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A NOTE ON THE CONNECTEDNESS OF THE BRANCH LOCUS OF RATIONAL MAPS

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Abstract. Milnor proved that the moduli space M_d of rational maps of degree $d \geq 2$ has a complex orbifold structure of dimension $2(d-1)$. Let us denote by $\mathcal{S}_d$ the singular locus of M_d and by $\mathcal{B}_d$ the branch locus, that is, the equivalence classes of rational maps with non-trivial holomorphic automorphisms. Milnor observed that we may identify M_2 with $\mathbb{C}^2$ and, within that identification, that $\mathcal{B}_2$ is a cubic curve; so $\mathcal{B}_2$ is connected and $\mathcal{S}_2 = \emptyset$. If $d \geq 3$, then it is well known that $\mathcal{S}_d = \mathcal{B}_d$. In this paper, we use simple arguments to prove the connectivity of $\mathcal{S}_d$.

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1. Introduction. The space Rat_d of complex rational maps of degree $d \geq 2$ can be identified with a Zariski open set of the $(2d+1)$-dimensional complex projective space $\mathbb{P}^{2d+1}_{\mathbb{C}}$; this is the complement of the algebraic hypersurface defined by the resultant of two polynomials of degree at most d.

The group of Möbius transformations $\mathrm{PSL}_2(\mathbb{C})$ acts on Rat_d by conjugation: $\phi, \psi \in \mathrm{Rat}_d$ are said to be equivalent if there is some $T \in \mathrm{PSL}_2(\mathbb{C})$ so that $\psi = T \circ \phi \circ T^{-1}$. The $\mathrm{PSL}_2(\mathbb{C})$-stabilizer of $\phi \in \mathrm{Rat}_d$, denoted as $\mathrm{Aut}(\phi)$, is the group of holomorphic automorphisms of ϕ. As the subgroups of $\mathrm{PSL}_2(\mathbb{C})$ keeping invariant a finite set of cardinality at least 3 must be finite, it follows that $\mathrm{Aut}(\phi)$ is finite. Levy [**6**] observed that the order of $\mathrm{Aut}(\phi)$ is bounded above by a constant depending on d.

The quotient space $\mathrm{M}_d = \mathrm{Rat}_d/\mathrm{PSL}_2(\mathbb{C})$ is the moduli space of rational maps of degree d. Silverman [**10**] obtained that M_d carries the structure of an affine geometric quotient, Milnor [**9**] proved that it also carries the structure of a complex orbifold of dimension $2(d-1)$ (Milnor also obtained that $\mathrm{M}_2 \cong \mathbb{C}^2$) and Levy [**6**] noted that M_d is a rational variety. Let us denote by $\mathcal{S}_d \subset \mathrm{M}_d$ the singular locus of M_d, that is, the set of points over which M_d fails to be a topological manifold. The branch locus of M_d is the set $\mathcal{B}_d \subset \mathrm{M}_d$ consisting of those (classes of) rational maps with non-trivial group of holomorphic automorphisms.

As $\mathrm{M}_2 \cong \mathbb{C}^2$, clearly $\mathcal{S}_2 = \emptyset$. Using this identification, the locus $\mathcal{B}_2$ corresponds to the cubic curve [**4**]:

$$2x^3 + x^2y - x^2 - 4y^2 - 8xy + 12x + 12y - 36 = 0,$$

where the cuspid $(-6, 12)$ corresponds to the class of a rational map $\phi(z) = 1/z^2$ with $\mathrm{Aut}(\phi) \cong D_3$ (dihedral group of order 6) and all other points in the cubic corresponds to those classes of rational maps with the cyclic group C_2 as full group of holomorphic automorphisms. In this way, $\mathcal{B}_2$ is connected.

If $d \geq 3$, then it was proved in [**7**] that $\mathcal{S}_d = \mathcal{B}_d$. In [**8**], Manes proved that the sublocus of $\mathcal{S}_d$ consisting of those classes having a point of formal period N is geometrically reducible for infinitely many N. In [**4**], Fujimura proved that the singular locus of the moduli space of polynomial maps of degree three is connected (this being an irreducible algebraic curve of degree three). At this point, one may wonder for the connectivity of $\mathcal{S}_d$. To the authors knowledge, this question has not been considered in the literature (see Remark 2 below for the genesis of this question) with the exception of the polynomial case in [**4**]. The aim of this paper is to provide an affirmative answer.

THEOREM 1. *If $d \geq 3$, then the singular locus $\mathcal{S}_d = \mathcal{B}_d$ is connected.*

If we denote by $\mathcal{B}_d(C_n)$ the locus in moduli space M_d of classes of rational maps of degree d admitting a holomorphic automorphism of order n, then $\mathcal{B}_d$ is union of these loci. So in order to prove the above one needs to see how these loci intersect.

To prove Theorem 1, we first provide a description of those rationals maps admitting a given cyclic group of holomorphic automorphisms; which we state as Theorem 2. We had realized that such a description was previously obtained in [**7**]. Ours description is more explicit and more adequate for our needs and a proof is provided in Section 2; our arguments are a little different, but follows the same general idea. In fact, our description permits to see explicitly $\mathcal{B}_d(C_n)$ as a Zariski open subset of Rat_r for a suitable r (see the proof of Corollary 2). Consequences of such a description are that $\mathcal{B}_d(C_2)$ is non-empty for every degree $d \geq 2$ (Corollary 1) and that $\mathcal{B}_d(C_n)$ (if non-empty) is connected (Corollary 2); we should say that this was also observed in Proposition 3 of [**7**].

The final point of the proof of Theorem 1 is Lemma 1, which asserts that if $\mathcal{B}_d(C_n) \neq \emptyset$, then $\mathcal{B}_d(C_2) \cap \mathcal{B}_d(C_n) \neq \emptyset$; this done by explicit constructions of rational maps admitting a dihedral group of order $2n$ as group of holomorphic automorphisms (again, this is due to the fact that we have presented a more detailed description of those rational maps admitting such kind of groups of automorphisms). It seems that this fact was not observed in [**7**].

REMARK 1. Theorem 1 states that given any two rational maps $\phi, \psi \in \mathrm{Rat}_d$, both with non-trivial group of holomorphic automorphisms, there is some $\rho \in \mathrm{Rat}_d$ which is equivalent to ψ and there is a continuous family $\Theta : [0, 1] \to \mathrm{Rat}_d$ with $\Theta(0) = \phi$, $\Phi(1) = \rho$ and $\mathrm{Aut}(\Theta(t))$ non-trivial for every t. At this point, we need to observe that if $\mathrm{Aut}(\phi) \cong \mathrm{Aut}(\psi)$, we may not ensure that $\mathrm{Aut}(\Theta(t))$ stay in the same isomorphic class; this comes from the existence of rigid rational maps [**3**] (in the non-cyclic situation).

REMARK 2 (On the genesis of this paper). In the 80's, Sullivan provided a dictionary between dynamic of rational maps and the dynamic of Kleinian groups [**11**]. If we restrict to Klenian groups being co-compact Fuchsian groups of a fixed genus $g \geq 2$, then we are dealing with closed Riemann surfaces of genus g whose moduli space $\mathcal{M}_g$ has the structure of an orbifold of complex dimension $3(g-1)$. The branch locus in $\mathcal{M}_g$, that is, the set of isomorphic classes of Riemann surfaces with non-trivial holomorphic automorphisms, is in general non-connected [**1**]. After attending a talk given by one of the authors of the previous paper, we were wondering about the connectivity of the singular locus of moduli spaces of Kleinian groups. In [**5**], Izquierdo and the first author proved that the singular locus of Schottky space was connected for odd rank and that it has two connected components for even rank. It was then natural to ask for the connectedness problem for the singular locus of moduli spaces of rational maps and this was the genesis of this paper. The techniques we use in this paper are quite similar to those used in [**1,5**], adapted to the case of rational maps, together with the description of rational maps with extra automorphisms as done in [**7**].

2. Rational maps with non-trivial group of holomorphic automorphisms. It is well known that a non-trivial finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is either isomorphic to a cyclic group C_n or the dihedral group D_n (of order $2n$) or one of the alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ or the symmetric group $\mathfrak{S}_4$ (see, for instance, [**2**]). So, the group of holomorphic automorphisms of a rational map of degree at least two is isomorphic to one of the previous ones. Moreover, for each such finite subgroup there is a rational map admitting it as group of holomorphic automorphisms [**3**].

Let G be either C_n $(n \geq 2)$, D_n $(n \geq 2)$, $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathfrak{S}_4$. Let us denote by $\mathcal{B}_d(G) \subset \mathrm{M}_d$ the locus of classes of rational maps ϕ with $\mathrm{Aut}(\phi)$ containing a subgroup isomorphic to G. We say that G is admissible for d if $\mathcal{B}_d(G) \neq \emptyset$.

If G is either C_n or D_n or $\mathcal{A}_4$, then there may be some elements in $\mathcal{B}_d(G)$ with full group of holomorphic automorphisms non-isomorphic to G (i.e., they admit more holomorphic automorphisms than G). If G is either isomorphic to $\mathfrak{S}_4$ or $\mathcal{A}_5$, then every element in $\mathcal{B}_d(G)$ has G as its full group of holomorphic automorphisms and it may have isolated points [**3**], so it is not connected in general.

In this section we recall a description of those values of d for which G is admissible and the dimensions of $\mathcal{B}_d(G)$ obtained in [**7**]. Since our main interest will be in the cyclic and dihedral cases, we present the explicit computations in those cases; in fact, we provide a more complete description as done in Lemmas 2 and 5 of [**7**] (see Theorems 2 and 3). As a matter of completeness we write down the cases of solid Paltonics without proofs (which can be found in [**7**])

2.1. Admissibility in the cyclic case. In the case of $G = C_n, n \geq 2$, the admissibility will depend on d. First, let us observe that if a rational map admits C_n as a group of holomorphic automorphisms, then we may conjugate it by a suitable Möbius transformation so that we may assume C_n to be generated by the rotation $T(z) = \omega_n z$, where $\omega_n = e^{2\pi i/n}$.

THEOREM 2. *Let $d, n \geq 2$ be integers. The group C_n is admissible for d if and only if d is congruent to either $-1, 0, 1$ modulo n. Moreover, for such values, every rational map of degree d admitting C_n as a group of holomorphic automorphisms is equivalent to one of the form $\phi(z) = z\psi(z^n)$, where*

$$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \mathrm{Rat}_r,$$

satisfies that

(*a*) $a_r b_0 \neq 0$, *if* $d = nr + 1$;
(*b*) $a_r \neq 0$ *and* $b_0 = 0$, *if* $d = nr$;
(*c*) $a_r = b_0 = 0$ *and* $b_r \neq 0$, *if* $d = nr - 1$.

In the above case, C_n is generated by the rotation $T(z) = \omega_n z$.

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Proof. Let ϕ be a rational map admitting a holomorphic automorphism of order n. By conjugating it by a suitable Möbius transformation, we may assume that such automorphism is the rotation $T(z) = \omega_n z$.

(1) Let us write $\phi(z) = z\rho(z)$. The equality $T \circ \phi \circ T^{-1} = \phi$ is equivalent to $\rho(\omega_n z) = \rho(z)$. Let

$$\rho(z) = \frac{U(z)}{V(z)} = \frac{\sum_{k=0}^{l} \alpha_k z^k}{\sum_{k=0}^{l} \beta_k z^k},$$

where either $\alpha_l \neq 0$ or $\beta_l \neq 0$ and $(U, V) = 1$.
The equality $\rho(\omega_n z) = \rho(z)$ is equivalent to the existence of some $\lambda \neq 0$ so that

$$\omega_n^k \alpha_k = \lambda \alpha_k, \quad \omega_n^k \beta_k = \lambda \beta_k.$$

By taking $k = l$, we obtain that $\lambda = \omega_n^l$. So, the above is equivalent to have, for $k < l$,

$$\omega_n^{l-k} \alpha_k = \alpha_k, \quad \omega_n^{l-k} \beta_k = \beta_k.$$

So, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $l - k \equiv 0 \mod (n)$. As $(U, V) = 1$, either $\alpha_0 \neq 0$ or $\beta_0 \neq 0$; so $l \equiv 0 \mod (n)$. In this way, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $k \equiv 0 \mod (n)$. In this way, $\rho(z) = \psi(z^n)$ for a suitable rational map $\psi(z)$.

(2) It follows from (1) that $\phi(z) = z\psi(z^n)$, for $\psi \in \mathrm{Rat}_r$ and suitable r. We next provide relations between d and r. Let us write

$$\psi(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k},$$

where $(P, Q) = 1$ and either $a_r \neq 0$ or $b_r \neq 0$. In this way,

$$\phi(z) = \frac{zP(z^n)}{Q(z^n)} = \frac{z\sum_{k=0}^{r} a_k z^{kn}}{\sum_{k=0}^{r} b_k z^{kn}}.$$

Let us first assume that $Q(0) \neq 0$, equivalently, $\psi(0) \neq \infty$. Then, $\phi(0) = 0$ and the polynomials $zP(z^n)$ and $Q(z^n)$ are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) \neq 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = 1 + nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr$.

Let us now assume that $Q(0) = 0$, equivalently, $\psi(0) = \infty$. Let us write $Q(u) = u^l \widehat{Q}(u)$, where $l \geq 1$ and $\widehat{Q}(0) \neq 0$; so, $\deg(Q) = l + \deg(\widehat{Q})$. In this case,

$$\phi(z) = \frac{P(z^n)}{z^{ln-1}\widehat{Q}(z^n)}$$

and the polynomials $P(z^n)$ (of degree $n\deg(P)$) and $z^{ln-1}\widehat{Q}(z^n)$ (of degree $n\deg(Q) - 1$) are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) \neq 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr - 1$.

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Summarizing all the above, we have the following situations:

(i) If $\phi(0) = 0$ and $\phi(\infty) = \infty$, then $\psi(0) \neq \infty$ and $\psi(\infty) \neq 0$; in particular, $d = nr + 1$. This case corresponds to have $a_r b_0 \neq 0$.
(ii) If $\phi(0) = \infty = \phi(\infty)$, then $\psi(0) = \infty$ and $\psi(\infty) \neq 0$; in which case $d = nr$. This case corresponds to have $a_r \neq 0$ and $b_0 = 0$.
(iii) If $\phi(0) = 0 = \phi(\infty)$, then $\psi(0) \neq \infty$ and $\psi(\infty) = 0$; in particular, $d = nr$. This case corresponds to have $a_r = 0$ and $b_0 \neq 0$. But in this case, we may conjugate ϕ by $A(z) = 1/z$ (which normalizes $\langle T \rangle$) in order to be in case (ii) above.
(iv) If $\phi(0) = \infty$ and $\phi(\infty) = 0$, then $\psi(0) = \infty$ and $\psi(\infty) = 0$; in particular, $d = nr - 1$. This case corresponds to have $a_r = b_0 = 0$ (in which case $b_r \neq 0$ as ψ has degree r).

□

COROLLARY 1. *C_2 is admissible for every $d \geq 2$.*

The explicit description provided in Theorem 2 permits to obtain the connectivity of $\mathcal{B}_d(C_n)$ and its dimension (see Proposition 3 in [7]).

COROLLARY 2. *If $n \geq 2$ and C_n is admissible for d, then $\mathcal{B}_d(C_n)$ is connected and*

$$\dim_{\mathbb{C}}(\mathcal{B}_d(C_n)) = \begin{cases} 2(d-1)/n, & d \equiv 1 \mod n \\ (2d-n)/n, & d \equiv 0 \mod n \\ 2(d+1-n)/n, & d \equiv -1 \mod n. \end{cases}$$

Proof.

(1) By Theorem 2, the rational maps in Rat_d admitting a holomorphic automorphism of order $n \geq 2$ are conjugated those of the form $\phi(z) = z\psi(z^n) \in \mathrm{Rat}_d$ for $\psi \in \mathrm{Rat}_r$ as described in the same theorem.
Let us denote by $\mathrm{Rat}_d(n, r)$ the subset of Rat_d formed by all those rational maps of the $\phi(z) = z\psi(z^n)$, where ψ satisfies the conditions in Theorem 2.
If $d = nr + 1$, then we may identify $\mathrm{Rat}_d(n, r)$ with an open Zariski subset of Rat_r; if $d = nr$, then it is identified with an open Zariski subset of a linear hypersurface of Rat_r; and if $d = nr - 1$, then it is identified with an open Zariski subspace of a linear subspace of co-dimension two of Rat_r. In each case, we have that $\mathrm{Rat}_d(n, r)$ is connected. As the projection of $\mathrm{Rat}_d(n, r)$ to M_d is exactly $\mathcal{B}_d(C_n)$, we obtain its connectivity.
(2) The dimension counting. We may see that, if $d = nr + 1$, then ψ depends on $2r + 1$ complex parameters; if $d = nr$, then ψ depends on $2r$ complex parameters; and if $d = nr - 1$, then ψ depends on $2r - 1$ complex parameters. The normalizer in $\mathrm{PSL}_2(\mathbb{C})$ of $\langle T \rangle$ is the 1-complex dimensional group $N_n = \langle A_\lambda(z) = \lambda z, B(z) = 1/z : \lambda \in \mathbb{C} - \{0\} \rangle$. If $U \in N_n$, then $U \circ \phi \circ U^{-1}$ will also have T as a holomorphic automorphism. In fact,

$$A_\lambda \circ \phi \circ A_\lambda^{-1}(z) = z\psi(z^n/\lambda^n),$$

$$B \circ \phi \circ B(z) = z/\psi(1/z^n).$$

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In this way, there is an action of N_n over Rat_r so that the orbit of $\psi(u)$ is given by the rational maps $\psi(u/t)$, where $t \in \mathbb{C} - \{0\}$, and $1/\psi(1/u)$. In this way, we obtain the desired dimensions. □

2.2. Admissibility in the dihedral case. Let us now assume $\phi \in \mathrm{Rat}_d$ admits the dihedral group D_n, $n \geq 2$, as a group of holomorphic automorphisms. Up to conjugation, we may assume that D_n is generated by $T(z) = \omega_n z$ and $A(z) = 1/z$. By Theorem 2, we may assume that $\phi(z) = z\psi(z^n)$, where

$$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \mathrm{Rat}_r,$$

where either

(a) $a_r b_0 \neq 0$, if $d = nr + 1$;
(b) $a_r \neq 0$ and $b_0 = 0$, if $d = nr$;
(c) $a_r = b_0 = 0$ and $b_r \neq 0$, if $d = nr - 1$;

with the extra condition that $\psi(z) = 1/\psi(1/z)$. This last condition is equivalent to the existence of some $\lambda \neq 0$ so that

$$\lambda a_k = b_{r-k}, \quad \lambda b_k = a_{r-k}, \quad k = 0, 1, ..., r.$$

The above is equivalent to have $\lambda \in \{\pm 1\}$ and $b_k = \lambda a_{r-k}$, for $k = 0, 1, ..., r$. In particular, this asserts that $a_r = 0$ if and only if $b_0 = 0$ (so case (b) above does not hold). Also, as the normalizer of the dihedral group $D_n = \langle T(z) = \omega_n z, A(z) = 1/z \rangle$ is a finite group, the dimension of $\mathcal{B}_d(D_n)$ is the same as half the projective dimension of those rational maps ψ satisfying (a) or (c). So, we may conclude the following result (this is a more complete description as done in Lemma 5 of **[7]** which permits to construct explicit examples as we will need in the proof of Theorem 1).

THEOREM 3. *Let $d, n \geq 2$ be integers. The dihedral group D_n is admissible for d if and only if d is congruent to either ± 1 modulo n. Moreover, for such values, every rational map of degree d admitting D_n as a group of holomorphic automorphisms is equivalent to one of the form $\phi(z) = z\psi(z^n)$, where*

$$\psi(z) = \pm \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} a_{r-k} z^k} \in \mathrm{Rat}_r,$$

satisfies that

(i) $a_r \neq 0$, if $d = nr + 1$;
(ii) $a_r = 0$ and $a_0 \neq 0$, if $d = nr - 1$.

In the above case, D_n is generated by the rotation $T(z) = \omega_n z$ and the involution $A(z) = 1/z$.

If $n \geq 2$ and D_n is admissible for d, then

$$\dim_{\mathbb{C}}(\mathcal{B}_d(D_n)) = \begin{cases} (d-1)/n, & d \equiv 1 \mod n \\ (d+1-n)/n, & d \equiv -1 \mod n. \end{cases}$$

REMARK 3.

(a) If we are in case (i) and "+" sign for ψ, then ϕ fixes both fixed points of T and both fixed points of A. But, if we are in case (i) and "−" sign

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for ψ, then ϕ fixes both fixed points of T and permutes both fixed points of A.

(b) If we are in case (ii) and "+" sign for ψ, then ϕ permutes both fixed points of T and fixes both fixed points of A. But, if we are in case (ii) and "−" sign for ψ, then ϕ permutes both fixed points of T and also both fixed points of A.

(c) If $n \geq 3$, then cases (i) and (ii) cannot happen simultaneously. Also, in either case, we obtain that $\mathcal{B}_d(D_n)$ has two connected components (they correspond to the choices of the sign "+" or "−").

2.3. Admissibility of the platonic cases. Let us now assume that $\phi \in \mathrm{Rat}_d$ admits as group of holomorphic automorphisms either $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathfrak{S}_4$. We may assume, up to conjugation, that (see, for instance, [**2**])

(1) $\langle T_3, B : T_3^3 = B^2 = (T_3 \circ A)^3 = I\rangle \cong \mathcal{A}_4$;
(2) $\langle T_4, C : T_4^4 = C^2 = (T_4 \circ C)^3 = I\rangle \cong \mathfrak{S}_4$.
(3) $\langle T_5, D : T_5^5 = D^2 = (T_5 \circ D)^3 = I\rangle \cong \mathcal{A}_5$;

where

$$T_n(z) = \omega_n z, \quad \omega_n = e^{2\pi i/n},$$

$$A(z) = 1/z,$$

$$B(z) = \frac{(\sqrt{3}-1)\left(z + (\sqrt{3}-1)\right)}{2z - (\sqrt{3}-1)},$$

$$C(z) = \frac{(\sqrt{2}+1)\left(-z + (\sqrt{2}+1)\right)}{z + (\sqrt{2}+1)},$$

$$D(z) = \frac{\left(1 + \sqrt{2 - \omega_5 - \omega_5^4}\right)\left(-z + \left(1 + \sqrt{2 - \omega_5 - \omega_5^4}\right)\right)}{(1 - \omega_5 - \omega_5^4)z + \left(1 + \sqrt{2 - \omega_5 - \omega_5^4}\right)}.$$

Working in a similar fashion as done for the dihedral situation, one may obtains the following.

THEOREM 4 ([**7**]). *Let $d \geq 2$.*

(1) *$\mathcal{A}_4$ is admissible for d if and only if d is odd.*
(2) *$\mathcal{A}_5$ is admissible for d if and only if d is congruent modulo* 30 *to either* 1*,* 11*,* 19*,* 21.
(3) *$\mathfrak{S}_4$ is admissible for d if and only if d is co-prime to* 6.

3. Proof of Theorem 1. It is clear that $\mathcal{B}_d$ is equal to the union of all $\mathcal{B}_d(G)$, where G runs over the admissible finite groups for d.

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If G is admissible for d and p is a prime integer dividing the order of G (so that the cyclic group C_p is a subgroup of G), then C_p is admissible for d and $\mathcal{B}_d(G) \subset \mathcal{B}_d(C_p)$. In this way, $\mathcal{B}_d$ is equal to the union of all $\mathcal{B}_d(C_p)$, where p runs over all integer primes with C_p admissible for d. Corollary 2 asserts that each $\mathcal{B}_d(C_p)$ is connected. Now, the connectivity of $\mathcal{B}_d$ will be consequence of Lemma 1 below.

LEMMA 1. *If $p \geq 3$ is a prime and C_p is admissible for d, then $\mathcal{B}_d(C_p) \cap \mathcal{B}_d(C_2) \neq \emptyset$.*

Proof. We only need to check the existence of a rational map $\phi \in \mathrm{Rat}_d$ admitting a holomorphic automorphism of order p and also a holomorphic automorphism of order 2.

First, let us consider those rational maps of the form $\phi(z) = z\psi(z^p)$, where (by Theorem 2) we may assume to be of the form

$$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \mathrm{Rat}_r,$$

with

(a) $a_r b_0 \neq 0$, if $d = pr + 1$;
(b) $a_r \neq 0$ and $b_0 = 0$, if $d = pr$;
(c) $a_r = b_0 = 0$, if $d = pr - 1$.

Assume we are in either case (a) or (c). By considering $b_k = a_{r-k}$, for every $k = 0, 1, \ldots, r$, we see that ψ satisfies the relation $\psi(1/z) = 1/\psi(z)$; so ϕ also admits the holomorphic automorphism $A(z) = 1/z$. The automorphisms $T(z) = \omega_p z$ and A generate a dihedral group of order $2p$.

In case (b), we can consider ψ so that $\psi(-z) = \psi(z)$, which is possible to find if we assume that $(-1)^k a_k = (-1)^r a_k$ and $(-1)^k b_k = (-1)^r b_k$ (which means that $a_k = b_k = 0$ if k and r have different parity). In this case T and $V(z) = -z$ are holomorphic automorphisms of ϕ, generating the cyclic group of order $2p$. □

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