**4**, Chapter 11] [**14**]).

THEOREM 2. *If $A \in \mathcal{L}(X)$ is n-multicyclic, then* $\operatorname{ind}(\lambda - A) \geqq -n$ *for all* $\lambda \in \rho_{s-F}(A)$*; moreover, if* Φ *is a component of*

$$\rho_{s-F}^{-n}(A) := \{\lambda \in \rho_{s-F}(A) : \quad \operatorname{ind}(\lambda - A) = -n\},$$

then

(i) $\operatorname{nul}(\lambda - A) = 0$ *and* $\operatorname{nul}(\lambda - A)^* = n$ *for all* $\lambda \in \Phi$, *and*
(ii) Φ *is a simply connected, bounded, open set.*

3. Proof of theorem 1. With the notation of the above claim:
Case 0. $\partial\Omega \cap \sigma(T) = \emptyset$.
Since $\lambda - S$ is Fredholm, and

$$\operatorname{nul}(\lambda - S) = \operatorname{nul}(\lambda - T) = \operatorname{nul}(\lambda - S)^* = \operatorname{nul}(\lambda - T)^* = 0$$

for all $\lambda \in \partial\Omega$, we deduce that $\partial\Omega \cap \sigma(S) = \emptyset$. Therefore,

$$X(\lambda - T)^{-1} = (\lambda - S)^{-1}X \quad \text{for all } \lambda \in \partial\Omega,$$

and

$$XE(T) = E(S)X,$$

where $E(T) \in \mathcal{L}(X)$ is the idempotent operator defined by

$$E(T) = \frac{1}{2\pi i}\int_{\partial\Omega}(\lambda - T)^{-1}d\lambda,$$

and $E(S) \in \mathcal{L}(\mathcal{Y})$ is similarly defined.

Since $\Omega^- \cap \sigma_e(T) = \emptyset, E(T)$ has finite rank. Since X has dense range, we infer that $E(S)$ is also a finite rank operator; therefore $E(S)$ is compact.

But this is clearly equivalent to saying that $\Omega^- \cap \sigma_e(S) = \emptyset$. (Consider the canonical projection of the contour integral defining $E(S)$ in the Calkin algebra $\mathcal{L}(\mathcal{Y})/\mathcal{K}(\mathcal{Y})$). This was, basically, the argument used in [**12**].)

Case 1. $\operatorname{nul}(\lambda - T) \equiv 0$, but $\operatorname{ind}(\lambda - T) = -\operatorname{nul}(\lambda - T)^* \equiv -n \neq 0$ for $\lambda \in \partial\Omega$.

Fix some μ in $\partial\Omega$; then $\operatorname{ran}(\mu - T)$ and $\operatorname{ran}(\mu - S)$ are subspaces and have codimension n in X and, respectively, in $\mathcal{Y}$. Let $\{f_1, f_2, \ldots, f_n\} \subset \mathcal{Y}$ be any set of (necessarily linearly independent) n vectors in $\mathcal{Y}$ such that $\vee\{f_j\}_{j=1}^n$ complements $\operatorname{ran}(\mu - S)$. Since X has dense range, the f_j's can be chosen so that

$f_j = Xe_j$ for a suitable (necessarily linearly independent) set $\{e_1, e_2, \ldots, e_n\}$ of n vectors in $\mathcal{X}$; moreover, it is easily seen that these vectors can be chosen so that $\vee\{e_j\}_{j=1}^n$ complements $\mathrm{ran}(\mu - T)$. Let

$$\mathcal{M} = \vee\{T^k e_j : j = 1, 2, \ldots, n\}_{k=0}^{\infty} \quad \text{and}$$

$$\mathcal{N} = \vee\{S^k(Xe_j) : j = 1, 2, \ldots, n\}_{k=0}^{\infty}.$$

Clearly, $T\mathcal{M} \subset \mathcal{M}$, $S\mathcal{N} \subset \mathcal{N}$ and the multiplicities of $T|\mathcal{M}$ and $S|\mathcal{N}$ cannot exceed n; moreover, $(\mu - T)\mathcal{M}$ and $(\mu - S)\mathcal{N}$ are subspaces, and

$$(\mu - T)\mathcal{M} \subset \mathcal{M} \cap \mathrm{ran}(\mu - T) \quad \text{and}$$

$$(\mu - S)\mathcal{N} \subset \mathcal{N} \cap \mathrm{ran}(\mu - S),$$

so that

$$(\mu - S)\mathcal{N} \cap (\vee\{Xe_j\}_{j=1}^n) = \{0\}.$$

It readily follows from general properties of the semi-Fredholm operators and Theorem 2 that

$$\mathcal{N} = (\vee\{Xe_j\}_{j=1}^n) \dot{+} (\mu - S)\mathcal{N}$$

(where $\dot{+}$ denotes direct sum of Banach spaces), $S|\mathcal{N}$ is an n-multicyclic operator and $\lambda - S|\mathcal{N}$ is a Fredholm operator such that

$$\mathrm{nul}(\lambda - S|\mathcal{N}) = 0 \quad \text{and}$$

$$\mathrm{ind}(\lambda - S|\mathcal{N}) = -\mathrm{nul}(\lambda - S|\mathcal{N})^* = -n$$

for all $\lambda \in \Omega^{\wedge} :=$ the smallest simply connected set including Ω^-.

Observe that

$$\begin{aligned} [X\mathcal{M}]^- &= [X(\vee\{T^k e_j : \quad j = 1, 2, \ldots, n\}_{k=0}^{\infty})]^- \\ &= \vee\{XT^k e_j : \quad j = 1, 2, \ldots, n\}_{k=0}^{\infty} \\ &= \vee\{S^k X e_j : \quad j = 1, 2, \ldots, n\}_{k=0}^{\infty} = \mathcal{N} \end{aligned}$$

and (by the same computation)

$$[X(\mu - T)\mathcal{M}]^- = (\mu - S)\mathcal{N}.$$

Clearly,

$$(\vee\{e_j\}_{j=1}^n \cap (\mu - T)\mathcal{M} = \{0\} \quad \text{and}$$

$$\mathcal{M} = (\vee\{e_j\}_{j=1}^n) + (\mu - T)\mathcal{M},$$

whence we conclude as in the case of S and $\mathcal{N}$ that

$\mathcal{M} = (\vee\{e_j\}_{j=1}^n) \dot{+} (\mu - T)\mu, T|\mathcal{M}$ is an n-multicyclic operator and $\lambda - T|\mathcal{M}$ is a Fredholm operator such that $\mathrm{nul}(\lambda - T|\mathcal{M}) = 0$ and $\mathrm{ind}(\lambda - T|\mathcal{M}) = -\mathrm{nul}(\lambda - T|\mathcal{M})^* = -n$ for all $\lambda \in \Omega^\wedge$.

In other words, $T|\mathcal{M}$ and $S|\mathcal{N}$ have exactly the same characteristics, and $\mathcal{N} = \{X\mathcal{M}\}^-$. If $\bar{T} \in \mathcal{L}(\mathcal{X}/\mathcal{M})$ and $\bar{S} \in \mathcal{L}(\mathcal{Y}/\mathcal{N})$ are the operators induced by $T \in \mathcal{L}(\mathcal{X})$ and, respectively, $S \in \mathcal{L}(\mathcal{Y})$ on the corresponding quotient spaces, then

$$\bar{X}\bar{T} = \bar{S}\bar{X}.$$

(Here $\bar{T}\bar{X} = (TX)^-$, $\bar{S}\bar{y} = (Sy)^-$ and $\bar{X}\bar{x} = (Xx)^-$, where $\bar{x}$ and $\bar{y}$ denote the cosets of $x \in \mathcal{X}$ in $\mathcal{X}/\mathcal{M}$ and, respectively, of $y \in \mathcal{Y}$ in $\mathcal{Y}/\mathcal{N}$, and $\bar{X} : \mathcal{X}/\mathcal{M} \to \mathcal{Y}/\mathcal{N}$ is the operator induced by $X : \mathcal{X} \to \mathcal{Y}$.)

It is easily seen that $(\mathrm{ran}\, X)^- = \mathcal{Y}$ implies $(\mathrm{ran}\, \bar{X})^- = \mathcal{Y}/\mathcal{N}$, so that

$$\dim \mathcal{Y}/\mathcal{N} \leqq \dim \mathcal{X}/\mathcal{M}.$$

If $\mathcal{Y}/\mathcal{N}$ is finite dimensional, then it readily follows that $\mathcal{N}$ has a finite dimensional complement in $\mathcal{Y}$, and

$$\Omega \cap \sigma_e(S) = \Omega \cap \sigma_e(S|\mathcal{N}) = \emptyset.$$

If $\mathcal{Y}/\mathcal{N}$ is infinite dimensional, then so is $\mathcal{X}/\mathcal{M}$. Since $\Omega \cap \sigma_e(T) = \emptyset$, it is not difficult to deduce that

$$\Omega \cap \sigma_e(\bar{T}) = \emptyset.$$

Furthermore, since

$$\mathrm{ind}(\lambda - T) = \mathrm{ind}(\lambda - T|\mathcal{M}) = -n (\lambda \in \Omega^-),$$

$\lambda - \bar{T}$ is a Fredholm operator of index 0 for all $\lambda \in \Omega^-$ **[11]** **[20]**. Moreover, if $(\mu - \bar{T})\bar{x} = \bar{0}$ for some $x \in \mathcal{X}$, then

$$(\mu - T)\, x \in \mathcal{M} \cap (\mu - T)\mathcal{X} = (\mu - T)\mathcal{M};$$

therefore, there is a y in $\mathcal{M}$ such that

$$(\mu - T)x = (\mu - T)y.$$

Since $(\mu - T)(x - y) = 0$ and $(\mu - T)$ is injective, we deduce that

$$x = y \in \mathcal{M} \quad \text{and therefore} \quad \bar{x} = \bar{0};$$

that is, $(\mu - \bar{T})$ is injective.

Since Ω^- is a connected, compact subset of $\{\lambda \in \mathbf{C} : (\lambda - \bar{T})$ is Fredholm of index $0\}$, we infer that $\ker(\lambda - \bar{T}) = \{\bar{0}\}$ for all but finitely many λ's in Ω^-, and that $\Omega^- \cap \sigma(\bar{T})$ is a finite (possibly empty) subset of normal eigenvalues of the operator $\bar{T}$.

By replacing, if necessary, Ω by a slightly smaller open set Ω' with the same characteristics as Ω ($\mu \in \partial\Omega'$), we can directly assume that $\lambda - \bar{T}$ is invertible for all $\lambda \in \partial\Omega$. It readily follows that $E(\bar{T})$ is a well-defined finite rank operator.

We conclude as in Case 0, that $E(\bar{S})$ also has finite rank, and

$$\Omega \cap \sigma_e(S) = \Omega \cap \sigma_e(\bar{S}) = \emptyset.$$

Case 1*. $\mathrm{nul}(\lambda - T)^* \equiv 0$, but $\mathrm{ind}(\lambda - T) = \mathrm{nul}(\lambda - T) \equiv n \neq 0$ for $\lambda \in \partial\Omega$.

This follows from Case 1 by taking adjoints. (If either $\mathcal{X}$ or $\mathcal{Y}$ is not a reflexive Banach space, then minor adjustments are necessary here and there; for instance, we can only assume that X^* and Y^* have weak * dense range, etc. The details are left to the reader, if any.)

Case 2. $\mathrm{nul}(\lambda - T) \equiv m \neq 0$ and $\mathrm{nul}(\lambda - T)^* \equiv n \neq 0$ for $\lambda \in \partial\Omega$. Define

$$\mathcal{M} = \vee\{\ker(\lambda - T) : \lambda \in \partial\Omega\}, \mathcal{N} = \vee\{\ker(\lambda - S) : \lambda \in \partial\Omega\}.$$

Clearly, $T\mathcal{M} \subset \mathcal{M}$ and $S\mathcal{N} \subset \mathcal{N}$.

Since $XT = SX$ and $X\ker(\lambda - T) = \ker(\lambda - S)$ for all $\lambda \in \partial\Omega$, we have $(X\mathcal{M})^- = \mathcal{N}$. Similarly, $(Y\mathcal{N})^- = \mathcal{M}$.

Since $\mathcal{M}$ is the span of a family of kernels, $T|\mathcal{M}$ is a "triangular operator", and therefore:

$\lambda - T|\mathcal{M}$ is a Fredholm operator such that

$$\mathrm{nul}(\lambda - T|\mathcal{M}) = \mathrm{ind}(\lambda - T|\mathcal{M}) = m$$

and $\mathrm{nul}(\lambda - T|\mathcal{M})^* = 0$ for all $\lambda \in \Omega^-$ (use [**12**] [**16**, Chapter 3] [**18**], and the fact that $\Omega^- \cap \sigma_e(T) = \emptyset$).

Similarly,

$\lambda - S|\mathcal{N}$ is a Fredholm operator such that

$$\mathrm{nul}(\lambda - S|\mathcal{N}) = \mathrm{ind}(\lambda - S|\mathcal{N}) = m$$

and $\mathrm{nul}(\lambda - S|\mathcal{N})^* = 0$ for all $\lambda \in \partial\Omega$.

Since $X\mathcal{M}$ is dense in $\mathcal{N}$ and $Y\mathcal{N}$ is dense in $\mathcal{M}$ it follows immediately that

$$X|\mathcal{M} : \mathcal{M} \longrightarrow \mathcal{N} \quad \text{and} \quad Y|\mathcal{N} : \mathcal{M} \longrightarrow \mathcal{M}$$

are quasiaffinities, and

$$(X|\mathcal{M})(T|\mathcal{M}) = (S|\mathcal{N})(X|\mathcal{M}) \text{ and } (T|\mathcal{M})(Y|\mathcal{N}) = (Y|\mathcal{N})(S|\mathcal{N}).$$

That is, $T|\mathcal{M}$ and $S|\mathcal{N}$ are quasisimilar.

It follows from Case 1* that

$$\Omega \cap \sigma_e(S|\mathcal{N}) = \emptyset.$$

Define $\bar{T}, \bar{S}, \bar{X}$ and $\bar{Y}$ exactly as in Case 1; then $\bar{X}$ and $\bar{Y}$ have dense ranges, and $\bar{X}\bar{T} = \bar{S}\bar{X}$ and $\bar{T}\bar{Y} = \bar{Y}\bar{S}$; moreover, $\mathcal{X}/\mathcal{M}$ and $\mathcal{Y}/\mathcal{N}$ are infinite dimensional spaces, $\lambda - \bar{T}$ is a Fredholm operator such that

$$\operatorname{ind}(\lambda - \bar{T}) = -\operatorname{nul}(\lambda - \bar{T})^* = \operatorname{ind}(\lambda - T) - m = (m - n) - m = -n$$

for all $\lambda \in \Omega^-$ and $\operatorname{nul}(\lambda - \bar{T}) = 0$ for all $\lambda \in \partial\Omega$. Indeed, if $(\lambda - \bar{T})\bar{x} = \bar{0}$ for some $\lambda \in \partial\Omega$ and some $x \in \mathcal{X}$, then

$$(\lambda - T)x \in \mathcal{M} = (\lambda - T)\mathcal{M},$$

because $(\lambda - T)|\mathcal{M}$ is onto. Therefore, $(\lambda - T)x = (\lambda - T)y$ for some $y \in \mathcal{M}$, and $(\lambda - T)(x - y) = 0$, so that

$$z = x - y \in \ker(\lambda - T) \subset \mathcal{M}.$$

It follows that $x = z + y \in \mathcal{M}$ and $\bar{x} = \bar{0}$. Hence, $(\lambda - \bar{T})$ is injective for all $\lambda \in \partial\Omega$.

Similarly, $\lambda - \bar{S}$ is a Fredholm operator such that

$$\operatorname{ind}(\lambda - \bar{S}) = -\operatorname{nul}(\lambda - \bar{S})^* = \operatorname{ind}(\lambda - S) - m = -n$$

and $\operatorname{nul}(\lambda - \bar{S}) = 0$ for all $\lambda \in \partial\Omega$.

Now it follows exactly as in Case 1 that

$$\Omega \cap \sigma_e(S) = \Omega \cap \sigma_e(\bar{S}) = \emptyset.$$

Indeed as remarked in the proof of Case 1), the only relevant property of $\bar{X}$ that we need here is the fact that this operator has dense range.

This completes the proof of our claim. As we have already observed, it readily follows from this fact that each component of $\sigma_e(S)$ intersects $\sigma_e(T)$. By a symmetric reasoning, each component of $\sigma_e(T)$ intersects $\sigma_e(S)$.

From the proof of Theorem 1, we immediately derive the following result.

COROLLARY 3. *Let $T \in \mathcal{L}(\mathcal{X}), S \in \mathcal{L}(\mathcal{Y})$, and let $X : \mathcal{X} \to \mathcal{Y}$ and $Y : \mathcal{Y} \to \mathcal{X}$ be operators such that $XT = SX$ and $TY = YS$.*

(i) *If* $(\operatorname{ran} X)^- = \mathcal{Y}$ *and Ω is an open subset of $\mathbf{C}\backslash\sigma_e(T)$ such that* $\operatorname{nul}(\lambda - T) = 0$ *for all $\lambda \in \Omega$ and $\partial\Omega \cap \sigma_e(S) = \emptyset$, then $\Omega \cap \sigma_e(S) = \emptyset$.*

(ii) *If* $\ker Y = \{0\}$ *and Ω is an open subset of $\mathbf{C}\backslash\sigma_e(T)$ such that $\lambda - T$ is onto for all $\lambda \in \Omega$ and $\partial\Omega \cap \sigma_e(S) = \emptyset$, then $\Omega \cap \sigma_e(S) = \emptyset$.*

Another consequence is the following mild improvement of the same theorem.

COROLLARY 4. *If $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are quasisimilar, then each component of $\sigma_{lre}(T)$ intersects $\sigma_e(S)$ and each component of $\sigma_{lre}(S)$ intersects $\sigma_e(T)$.*

Proof. Let σ be a component of $\sigma_{lre}(S)$, and let σ' be the component of $\sigma_e(S)$ including σ. If $\sigma = \sigma'$, the result follows from Theorem 1. If $\sigma \neq \sigma'$, then there exist $\alpha \in \partial\sigma$ and a Cauchy sequence $\{\alpha_k\}_{k=1}^{\infty}$ converging to α such that $\alpha_k - S$ is semi-Fredholm and

$$\operatorname{ind}(\alpha_k - S) = \pm\infty \quad \text{for all } k = 1, 2, \ldots.$$

It readily follows that $\alpha_k \in \sigma_e(T)$ for all $k = 1, 2, \ldots$, and therefore

$$\alpha = \lim_{k\to\infty} \alpha_k \in \sigma \cap \sigma_e(T).$$

4. Complementary results and examples. (a) If T and S are quasisimilar, then

$$\sigma_{lre}(T) \cap \sigma_{lre}(S) \neq \emptyset \quad [\mathbf{23}].$$

The Wolf spectrum of $T \in \mathcal{L}(\mathcal{X})$ is the intersection of the left essential spectrum,

$$\sigma_l e(T) = \{\lambda \in \mathbf{C} : \text{either } \lambda - T \text{ is not semi-Fredholm, or}$$

$$\operatorname{ind}(\lambda - T) = \infty\},$$

and the right essential spectrum,

$$\sigma_{re}(T) = \{\lambda \in \mathbf{C} : \text{either } \lambda - T \text{ is not semi-Fredholm, or}$$

$$\operatorname{ind}(\lambda - T) = -\infty\};$$

the essential spectrum, on the other hand, is equal to the union of $\sigma_{le}(T)$ and $\sigma_{re}(T)$. (If $\mathcal{X}$ is a Hilbert space, then $\sigma_{le}(T)$ and $\sigma_{re}(T)$ coincide with the left and, respectively, the right spectrum of the canonical projection of T in Calkin algebra.)

For $0 < r < 1$ let $H(r)$ be the compact diagonal positive operator defined by

$$H(r)e_n = r^n e_n \quad (n = 1, 2, \ldots)$$

with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of the Hilbert space $\mathcal{H}$. Let S^* be the backward shift with respect to the same basis (defined by $S^*e_1 = 0, S^*e_n = e_{n-1}$ for $n \geqq 2$); then a straightforward computation shows that

$$S^*H(r) = H(r)(rS^*),$$

and therefore

$$H(r)S = (rS)H(r).$$

By using this trivial example, it is very easy to construct examples of quasisimilar operators A and B such that not every component of $\sigma_{le}(A)(\sigma_{re}(A))$ intersects $\sigma_{le}(B)(\sigma_{re}(B)$, resp), and viceversa, not every component of $\sigma_{le}(B)(\sigma_{re}(B))$ intersects $\sigma_{le}(A)(\sigma_{re}(A)$, resp.): let $\{a_n\}_{m\in\mathbf{Z}}$ and $\{b_m\}_{n\in\mathbf{Z}}$ be any two strictly increasing sequences of real numbers in the interval [r,1] ($0 \leqq r < 1$) such that

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = 1, \quad \lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = r.$$

Define

$$A = \bigoplus_{m\in\mathbf{Z}} a_m S \quad \text{and} \quad B = \bigoplus_{n\in\mathbf{Z}} b_m S,$$

where $\bigoplus$ denotes orthogonal direct sum ($A, B \in \mathcal{L}(\bigoplus_{m\in\mathbf{Z}} \mathcal{H}_m)$, where $\mathcal{H}_m \cong \mathcal{H}$ for all $m \in \mathbf{Z}$). Clearly, we can always construct bijections $\omega : \mathbf{Z} \to \mathbf{Z}$ and $\psi : \mathbf{Z} \to \mathbf{Z}$ such that

$$a_m < b_{\omega(m)} \quad \text{and} \quad a_m > b_{\psi(m)} \quad \text{for all } m \in \mathbf{Z}.$$

It readily follows from the basic example $H(r)S = (rS)H(r)$ that A and B are quasisimilar; moreover, $\sigma(A) = \sigma(B) = \sigma_e(A) = \sigma_e(B) = \sigma_{re}(A) = \sigma_{re}(B) =$ the closed unit disk, but

$$\begin{aligned}\sigma_{le}(A) = \sigma_{lre}(A) &= \{\lambda \in \mathbf{C} : |\lambda| = r, 1, \text{ or } a_m \text{ for some } m \in \mathbf{Z}\}\\ &= \text{left spectrum of } A\end{aligned}$$

and

$$\begin{aligned}\sigma_{le}(B) = \sigma_{lre}(B) &= \{\lambda \in \mathbf{C} : |\lambda| = r, 1, \text{ or } b_m \text{ for some } m \in \mathbf{Z}\}\\ &= \text{left spectrum of } B.\end{aligned}$$

In particular, $\{a_m\}_{m\in\mathbf{Z}}$ and $\{b_m\}_{m\in\mathbf{Z}}$ can be chosen so that $\{a_m\} \cap \{b_n\} = \emptyset$. By taking adjoints, we have analogous examples with the roles of $\sigma_{le}(\cdot)$ and $\sigma_{re}(\cdot)$ interchanged (see also [**27**], [**28**]).

These examples show that $\sigma_e(T)$ cannot be replaced by $\sigma_{le}(T)$ or $\sigma_{re}(T)$ in Corollary 4. They also illustrate the difficulties that we may find in proving Stampfli's result [**23**, Theorem 9]: $\sigma_{lre}(T) \cap \sigma_{lre}(S) \neq \emptyset$ whenever T and S are quasisimilar.

An alternative proof of this result (indeed, a mild improvement of it) can be easily obtained as a corollary of Theorem 1:

THEOREM 5. *If $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{L}(\mathcal{Y})$ are quasisimilar and Ω is a bounded open set such that*

$$\sigma_e(T) \cap \Omega \neq \emptyset, \quad \textit{but} \quad \sigma_e(T) \cap \partial\Omega = \emptyset,$$

then

$$\sigma_{lre}(T) \cap \sigma_{lre}(S) \cap \Omega \neq \emptyset.$$

Proof. Let α be any point of $\partial \sigma_e(T) \cap \Omega$, and let σ be the component of α in $\sigma_e(T)$. Clearly, $\alpha \in \sigma_{lre}(T)$. By Theorem 1, $\sigma \cap \sigma_e(S) \neq \emptyset$.

Let β be any point of $\sigma \cap \sigma_e(S)$. If $\beta \in \sigma_{lre}(T) \cap \sigma_{lre}(S)$, then we are done. If $\beta \notin \sigma_{lre}(T) \cap \sigma_{lre}(S)$, then either $\beta - T$ is a semi-Fredholm operator of index $\pm\infty$, or $\beta - S$ has that form. If $\operatorname{ind}(\beta - T) = \infty$, then $\operatorname{ind}(\lambda - T) = \infty$ for all $\lambda \in \mathbf{C}$ such that $|\lambda - \beta| < \epsilon$ (for some $\epsilon > 0$ small enough), and therefore

$$\begin{aligned}\{\lambda \in \mathbf{C} : |\lambda - \beta| < \varepsilon\} &\subset \{\lambda \in \mathbf{C} : \operatorname{nul}(\lambda - T) = \operatorname{nul}(\lambda - S) = \infty\} \\ &\subset \sigma_e(T) \cap \sigma_e(S)\end{aligned}$$

(and similarly for the cases when $\operatorname{ind}(\lambda - T) = -\infty$ or $\operatorname{ind}(\lambda - S) = \pm\infty$). In particular, $\beta \neq \alpha$.

Let γ be a smooth Jordan arc in Ω, joining α with β, and let μ be the first point of γ (when moving from α to β) such that

$$\mu \in \sigma \cap \sigma_e(S).$$

Then $\mu \in \sigma_e(T) \cap \sigma_e(S)$; but $\mu \notin \sigma_{lre}(T) \cap \sigma_{lre}(S)$ implies (exactly as above with β) that $\sigma_e(T) \cap \sigma_e(S)$ includes an open disk centered at μ, which contradicts the definition of μ.

Hence,

$$\mu \in \sigma_{lre}(T) \cap \sigma_{lre}(S) \cap \Omega.$$

(b) Even more striking examples can be constructed on the same lines as in (a). Let $L \in \mathcal{L}(\mathcal{H})$ and assume $\Omega \subset \sigma(L)$ is a component of the left resolvent set of L (that is, $\lambda - L$ has a left inverse, but not an inverse for each $\lambda \in \Omega$; $\mathcal{H}$ is a Hilbert space). Let Γ and Δ be two relatively closed subsets of Ω; then there exist $T, S \in \mathcal{L}(\mathcal{H}^{(\infty)})$ such that T, S and $L^{(\infty)}$ are quasisimilar, $\sigma_{re}(T) = \sigma_{re}(S) = \sigma_{re}(L^{(\infty)})$,

$$\sigma_{le}(T) = \sigma_{le}(L^{(\infty)}) \cup \Gamma \quad \text{and} \quad \sigma_{le}(S) = \sigma_{le}(L^{(\infty)}) \cup \Delta.$$

Proof. If $\alpha \in \Omega$, then we have a decomposition

$$L = \begin{pmatrix} L_\alpha & P_\alpha \\ 0 & \alpha \end{pmatrix} \begin{matrix} \mathcal{M}_\alpha \\ \mathcal{H} \ominus \mathcal{M}_\alpha \end{matrix},$$

where $\mathcal{H} \ominus \mathcal{M}_\alpha = \ker(\alpha - L)^* \neq \{0\}$ and L_α has the same spectral properties as L; moreover, it is clear that $L_\alpha - \lambda$ and $(L_\alpha - \lambda)^*$ admit the same lower bounds as $L - \lambda$ and, respectively, $(L - \lambda)^*$, for each $\lambda \in \rho_{s-F}(L)$.

Define

$$L(\alpha, n) = \begin{pmatrix} L_\alpha & (1/n)P_n \\ 0 & \alpha \end{pmatrix}, \quad n = 1, 2, \dots.$$

The operator $L(\alpha, n)$ is similar to L, and therefore

$$L(\alpha) := \bigoplus_{n=1}^{\infty} L(\alpha, n) \quad \text{and} \quad L^{(\infty)} \text{ are quasisimilar [}\mathbf{15}\text{]}.$$

Let $\{\alpha_j\}$ and $\{\beta_k\}$ be countable dense subsets of Γ and, respectively, Δ, and define

$$T = \bigoplus_j L(\alpha_j) \quad \text{and} \quad S = \bigoplus_k L(\beta_k).$$

Since $(L^{(\infty)})^{(m)}$ is unitarily equivalent to $L^{(\infty)}$ for all cardinals m, $1 \leqq m \leqq \infty$, it is straightforward to check that T, S and $L^{(\infty)}$ are, indeed, quasisimilar, and that $\sigma_{re}(T) = \sigma_{re}(S) = \sigma_{re}(L^{(\infty)})$, but

$$\sigma_{le}(T) = \sigma_{le}(L^{(\infty)}) \cup \Gamma \quad \text{and} \quad \sigma_{le}(S) = \sigma_{le}(L^{(\infty)}) \cup \Delta.$$

By taking adjoints, we have similar examples with $\sigma_{le}(\cdot)$ and $\sigma_{re}(\cdot)$ interchanged.

(c) In [**10**], L. A. Fialkow used the spectral characterization of the closure of a similarity orbit of C. Apostol, D. A. Herrero and D. Voiculescu [**4**, Chapter 9, Theorem 9.2] in order to analyze the following kinds of problems: for which operators $T \in \mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a complex, separable, infinite dimensional Hilbert space) does

$$A \in \mathcal{L}(\mathcal{H}), \quad A \text{ quasisimilar to } T \Rightarrow A \in \mathcal{S}(T)^-?$$

(Here $\mathcal{S}(T) = \{WTW^{-1} : W \in \mathcal{L}(\mathcal{H}) \text{ is invertible}\}$ is the similarity orbit of T.) For which T's does

$$A \in \mathcal{L}(\mathcal{H}),\ A \text{ quasisimilar to } T \Rightarrow \sigma(T) \subset \sigma(A)?$$

As a corollary of Theorem 1, we have the following result along the lines of [**10**] (see also [**15**]).

THEOREM 6. *Suppose that $T, A \in \mathcal{L}(\mathcal{H})$ are quasisimilar operators, and T satisfies the conditions*

(1) *interior* $\sigma_{lre}(T) = \emptyset$,

(2) *if* $\mu \in \sigma_{lre}(T)$ *and* $\operatorname{ind}(\lambda - T)$ *is constant on* $\Omega \setminus \sigma_{lre}(T)$ *for some neighbourhood* Ω *of* μ, *then that index is finite and* $\{\mu\}$ *is a component of* $\sigma_e(T)$, *and* (3) *furthermore, if* μ *is an isolated point of* $\sigma_e(T)$, *and* k_μ *is the function defined by*

$$k_\mu(\lambda) = \begin{cases} \lambda - \mu, & \text{on some neighborhood of } \mu \\ 0, & \text{on some neighborhood of } \sigma_e(T) \setminus \{\mu\}, \end{cases}$$

then

$$k_\mu(\tilde{T})^m \neq 0 \quad \text{for all } m = 1, 2, \ldots,$$

where $\tilde{T}$ denotes the canonical projection T in the Calkin algebra.

Then $\sigma(T) \subset \sigma(A)$ and $A \in S(T)^-$.

Proof. Since T and A are quasisimilar,

$$\operatorname{nul}(\lambda - A)^k = \operatorname{nul}(\lambda - T)^k \quad \text{and} \quad \operatorname{nul}(\lambda - A)^{*k} = \operatorname{nul}(\lambda - T)^{*k}$$

for all $\lambda \in \mathbf{C}$ and all $k = 1, 2, \ldots$. In particular,

$$\operatorname{ind}(\lambda - A) = \operatorname{ind}(\lambda - T) \quad \text{for all } \lambda \in \rho_{s-p}(A).$$

On the other hand, (1) and (2) and Theorem 1 imply that $\sigma_{lre}(A) \supset \sigma_{lre}(T), \sigma_e(A) \supset \sigma_e(T)$, and each component of $\sigma_{lre}(A)$ intersects $\sigma_e(T)$.

It readily follows that $\sigma(A) \supset \sigma(T)$. Moreover, by using these observations and (3), it follows from [**4**, Theorem 9.2] that $A \in \mathcal{S}(T)^-$.

The result of Theorem 6 is "almost sharp", in the following sense: if T fails to satisfy (1), (2) or (3), then there exists $T' \in \mathcal{L}(\mathcal{H})$ with the same spectral characteristics as T, such that $A' \notin \mathcal{S}(T')^-$ for some $A' \in \mathcal{L}(\mathcal{H})$ quasisimilar to T'. (If T fails to satisfy (1) or (2), then we can find T' and A' as above so that $T' \in \mathcal{S}(A')^-$ and $\sigma_e(A')$ is a proper subset of $\sigma_e(T') = \sigma_e(T)$. If T satisfies (1) and (2), but not (3), then $\sigma(A) \supset \sigma(T)$ and $\sigma_e(A) \supset \sigma_e(T)$ for each $A \in \mathcal{L}(\mathcal{H})$ quasisimilar to T; in this case we can construct T' with $\sigma(T') = \sigma(T), \sigma_e(T') = \sigma_e(T)$, etc, and $k_\mu(\tilde{T}')^m \neq 0$ for all $m = 1, 2, \ldots$, for some function k_μ such that $k_\mu(\tilde{T})^p = 0$ for some $p \geqq 1$. The details of the construction of T' and A' follow easily from [**15**] and [**4**, Chapter 9].)

(d) In several results of [**10**], L. A. Fialkow uses the set

$$\sigma_{qn}(T) := \{\lambda \in \mathbf{C} : \|(\lambda - T)^n x\|^{1/n} \longrightarrow 0 (n \longrightarrow \infty)$$

for some nonzero x in $\mathcal{H}$ }

($\sigma_p(T) \subset \sigma_{qn}(T) \subset \sigma_{ap}(T)$, where $\sigma_p(T)$ and $\sigma_{ap}(T)$ denote the *point spectrum* and, respectively, the *approximate point* spectrum of the operator T). The same kinds of results can be achieved by using the "more elastic" subset

$$\sigma'_{qn}(T) := \{\lambda \in \mathbf{C} : \text{ for each } r > 0 \text{ there exists a nonzero } x_r \text{ in } \mathcal{H}$$

$$\text{such that } \liminf (n \longrightarrow \infty) \|(\lambda - T)^n x_r\|^{1/n} < r\}.$$

Once again, we have $\sigma_p(T) \subset \sigma_{qn}(T) \subset \sigma'_{qn}(T) \subset \sigma_{ap}(T)$. For every normal operator T on a Hilbert space, or even for every decomposable operator T on a Banach space, $\sigma(T) = \sigma'_{qn}(T)$. But $\sigma_{qn}(T) = \emptyset$ if, for instance, T is normal and its spectral measure has no atoms (see [**1**] and [**6**] for definition and properties of the decomposable operators).

(e) Given a Hilbert space operator T, let

$$(T)_{qs} = \{A \in \mathcal{L}(\mathcal{H}) : \quad A \text{ is quasisimilar to } T\}$$

denote the *quasisimilarity orbit* of T.

In [**10**, Question 2.16], L. A. Fialkow asked which nilpotent or quasinilpotent operators Q on $\mathcal{H}$ satisfy the inclusion $(Q)_{qs} \subset S(Q)^-$. Here is a partial answer to this question.

PROPOSITION 7 . (i) *If $Q \in \mathcal{L}(\mathcal{H})$ is a finite rank nilpotent, then*

$$(Q)_{qs} = S(Q) = \{A \in \mathcal{L}(\mathcal{H}) : \quad \mathcal{S}(A)^- = \mathcal{S}(Q)^-\}.$$

(ii) *If Q is a nilpotent of infinite rank, then $(Q)_{qs} \not\subset S(Q)^-$.*

(iii) *If Q_u is a universal quasinilpotent, then $(Q_u)_{qs} \subset \mathcal{S}(Q_u)^-$. (A quasinilpotent operator Q_u is "universal" if $(Q_u)^k \notin \mathcal{K}(\mathcal{H})$ for $k = 1, 2, \ldots$; see* [**4**, Chapter 9] [**16**, Chapter 8].*)*

(iv) *Given a quasinilpotent operator Q and $\epsilon > 0$, there exists $K_\epsilon \in \mathcal{K}(\mathcal{H})$, with $\|K_\epsilon\| < \epsilon$, such that $Q_\epsilon = Q + K_\epsilon$ is quasisimilar to a universal quasinilpotent Q_u. In this case, $(Q_\epsilon)_{qs} \subset S(Q_\epsilon)^-$ if and only if Q (and therefore Q_ϵ) is universal.*

Proof. (i) follows immediately from our observations in Section 2 about the elementary properties of quasisimilarity, and the results of [**16**, Section 2.1]. On the other hand, the proof of (iii) can be found in Fialkow's article [**10**, Corollary 2.6].

(ii) Let q_n denote the $n \times n (n \geqq 1)$ Jordon nilpotent cell in $\mathcal{L}(\mathbf{C}^n)$. If Q has the form of (ii), then it follows from [**3**] that for some $m \geqq 2$, all the operators

$$Q, \quad q_m^{(\infty)} \oplus F, \quad q_1 \oplus q_m^{(\infty)} \oplus F, \ldots, q_{m-1} \oplus q_m^{(\infty)} \oplus F$$

are quasisimilar. Here F is the only finite direct sum of Jordan cells of order strictly larger than m such that

$$\text{rank } F^k = \text{rank } Q^k \quad \text{for all } k \geqq m.$$

(Of course, the direct summand F acts on a finite dimensional space, and can be absent.)

If Q is not similar to $q_m^{(\infty)} \oplus F$ or $q_j \oplus q_m^{(\infty)} \oplus F$ (for some j, $1 \leqq j \leqq m-1$), then

$$Q \in S(q_m^{(\infty)} \oplus F)^- \cap \left\{ \bigcap_{j=1}^{m-1} S(q_j \oplus q_m^{(\infty)} \oplus F)^- \right\},$$

but

$$q_m^{(\infty)} \oplus F, \quad q_j \oplus q_m^{(\infty)} \oplus F \notin S(Q)^-$$

$j = 1, 2, \ldots, m-1$) [**16**, Proposition 8.5 and Corollary 8.17].

If Q is similar to $q_m^{(\infty)} \oplus F(q_j \oplus q_m^{(\infty)} \oplus F$ for some j, $1 \leqq j \leqq m-1$), then

$$q_1 \oplus q_m^{(\infty)} \oplus F \notin S(Q)^- \quad \text{and} \quad Q \notin S(q_1 \oplus q_m^{(\infty)} \oplus F)^-$$

$$(q_m^{(\infty)} \oplus F \notin S(Q)^- \quad \text{and} \quad Q \notin S(q_m^{(\infty)} \oplus F)^-, \text{ resp.})$$

[**16**, Proposition 8.19].

In either case, $(Q)_{qs} \not\subset S(Q)^-$.

(iv) If Q is universal, we are done (take $K_\epsilon = 0$). If Q is not universal, then Q^m is compact and Q^{m-1} is not compact (for some $m \geqq 1$), and there exists $K_1 \in \mathcal{K}(\mathcal{H})$, with $\|K_1\| < \epsilon/2$, such that

$$Q + K_1 \cong Q \oplus R^{(\infty)},$$

for some $R \in \mathcal{L}(\mathcal{H})$ such that $R^m = 0$, $R^{m-1} \neq 0$.

According to [**16**, Lemmas 7.8 and 7.9], R is similar to

$$R' = R_0 \oplus q_m.$$

Therefore, there exists W invertible such that

$$Q + K_1 = W[Q \oplus R_0^{(\infty)} \oplus q_m^{(\infty)}]W^{-1}.$$

Clearly, we can find $K_2 \in \mathcal{K}(\mathcal{H})$, with $\|K_2\| < \epsilon/2$, such that $K_\epsilon = K_1 + K_2$ satisfies

$$Q_\epsilon := Q + K_\epsilon = W\left[Q \oplus R_0^{(\infty)} \oplus \left\{\bigoplus_{k=1}^{\infty} q'_{km}\right\}\right]W^{-1},$$

where

$$\|q'_{km} - q_m^{(k)}\| < 2^{-(k+1)}\epsilon/\|W\| \cdot \|W^{-1}\|$$

It readily follows that if q''_{km} is similar to q_{km}, then Q_ϵ is quasisimilar to

$$Q'' = Q \oplus R_0^{(\infty)} \oplus \left\{ \bigoplus_{k=1}^{\infty} q''_{km} \right\}.$$

If $q''_{km}e_1 = 0$, $q''_{km}e_n = 1/n(e_{n-1})(n = 2, 3, \dots, km; \{e_n\}_{n=1}^{km}$ is the canonical orthonormal basis of $\mathbf{C}^{km}$), then it is easy to check that Q'' is a universal quasinilpotent quasisimilar to Q_ϵ. Clearly, $Q'' \notin S(Q_\epsilon)^-$ [**16**, Chapter 8].

A complete answer to Fialkow's question for the case when Q is a non universal quasinilpotent that is not nilpotent is, most probably, a hopeless task. Proposition 7 (iv) illustrates some of the difficulties involved in the problem. On the other hand, the proof of Proposition A3.1 of [**4**] shows that, in many cases, given a quasinilpotent operator Q and $\epsilon > 0$, it is possible to find $K_\epsilon \in \mathcal{K}(\mathcal{H})$, with $\|K_\epsilon\| < \epsilon$, such that $Q_\epsilon = Q + K_\epsilon$ is a strictly cyclic operator. The author conjectures that this is true for all quasinilpotent operators. According to [**15**], if Q_ϵ is strictly cyclic, or even if the double commutant of Q_ϵ has finite strict multiplicity, then

$$(Q_\epsilon)_{qs} = S(Q_\epsilon) \subset S(Q_\epsilon)^-.$$

(The reader is referred to this last reference for definition and properties of algebras of finite strict multiplicity.)

Thus, if the above conjecture is true, and Q has the above described form, then arbitrarily small compact perturbations of Q will produce quasinilpotent operators with radically different behaviour with respect to the relation between $(Q)_{qs}$ and $S(Q)^-$.

(f) In [**10**, question 2.11], L. A. Fialkow raised the following question: Suppose $T, S \in \mathcal{L}(\mathcal{H})$ are quasisimilar operators and $\sigma(T)$ is the disjoint union of two compact subsets, σ_1 and σ_2; then (by the Riesz decomposition theorem) $\mathcal{H}$ is the algebraic direct sum of two complementary invariant subspaces, $\mathcal{M}_1$ and $\mathcal{M}_2$, of $T(\sigma(T|\mathcal{M}_j) = \sigma_j, \quad j = 1, 2)$. Does S always have two complementary invariant subspaces, $\mathcal{N}_1$ and $\mathcal{N}_2$, such that $S|\mathcal{N}_j$ is quasisimilar to $T|\mathcal{M}_j \quad (j = 1, 2)$?

The following example provides a negative answer to this question.

Let $T = M \bigoplus N$, where M and N are normal operators such that

$$\sigma(M) = \{\lambda : |\lambda + 2| \leqq 1\} \quad \text{and} \quad \sigma(N) = \{\lambda : |\lambda - 2| \leqq 1\};$$

then $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ denote the space of M and, respectively, of N.

We can write $M = \bigoplus_{k=1}^{\infty} M_k$, where M_k is a normal operator with perfect spectrum. Let $L \cong L \oplus 0$ be a normal operator such that

$$\sigma(L) = \{\lambda \in \mathbf{C} : |\lambda| \leqq 4\}.$$

According to [**4**, Theorem 9.1] there exists an operator M_k' similar to M_k such that

$$\|L - M_k'\| < 1/k \quad (k = 1, 2, \ldots).$$

Similarly, we can write $N = \bigoplus_{k=1}^{\infty} N_k$ and find N_k' similar to N_k such that

$$\|L - N_k'\| < 1/k \quad (k = 1, 2, \ldots).$$

Therefore, there exist unit vectors e_k, f_k such that

$$\max\{\|M_k' e_k\|, \|(M_k')^* e_k\|, \|N_k' f_k\|, \|(N_k')^* f_k\|\} < 1/k$$

for each $k = 1, 2, \ldots$.

Thus, we can write

$$M_k' = \begin{pmatrix} M_k'' & C_k \\ D_k & E_k \end{pmatrix} \begin{matrix} (e_k)^{\perp} \\ e_k \end{matrix}, \quad N_k' = \begin{pmatrix} F_k & G_k \\ H_k & N_k'' \end{pmatrix} \begin{matrix} f_k \\ (f_k)^{\perp} \end{matrix}$$

where

$$\max[\|C_k\|, \|D_k\|, \|E_k\|, \|F_k\|, \|G_k\|, \|H_k\|] < 1/k.$$

Define

$$S = \bigoplus_{k=1}^{\infty} \begin{pmatrix} M_k'' & C_k & 0 & 0 \\ D_k & E_k & 1 & 0 \\ 0 & 0 & F_k & G_k \\ 0 & 0 & H_k & N_k'' \end{pmatrix} \begin{matrix} (e_k)^{\perp} \\ e_k \\ f_k \\ (f_k)^{\perp} \end{matrix} = \bigoplus_{k=1}^{\infty} \begin{pmatrix} M_k' & P_k \\ 0 & N_k' \end{pmatrix};$$

then $S f_k = e_k + g_k$, where $\|g_k\| \leqq \|F_k\| + \|H_k\| < 2/k$.

Each of the direct summands of S is similar to $M_k \oplus N_k$, and therefore S is quasisimilar to T [**15**].

Suppose that S has two complementary invariant subspaces, $\mathcal{M}'$ and $\mathcal{N}'$ such that $S|\mathcal{M}'$ and $S|\mathcal{N}'$ are quasisimilar to $T|\mathcal{M} = M$ and, respectively, to $T|\mathcal{N} = N$. It follows from [**2**] that $S|\mathcal{M}'$ admits a family $\{\mathcal{R}_j\}_{j=1}^{\infty}$ of invariant subspaces such that $\mathcal{R}_h$ complements

$$\mathcal{R}_h' = \vee\{\mathcal{R}_j : j \neq h\} \quad (h = 1, 2, \ldots)$$

and $\bigcap_{j=1}^{\infty}\mathcal{R}_j' = \{0\}, S|\mathcal{R}_j$ is similar to a direct summand R_j of M, and $M = \bigoplus_{j=1}^{\infty} R_j$. Similarly, $S|\mathcal{N}'$ admits a family $\{\mathcal{S}_j\}_{j=1}^{\infty}$ of invariant subspaces such that $\mathcal{S}_h$ complements $\mathcal{S}_h' = \vee\{\mathcal{S}_j : j \neq h\}$ $(h = 1,2,\ldots)$ and $\bigcap_{j=1}^{\infty}\mathcal{S}_j' = \{0\}, S|\mathcal{S}_j$ is similar to a direct summand S_j of N, and $N = \bigoplus_{j=1}^{\infty} S_j$. Clearly, we can split $\mathcal{R}_j$ and $\mathcal{S}_j$ (if necessary) in such a way that M_k is a direct sum of a subfamily of $\{S_j\}_{j=1}^{\infty}$ and N_k is a direct sum of a subfamily of $\{S_j\}_{j=1}^{\infty}$. It is easily seen that $\mathcal{M}'$ includes $\mathcal{M}$, and therefore $e_k \in \mathcal{M}'$ for all $k = 1,2,\ldots$; moreover,

$$\begin{pmatrix} M_k' & P_k \\ 0 & N_k' \end{pmatrix} = \begin{pmatrix} 1 & X_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_k' & 0 \\ 0 & N_k' \end{pmatrix} \begin{pmatrix} 1 & X_k \\ 0 & 1 \end{pmatrix}^{-1},$$

where $P_k = X_k N_k' - M_k' X_k$.

It follows from our construction that

$$\mathcal{N}' \supset \vee\{V_k \mathcal{N}_k\}_{k=1}^{\infty},$$

where

$$V_k = \begin{pmatrix} 1 & X_k \\ 0 & 1 \end{pmatrix}$$

and $\mathcal{N}_k$ is the space of N_k. In particular,

$$V_k f_k = \begin{pmatrix} X_k f_k \\ f_k \end{pmatrix} \in \mathcal{N}',$$

whence it readily follows that the distance from the unit sphere of $\mathcal{N}'$ to $\mathcal{M}'$ cannot exceed

$$\|f_k\|/\|V f_k\| = (1 + \|X_k f_k\|^2)^{-1/2}.$$

If $\sup_k \|X_k f_k\| = \infty$, it is immediate that $\mathcal{M}'$ and $\mathcal{N}'$ cannot be complementary.

Assume that $\|X_k f_k\| \leqq C$ for all $k \geqq 1$ (for some $C > 0$); then $\|V_k f_k\| \leqq 1 + C$, and

$$S(V_k f_k) = S(X_k f_k) + e_k + g_k \in \mathcal{N}'$$

satisfies

$$\|S(V_k f_k)\| \leqq C + 1 + 1/k \leqq C + 2.$$

Since $S(X_k f_k) + e_k \in \mathcal{M} \subset \mathcal{M}'$, the distance from the unit sphere of $\mathcal{N}'$ to $\mathcal{M}'$ cannot exceed

$$\|g_k\|/\|S(V_k f_k)\| < (C + 2)/k \longrightarrow 0 (k \longrightarrow \infty).$$

Once again, we conclude that $\mathcal{M}'$ and $\mathcal{N}'$ cannot be complementary subspaces.

(g) If $T, E \in \mathcal{L}(\mathcal{H})$ and $E = E^2$ belongs to the double commutant of T, then $\mathcal{X} = \mathcal{M} \dotplus \mathcal{N}$ (= direct sum of the Banach spaces $\mathcal{M}$ and $\mathcal{N}$), where $\mathcal{M} = \operatorname{ran} E$ and $\mathcal{N} = \ker E$ are invariant under every operator $A \in \mathcal{L}(\mathcal{H})$ commuting with T, that is, $\mathcal{M}$ and $\mathcal{N}$ are hyperinvariant. It follows from [**8**, Proposition 4.1] that if $S \in \mathcal{L}(\mathcal{Y})$, and $TX = XS$ and $YT = SY$ (X, Y quasiaffinities), then

$$\mathcal{M}'' = \vee\{RX\mathcal{M} : RS = SR\} \quad \text{and} \quad \mathcal{N}'' = \vee\{RX\mathcal{N} : RS = SR\}$$

are hyperinvariant subspaces of S; moreover, $\mathcal{M}''$ and $\mathcal{N}''$ are *quasicomplementary*, that is, $\mathcal{M}'' \cap \mathcal{N}'' = \{0\}$ and $\mathcal{M}'' + \mathcal{N}''$ is dense in $\mathcal{Y}$.

In particular, if $T, S, \mathcal{M}$ and $\mathcal{N}$ are defined as in Example (f), and E is the Riesz idempotent corresponding to the clopen subset $\sigma(M)$ of $\sigma(T)$, then $E = E^2$ belongs to the double commutant of T, and $\mathcal{M}'' = \mathcal{M}$ and $\mathcal{N}'' = \vee\{V_k\mathcal{N}_k\}_{k=1}^{\infty}$ are the corresponding quasicomplementary hyperinvariant subspaces of S. (It readily follows from (f) that $\mathcal{M}''$ *and* $\mathcal{N}''$ *are* not complementary.)

This example and the results of [**15**] might suggest that if $T \in \mathcal{L}(\mathcal{H})$ has two quasicomplementary invariant subspaces, $\mathcal{M}$ and $\mathcal{N}$, then T is quasisimilar to

$$S = (T|\mathcal{M}) \oplus (T|\mathcal{N}).$$

Nevertheless, this is utterly false: if T is the bilateral shift "multiplication by $e^{i\theta}$" on $L^2(\mathbf{T})$($\mathbf{T}$ is the unit circle), then $\mathcal{M} = H^2(\mathbf{T})$ and $\mathcal{N} = L^2$ ("upperhalf-circle") are quasicomplementary invariant subspaces of T. But there is no quasiaffinity Y such that $YT = SY$. Indeed $YT = SY$ is equivalent to $T^*Y^* = Y^*S^*$, and this last equation implies that each point of the open unit disk is an eigenvalue of T^*, because all those points are eigenvalues of S^*, and Y^* is injective. But T^* does not have any eigenvalue.

In this example, $\mathcal{N}$ is hyperinvariant, but $\mathcal{M}$ is not. A similar example can be constructed, where both $\mathcal{M}$ and $\mathcal{N}$ are hyperinvariant subspaces.

Let $\mathbf{A}^2(\mathbf{D})$ denote the norm-closure of the polynomials in $L^2(\mathbf{D}, dA)$, where $\mathbf{D}$ denotes the open unit disk and dA is the planar Lebesgue measure. $\mathbf{A}^2(\mathbf{D})$ is the *Bergman space* of the disk, and the operator B = "multiplication by λ" on $\mathbf{A}^2(\mathbf{D})$ is the Bergman shift. In [**21**, Corollary 2.3], J. H. Shapiro proved that there is a function f in $\mathbf{A}^2(\mathbf{D})$ such that if $\mathcal{M}_+(\mathcal{M}_-) = \{g \in \mathbf{A}^2(\mathbf{D}) : f(\lambda) = 1$ $(= -1$, resp.) $\Rightarrow g(\lambda) = 0\}$, then $\mathcal{M}_+ \cap \mathcal{M}_- = \{0\}$.

Clearly, $\mathcal{M}_+$ and $\mathcal{M}_-$ are invariant under B; furthermore, since B is a unilateral weighted shift, the commutant of B,

$$\mathcal{A}'(B) = \{R \in \mathcal{L}(\mathbf{A}^2(\mathbf{D})) : RB = BR\},$$

coincides with the weak closure of the polynomials in B[**22**], and therefore $\mathcal{M}_+$ and $\mathcal{M}_-$ are actually hyperinvariant subspaces of B.

Observe that, for each polynomial p,

$$p = \frac{1}{2}(f + 1)p - \frac{1}{2}(f - 1)p \in \mathcal{M}_+ = \mathcal{M}_-,$$

so that $(\mathcal{M}_+ + \mathcal{M}_-)^- = \mathbf{A}^2(\mathbf{D})$, whence we infer that $\mathcal{M}_+$ and $\mathcal{M}_-$ are quasi-complementary subspaces.

It is easy to check that $\sigma(B) = \mathbf{D}^-$, $\sigma_e(B) = \partial\mathbf{D}$ and $\lambda - B$ is a Fredholm operator of trivial kernel and index -1 for all $\lambda \in \mathbf{D}$ (see, e.g., [**16**, Chapter 3]); moreover,

$$\|B\| = 1 \quad \text{and} \quad \sigma(B|\mathcal{M}_+) = \sigma(B|\mathcal{M}_-) = \mathbf{D}^-.$$

Since $\lambda - B$ is bounded below for all λ in the unit disk, so are $\lambda - B|\mathcal{M}_+$ and $\lambda - B|\mathcal{M}_-$, so that these two operators are semi-Fredholm of negative index.

Hence, for $\lambda = 0$, we have

$$\begin{aligned}\operatorname{ind} B &= -1 < -2 \leqq \operatorname{ind} B|\mathcal{M}_+ + \operatorname{ind} B|\mathcal{M}_- \\ &= \operatorname{ind} (B|\mathcal{M}_+) \oplus (B|\mathcal{M}_-).\end{aligned}$$

Therefore, B cannot be quasisimilar to $(B|\mathcal{M}_+) \bigoplus (B|\mathcal{M}_-)$.

(h) We close this article with an example that "has been in the air" for several years. Define

$$T = \bigoplus_{n=1}^{\infty} [1/n = (1 - 1/n)q_n], \quad S = \bigoplus_{n=1}^{\infty} [1/n + (1/n)q_n];$$

then $T, S \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbf{C}^n$, and T and S are quasisimilar because $1/n + (1 - 1/n)q_n$ is similar to $1/n + 1/nq_n$ for each $n = 1, 2, \ldots$ (use [**15**]).

It is not difficult to check that T is a contraction and

$$\sigma(T) = \sigma_{lre}(T) = \{\lambda \in \mathbf{C} : |\lambda| \leqq 1\},$$

so that, T is a (BCP) operator, and therefore reflexive [**5**]. On the other hand, S is compact, $\sigma(S) = \{0\} \cup \{1/n\}_{n=1}^{\infty}$ does not disconnect the plane, and for $m \geqq 2$,

$$S = (1/m + 1/mq_m) \oplus S_m.$$

Since q_m is not reflexive and the invariant subspace lattice of S "splits" (that is, $S\mathcal{M} \subset \mathcal{M}$ if and only if $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathcal{M}_n$, where $\mathcal{M}_n$ is invariant under q_n for all $n = 1, 2, \ldots$), S cannot be reflexive.

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