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On a random solution of a nonlinear perturbed stochastic integral equation of the Volterra type

J. Susan Milton and Chris P. Tsokos

The object of this present paper is to study a nonlinear perturbed stochastic integral equation of the form

$$x(t;\ \omega) = h\big(t,\ x(t;\ \omega)\big) + \int_0^t k\big(\tau,\ x(\tau;\ \omega);\ \omega\big)d\tau\ ,\quad t \geq 0\ ,$$

where $\omega \in \Omega$, the supporting set of the complete probability measure space $(\Omega,\ A,\ \mu)$. We are concerned with the existence and uniqueness of a random solution to the above equation. A random solution, $x(t;\ \omega)$, of the above equation is defined to be a vector random variable which satisfies the equation μ almost everywhere.

1. Introduction

Stochastic integral equations play a major role in characterizing some very important problems in life sciences and engineering [*1*, *2*, *3*, *4*, *5*, *6*, *7*, *8*, *9*, *10*, *11*, *13*, *14*, *15*]. The object of the present study is concerned with a theoretical investigation of a class of nonlinear perturbed stochastic integral equations. More specifically, we consider a stochastic vector integral equation of the form

$$(1.1)\qquad x(t;\ \omega) = h\big(t,\ x(t;\ \omega)\big) + \int_0^t k\big(\tau,\ x(\tau;\ \omega);\ \omega\big)d\tau\ ,\quad t \geq 0\ ,$$

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where

(i) $\omega \in \Omega$, the supporting set of the complete probability measure space (Ω, A, μ);

(ii) $x(t; \omega)$ is the unknown m-dimensional vector valued random function defined on R_+, the non-negative real numbers;

(iii) under appropriate conditions the stochastic kernel $k(\tau, x(\tau; \omega); \omega)$ is an m-dimensional vector valued random function on R_+;

(iv) for each $t \in R_+$ and each m-dimensional vector valued random function $x(t; \omega)$, $h(t, x(t; \omega))$ is an m-dimensional vector valued random variable.

We shall be concerned with the existence and uniqueness of a random solution, a second order stochastic process, to random integral equation (1.1). The above equation is very important in the formulation of stochastic chemical kinetics models. The random equation (1.1) is a generalization of the recent study of Anderson [1] and Tsokos [12] in that both the stochastic kernel and the stochastic free term are functions of the unknown m-dimensional valued random function $x(t; \omega)$.

2. Preliminary concepts

We shall now define several spaces of functions and state lemmas which are essential in fulfilling the objectives of the present study.

DEFINITION 2.1. The random vectors

$$x(\omega) = \left(x_1(\omega), x_2(\omega), \ldots, x_m(\omega)\right)$$

and

$$y(\omega) = \left(y_1(\omega), y_2(\omega), \ldots, y_m(\omega)\right)$$

are said to be equal if and only if

$x_i(\omega) \equiv y_i(\omega)$ μ almost everywhere for each $i = 1, 2, \ldots, m$.

DEFINITION 2.2. Let $\Psi(\Omega, A, \mu)$ be the set of all random vectors of the form

$$z(\omega) = \left(z_1(\omega), z_2(\omega), \ldots, z_m(\omega)\right) ,$$

where for each $i = 1, 2, \ldots, m$, $z_i(\omega)$ is an element of $L_\infty(\Omega, A, \mu)$.

LEMMA 2.1. *$\Psi(\Omega, A, \mu)$ is a complete normed linear space over the reals with the usual definition of component-wise addition and scalar multiplication where the norm in $\Psi(\Omega, A, \mu)$ is given by*

$$\|z(\omega)\|_{\Psi(\Omega,A,\mu)} = \|z(\omega)\|_\Psi = \max_i |||\, z_i(\omega)\, ||| .$$

DEFINITION 2.3. Let $C_\Psi = C_\Psi\left(R_+, \Psi(\Omega, A, \mu)\right)$ be the set of all continuous functions from R_+ into $\Psi(\Omega, A, \mu)$.

Note that for each $t \in R_+$ we get an associated random vector $x(t; \omega) = \left(x_1(t; \omega), x_2(t; \omega), \ldots, x_m(t; \omega)\right)$. We shall be tacitly assuming that for each i the sample function $x_i(t; \omega)$ is continuous in t for each ω . Since we are dealing with a finite measure space, for each t and each i , $E|x_i(t; \omega)| < \infty$. The main purpose for defining the norm in $\Psi(\Omega, A, \mu)$ as it was done was to enable us to obtain a relatively simple norm defined in terms of the components of the vector involved.

LEMMA 2.2. *C_Ψ is a linear space over the reals with the usual definitions of addition and scalar multiplication for continuous functions.*

LEMMA 2.3. *Let*

$$F = \left\{\|x(t; \omega)\|_n : \|x(t; \omega)\|_n\right\} = \sup_{0\le t\le n} \left\{\|x(t; \omega)\|_\Psi\right\} ,$$

$n = 1, 2, 3, \ldots$. *F is a family of semi-norms defined on C_Ψ .*

LEMMA 2.4. *The space C_Ψ can be topologized by the family F of semi-norms defined in Lemma 2.3 and the topology obtained is locally convex and Hausdorff.*

LEMMA 2.5. *The topology τ on C_Ψ induced by the family F of semi-norms defined in Lemma 2.3 is metrizable where the metric ρ is* *defined by*

$$\rho\big(x(t;\ \omega),\ y(t;\ \omega)\big) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x(t;\omega)-y(t;\omega)\|_n}{1+\|x(t;\omega)-y(t;\omega)\|_n} .$$

The following lemma is important in that it characterizes the topology τ defined on C_Ψ in a convenient manner.

LEMMA 2.6. *The topology* τ *on* C_Ψ *induced by the family* F *of semi-norms (and hence also by* ρ *) is the topology of uniform convergence. That is,* $x^m(t;\ \omega) \xrightarrow{\tau} x(t;\ \omega)$ *if and only if* $\lim_{m\to\infty} \|x^m(t;\ \omega)-x(t;\ \omega)\| = 0$ *uniformly on every interval* $[0,\ M] \leq R_+$.

Throughout the paper T will represent a linear operator from $C_\Psi\big(R_+,\ \Psi(\Omega,\ A,\ \mu)\big) \to C_\Psi\big(R_+,\ \Psi(\Omega,\ A,\ \mu)\big)$ and B and D will represent Banach spaces contained in $C_\Psi\big(R_+,\ \Psi(\Omega,\ A,\ \mu)\big)$.

DEFINITION 2.4. The Banach space B is said to be stronger than the space C_Ψ if every sequence which converges in B with respect to its norm also converges in C_Ψ but the converse need not be true.

DEFINITION 2.5. The pair of spaces $(B,\ D)$ will be called admissible with respect to the operator T if and only if $TB \subseteq D$.

LEMMA 2.7. *Let* T *be a continuous linear operator from* $C_\Psi \to C_\Psi$. *If the pair of Banach spaces* B *and* D *are stronger than* C_Ψ *and if* $(B,\ D)$ *is admissible with respect to* T *then* T *is continuous from* B *to* D .

Note that since T is a continuous operator from B to D it is bounded and hence there exists a constant Q such that

$$\|(Tx)(t;\ \omega)\|_D \leq Q\|x(t;\ \omega)\|_B .$$

Thus we can define a norm on T by

$$M = \|T\|_0 = \sup\left\{\frac{\|(Tx)(t;\omega)\|_D}{\|x(t;\omega)\|_B} : x(t;\ \omega) \in B,\ \|x(t;\ \omega)\|_B \neq 0\right\} .$$

DEFINITION 2.6. By a random solution of equation 1.1 we shall mean

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the following: the random vector valued function $x(t;\omega)$ on R_+ is a random solution of equation 1.1 if for each fixed $t \geq 0$, $x(t;\omega)$ is a vector random variable and satisfies equation 1.1 μ almost everywhere.

LEMMA 2.8. *The operator* T *defined on* C_Ψ *by*

$$(Tx)(t;\omega) = \int_0^t x(\tau;\omega)d\tau$$

is a continuous linear operator from C_Ψ *into* C_Ψ.

DEFINITION 2.7. Let $C'_g = C'_g(R_+, \Psi(\Omega, A, \mu))$ be the collection of all continuous functions $x(t;\omega)$ from R_+ into $\Psi(\Omega, A, \mu)$ such that for g a positive valued continuous function on R_+ we have

$$\|x(t;\omega)\|_\Psi \leq Ag(t)$$

for some positive constant A.

LEMMA 2.9. C'_g *is a complete normed linear subspace of* C_Ψ *where the norm in* C'_g, *denoted* $\|x(t;\omega)\|_{C'_g}$, *is given by*

$$\|x(t;\omega)\|_{C'_g} = \sup_{0\leq t}\left\{\frac{\|x(t;\omega)\|_\Psi}{g(t)}\right\}.$$

DEFINITION 2.8. Let $C' = C'(R_+, \Psi(\Omega, A, \mu))$ be the collection of all continuous and bounded functions $x(t;\omega)$ from R_+ into $\Psi(\Omega, A, \mu)$.

LEMMA 2.10. C' *is a complete normed linear subspace of* C_Ψ *where the norm in* C', *denoted* $\|x(t;\omega)\|_{C'}$, *is given by*

$$\|x(t;\omega)\|_{C'} = \sup_{0\leq t}\left\{\|x(t;\omega)\|_\Psi\right\}.$$

LEMMA 2.11. *The Banach spaces* C'_g *and* C' *are stronger than* C_Ψ.

DEFINITION 2.9. Let E be an arbitrary metric space with metric ρ. A mapping Z of $E \to E$ is called a contraction if there exists a real number r, $0 \leq r < 1$ such that $\rho(Z(x), Z(y)) \leq r\rho(x, y)$ for all x, y in E.

THEOREM 2.1 (Banach's Fixed Point Theorem). *If a contraction operator* Z *is defined on a complete metric space* E *, then there exists a unique point* $x^* \in E$ *such that* $Zx^* = x^*$.

3. Existence of a random solution

With respect to the aims of the present study, we state and prove the following theorems.

THEOREM 3.1. *Assume that equation* (1.1) *satisfies the following conditions:*

(i) $B, D \subseteq C_\psi$ *are Banach spaces stronger than* C_ψ *and the pair* (B, D) *is admissible with respect to the operator* T *given by* $(Tx)(t; \omega) = \int_0^t x(\tau; \omega)d\tau$ *;*

(ii) $k(t, x(t; \omega); \omega)$ *is a mapping from the set*

$$W = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \le \rho\}$$

into the space B *for some* $\rho \ge 0$ *such that*

$$\|k(t, x(t; \omega); \omega)-k(t, y(t; \omega); \omega)\|_B \le \lambda\|x(t; \omega)-y(t; \omega)\|_D$$

for $x(t; \omega), y(t; \omega) \in W$ *and a constant* $\lambda \ge 0$ *;*

(iii) $x(t; \omega) \to h(t, x(t; \omega))$ *is a mapping from* W *into* D *such that*

$$\|h(t, x(t; \omega))-h(t, y(t; \omega))\|_D \le \gamma\|x(t; \omega)-y(t; \omega)\|_D$$

for some $\gamma \ge 0$.

Then there exists a unique random solution of equation (1.1), *an element of* W *, provided that* $\gamma + \lambda M < 1$ *, where* $M = \|T\|_0$ *and*

$$\|h(t, x(t; \omega))\|_D + M\|k(t, x(t; \omega); \omega)\|_B \le \rho .$$

Proof. Note that by Lemmas 2.7 and 2.8 the operator $(Tx)(t; \omega) = \int_0^t x(\tau; \omega)d\tau$ is continuous from B to D . Define the operator U from W into D by

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$$(Ux)(t;\,\omega) = h\big(t,\, x(t;\,\omega)\big) + \int_0^t k\big(\tau,\, x(\tau;\,\omega);\, \omega\big)d\tau\ .$$

We must show that $UW \subseteq W$ and that for some $r \in [0, 1)$,

$$\|(Ux)(t;\,\omega)-(Uy)(t;\,\omega)\|_D \le r\|x(t;\,\omega)-y(t;\,\omega)\|_D\ .$$

Let $x(t;\,\omega),\, y(t;\,\omega) \in W$. Since $(Ux)(t;\,\omega)$ and $(Uy)(t;\,\omega) \in D$ and D is a Banach space, $(Ux)(t;\,\omega) - (Uy)(t;\,\omega) \in D$. Thus,

$$\begin{aligned}\|(Ux)(t;\,\omega)-(Uy)(t;\,\omega)\|_D &= \Big\|h\big(t,\, x(t;\,\omega)\big) + \int_0^t k\big(\tau,\, x(\tau;\,\omega);\, \omega\big)d\tau \\ &\qquad - h\big(t,\, y(t;\,\omega)\big) - \int_0^t k\big(\tau,\, y(\tau;\,\omega);\, \omega\big)d\tau\Big\|_D \\ &= \Big\|h\big(t,\, x(t;\,\omega)\big)-h\big(t,\, y(t;\,\omega)\big) \\ &\qquad + \int_0^t \big[k\big(\tau,\, x(\tau;\,\omega);\, \omega\big) - k\big(\tau,\, y(\tau;\,\omega);\, \omega\big)\big]d\tau\Big\|_D \\ &\le \|h\big(t,\, x(t;\,\omega)\big)-h\big(t,\, y(t;\,\omega)\big)\|_D \\ &\qquad + \Big\|\int_0^t \big[k\big(\tau,\, x(\tau;\,\omega);\, \omega\big) - k\big(\tau,\, y(\tau;\,\omega);\, \omega\big)\big]d\tau\Big\|_D \\ &\le \gamma\|x(t;\,\omega)-y(t;\,\omega)\|_D \\ &\qquad + M\|k\big(t,\, x(t;\,\omega);\, \omega\big)-k\big(t,\, y(t;\,\omega);\, \omega\big)\|_B\end{aligned}$$

where the last inequality is due to the Lipschitz condition and the fact that T is continuous from B to D and therefore bounded. However,

$$\begin{aligned}\gamma\|x(t;\,\omega)-y(t;\,\omega)\|_D &+ M\|k\big(t,\, x(t;\,\omega);\, \omega\big)-k\big(t,\, y(t;\,\omega);\, \omega\big)\|_B \\ &\le \gamma\|x(t;\,\omega)-y(t;\,\omega)\|_D + M\lambda\|x(t;\,\omega)-y(t;\,\omega)\|_D \\ &= (\gamma+M\lambda)\|x(t;\,\omega)-y(t;\,\omega)\|_D\ .\end{aligned}$$

Since $\gamma + M\lambda < 1$, one condition of the definition of contraction map is satisfied.

We must now show inclusion. Let $x(t;\,\omega) \in W$. We have

$$\|(Ux)(t;\ \omega)\|_D = \left\|h\big(t,\ x(t;\ \omega)\big) + \int_0^t k\big(\tau,\ x(\tau;\ \omega);\ \omega\big)d\tau\right\|_D$$
$$\leq \|h\big(t,\ x(t;\ \omega)\big)\|_D + \left\|\int_0^t k\big(\tau,\ x(\tau;\ \omega);\ \omega\big)d\tau\right\|_D$$
$$\leq \|h\big(t,\ x(t;\ \omega)\big)\|_D + M\|k\big(t,\ x(t;\ \omega);\ \omega\big)\|_B$$
$$\leq \rho\ .$$

Hence, $(Ux)(t;\ \omega) \in W$ implying $UW \subseteq W$. Thus by Banach's fixed point theorem there exists a unique point $x(t;\ \omega) \in W$ such that

$$(Ux)(t;\ \omega) = h\big(t,\ x(t;\ \omega)\big) + \int_0^t k\big(\tau,\ x(\tau;\ \omega);\ \omega\big)d\tau = x(t;\ \omega)$$

and the proof is complete.

The following theorem is a special case of Theorem 3.1 which is useful in various applications.

THEOREM 3.2. *Assume that equation* (1.1) *satisfies the following conditions:*

(i) $k\big(t,\ x(t;\ \omega);\ \omega\big)$ *is a mapping from the set*

$$W = \{x(t;\ \omega)\ :\ x(t;\ \omega) \in C',\ \|x(t;\ \omega)\|_{C'} \leq \rho\}$$

into the space C'_g *for some* $\rho \geq 0$ *;*

$$\|k\big(t,\ x(t;\ \omega);\ \omega\big)-k\big(t,\ y(t;\ \omega);\ \omega\big)\|_{C'_g} \leq \lambda\|x(t;\ \omega)-y(t;\ \omega)\|_{C'}$$

for $x(t;\ \omega),\ y(t;\ \omega) \in W$ *,* $\lambda \geq 0$ *a constant;* g *is also integrable on* R_+ *;*

(ii) $x(t;\ \omega) \to h\big(t,\ x(t;\ \omega)\big)$ *is a mapping from* W *into* C' *such that*

$$\|h\big(t,\ x(t;\ \omega)\big)-h\big(t,\ y(t,\ \omega)\big)\|_{C'} \leq \gamma\|x(t;\ \omega)-y(t;\ \omega)\|_{C'}$$

for some $\gamma \geq 0$.

Then there exists a unique random solution of equation (1.1) $\in W$ *provided that* $\gamma + \lambda M < 1$ *, where* $M = \|T\|_0$ *(*T *as defined in Theorem* 3.1 *(i)) and*

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$$\|h(t, x(t; \omega))\|_{C'} + M\|k(t, x(t; \omega); \omega)\|_{C'_g} \le \rho .$$

The proof consists of showing that under the assumption g is admissible with respect to the operator T given by

$$(Tx)(t; \omega) = \int_0^t x(\tau; \omega)d\tau .$$

Let $x(t; \omega) \in C'_g$. Consider

$$\begin{aligned}
|(Tx_i)(t; \omega)| &= \left|\int_0^t x_i(\tau; \omega)d\tau\right| \\
&\le \int_0^t |x_i(\tau; \omega)|d\tau \\
&\le \int_0^t |||\, x_i(\tau; \omega)\, |||d\tau , \quad \mu \text{ almost everywhere} \\
&\le \int_0^t \frac{|||\, x_i(\tau;\omega)\, |||}{g(\tau)} g(\tau)d\tau \\
&\le \int_0^t \frac{\|x(\tau;\omega)\|}{g(\tau)} g(\tau)d\tau \\
&\le \int_0^\infty \frac{\|x(\tau;\omega)\|_\psi}{g(\tau)} g(\tau)d\tau \\
&\le \|x(\tau; \omega)\|_{C'_g} \int_0^\infty g(\tau)d\tau \\
&= \beta .
\end{aligned}$$

By definition of the norm in $L_\infty(\Omega, A, \mu)$, we can conclude that $|||\, (Tx_i)(t, \omega)\, ||| \le \beta$ for each i . This in turn implies that

$$\|(Tx)(t; \omega)\|_\psi = \max_i \{|||\, (Tx_i)(t; \omega)\, |||\} < \beta ,$$

which is the condition needed for $(Tx)(t; \omega)$ to be an element of C' . Since the remaining conditions are identical to those of Theorem 3.1 the proof is complete.

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Radford College,
Radford,
Virginia,
USA;

Department of Mathematics,
University of South Florida,
Tampa,
Florida,
USA.