ON THE REPRESENTATION OF MAPPINGS OF COMPACT METRIZABLE SPACES AS RESTRICTIONS OF LINEAR TRANSFORMATIONS

MICHAEL EDELSTEIN

1. Introduction. Let $f\colon X \to X$ be a continuous mapping of the compact metrizable space X into itself with $\bigcap_1^\infty f^n[X]$ a singleton. In [**3**] Janos proved that for any λ, $0 < \lambda < 1$, a metric ρ compatible with the topology of X exists such that $\rho(f(x), f(y)) \leqq \lambda\rho(x, y)$ for all $x, y \in X$. More recently, Janos [**4**] has shown that if, in addition, f is one-to-one, then a Hilbert space H and a homeomorphism $\mu\colon X \to H$ exist such that $\mu f \mu^{-1}$ is the restriction to $\mu[X]$ of the transformation sending $y \in H$ into λy. Our aim in this note is to show that in both cases a homeomorphism h of X into l_2 exists such that hfh^{-1} is the restriction of a linear transformation. (Apart from replacing, in the second case, an ad hoc constructed space H in [**4**] by l_2, our method of proof seems to be considerably simpler and shorter.)

Related results for the case when $\bigcap_1^\infty f^n[X]$ is a finite set are also treated.

2. The case where $\bigcap_1^\infty f^n[X]$ is a singleton.

THEOREM 1. *Let f be a continuous mapping of the compact metrizable space X into itself with $\bigcap_1^\infty f^n[X]$ a singleton and $P\colon l_2 \to l_2$ the (linear) transformation defined by $P(y) = (y_2, y_4, \ldots, y_{2n}, \ldots)$ for $y = (y_1, y_2, \ldots, y_n, \ldots) \in l_2$. Given λ, $0 < \lambda < 1$, there is a homeomorphism h of X into l_2 such that hfh^{-1} is the restriction of λP to $h[X]$.*

Proof. If X is a singleton, the result is obvious. Assuming then that $\emptyset \neq X \sim \bigcap_1^\infty f^n[X]$, let $\mathscr{B}$ be a countable base for the topology of this subset such that $\mathscr{B} = \bigcup_1^\infty \mathscr{B}_n$, where $\mathscr{B}_n$ is a base for $X \sim f^n[X]$. It is easily seen that an enumeration $\{(U_n, V_n)\colon n = 1, 2, \ldots\}$ of all members of $\mathscr{B} \times \mathscr{B}$ with $\bar{V}_n \subset U_n$ exists such that $U_n \in \mathscr{B}_n$. Thus for $n = 1, 2, \ldots,$ a continuous mapping $\varphi_{2n-1}\colon X \to [0, \lambda^{2n}]$ exists with

$$\varphi_{2n-1}[X \sim U_n] = 0 \quad \text{and} \quad \varphi_{2n-1}[\bar{V}_n] = \lambda^{2n}.$$

Recursively, define $\varphi_{2^m(2n-1)}\colon X \to [0, \lambda^{2n-m}]$ by setting

$$\varphi_{2^m(2n-1)}(x) = \frac{1}{\lambda}\varphi_{2^{m-1}(2n-1)}(f(x)).$$

Hence a family $\{\varphi_k\colon k = 1, 2, \ldots\}$ of functions is obtained satisfying

$$\varphi_k(f(x)) = \lambda\varphi_{2k}(x) \qquad (x \in X; k = 1, 2, \ldots).$$

Received January 27, 1969.

Since $U_n \subset X \sim f^n[X]$ implies $\varphi_{2n-1}[f^n[X]] = 0$, it follows that $\varphi_{2^m(2n-1)} = 0$ whenever $m \geqq n$. Thus,

$$\sum_{k=1}^{\infty} (\varphi_k(x))^2 = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (\varphi_{2^m(2n-1)}(x))^2 \leqq \sum_{m=0}^{\infty} \left(\lambda^{-2m} \sum_{n=m}^{\infty} \lambda^{4n}\right) < \infty,$$

and $y = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots) \in l_2$ for all $x \in X$. We now define h by setting $h(x) = y$. It is a straightforward matter to verify that h is continuous and one-to-one, and hence by compactness of X, a homeomorphism onto $h[X]$. Finally, if $y = h(x)$, we have

$$\begin{aligned}(hfh^{-1}(y))_k = (h(f(x)))_k = \varphi_k(f(x)) &= \lambda\varphi_{2k}(x) \\ &= \lambda(h(x))_{2k} = \lambda y_{2k} = \lambda(P(y))_k,\end{aligned}$$

completing the proof of the theorem.

COROLLARY. Since

$$h(f(x')) - h(f(x'')) = \lambda P(h(x')) - \lambda P(h(x'')) = \lambda P(h(x') - h(x''))$$

and $||P|| = 1$, we have

$$||h(f(x')) - h(f(x''))|| \leqq \lambda ||h(x') - h(x'')||.$$

Setting, then, $\rho(x', x'') = ||h(x') - h(x'')||$, we obtain a metric ρ on X satisfying the conclusion of the main theorem of [**3**].

THEOREM 2. *Let X, f, and λ be as in Theorem* 1 *and let, in addition, f be one-to-one; then a homeomorphism h of X into l_2 exists such that hfh^{-1} is the restriction of $g: l_2 \to l_2$ defined by $g(y) = \lambda y$ for all $y \in l_2$.*

Proof. We may clearly assume that $\emptyset \neq X \sim f[X]$. Let then $\mathscr{B}$ be a countable base for this set and $\{(U_n, V_n): n = 1, 2, \ldots\}$ the collection of all members of $\mathscr{B} \times \mathscr{B}$ such that $\bar{V}_n \subset U_n$. Let $\varphi_{2n-1}: X \to [0, 1]$ be continuous and such that

$$\varphi_{2n-1}[X \sim U_n] = 0 \quad \text{and} \quad \varphi_{2n-1}[\bar{V}_n] = 1 \qquad (n = 1, 2, \ldots).$$

Further, let,

$$\overline{\varphi_{2n-1}}: f[X] \cup \bar{V}_n \to [0, 1]$$

be identically 1 on $\bar{V}_n$ and coincide with $\varphi_{2n-1}f^{-1}$ on $f[X]$. Then the Tietze extension theorem applies to the effect that a continuous mapping

$$\varphi_{2(2n-1)}: X \to [0, 1]$$

exists which extends $\overline{\varphi_{2n-1}}$. Thus

$$\varphi_{2(2n-1)}(f(x)) = \varphi_{2n-1}(x) \qquad (n = 1, 2, \ldots; x \in X)$$

and

$$\varphi_{2(2n-1)}[\bar{V}_n] = 1 \qquad (n = 1, 2, \ldots).$$

Proceeding by induction we obtain a family

$$\{\varphi_{2^m(2n-1)}: m = 0, 1, \ldots; n = 1, 2, \ldots\}$$

of mappings with the following properties:

$$(1) \qquad \varphi_{2^m(2n-1)}(f(x)) = \varphi_{2^{m-1}(2n-1)}(x) \qquad (m, n = 1, 2, \ldots; x \in X),$$

$$(2) \qquad \varphi_{2n-1}(f(x)) = 0 \qquad (n = 1, 2, \ldots; x \in X),$$

$$(3) \qquad \varphi_{2^m(2n-1)}[\bar{V}_n] = 1 \qquad (m = 0, 1, \ldots; n = 1, 2, \ldots).$$

Next, we define $\psi_n: X \to [0, (1 - \lambda)^{-1}]$ by setting

$$(4) \qquad \psi_n(x) = \sum_{m=0}^{\infty} \lambda^m \varphi_{2m(2n-1)}(x).$$

Thus, by (1) and (2),

$$(5) \quad \psi_n(f(x)) = \sum_{m=0}^{\infty} \lambda^m \varphi_{2m(2n-1)}(f(x)) = \lambda \sum_{m=0}^{\infty} \lambda^m \varphi_{2m(2n-1)}(x) = \lambda\psi_n(x).$$

To define h we set

$$h(x) = \left(\frac{\psi_1(x)}{1}, \frac{\psi_2(x)}{2}, \ldots, \frac{\psi_n(x)}{n}, \ldots\right).$$

Clearly, $h(x) \in l_2$ for all $x \in X$ and $h: X \to l_2$ is continuous. To conclude the proof, it suffices to show that h is one-to-one. Let then u and v be distinct points of X. We may clearly assume that $\{u, v\} \subset X \sim \bigcap f^n[X]$ so that non-negative integers m, n exist with $\{f^{-m}(u), f^{-n}(v)\} \subset X \sim f[X]$. If $f^{-m}(u) = f^{-n}(v)$, then u and v are (distinct) iterates of the same point and, from (5), $h(v) = \lambda^k h(u) \neq h(u)$. Suppose that this is not the case and $m \leqq n$. Then there is a positive integer j such that $f^{-m}(u) \in V_j$ and $f^{-n}(v) \in X \sim U_j$. It follows from (5) that

$$\psi_j(u) = \frac{\lambda^m}{1 - \lambda} \geqq \frac{\lambda^n}{1 - \lambda} > \psi_j(v).$$

Remark. The construction of h in the above proof is a refinement of a similar procedure used in [**1**; **2**].

3. The case where $\bigcap_1^\infty f^n[X]$ is a finite set.

THEOREM 3. *Let $f: X \to X$ be a continuous mapping of the compact metrizable space X into itself with $\bigcap f^n[X] = \{\xi_1, \xi_2, \ldots, \xi_k\}$ and let λ and P be as in Theorem* 1. *Let p denote the permutation of* $(1, 2, \ldots, k)$ *with the property that $p(i) = j$ if and only if $f(\xi_i) = \xi_j$. Then a homeomorphism h of X into $E^k \times l_2$, where E^k is the Euclidean k-dimensional space, exists such that hfh^{-1} is the restriction to $h[X]$ of the transformation which assigns to $((x_1, x_2, \ldots, x_k), y)$ the element $((x_{p(1)}, x_{p(2)}, \ldots, x_{p(k)}), \lambda P(y))$.*

Proof. We first show that a cover of X, consisting of k disjoint closed and open sets $\{X_1, X_2, \ldots, X_k\}$, exists such that $f(\xi_i) = \xi_j$ implies $f[X_i] \subset X_j$. This being the case with $X = X_1$, for $k = 1$, assume the truth of the statement for $k = l - 1$ and let $\bigcap_1^\infty f^n[X] = \{\xi_1, \xi_2, \ldots, \xi_l\}$. If two non-empty closed and open disjoint sets Y_1 and Y_2 exist such that $X = Y_1 \cup Y_2$ and $f[Y_i] \subset Y_i$, $i = 1, 2$, then both $\bigcap_1^\infty f^n[Y_1]$ and $\bigcap_1^\infty f^n[Y_2]$ are of cardinality smaller than l and the statement easily follows from the inductive assumption. Otherwise, let X_i be the set of all $x \in X$ with the property that

$$\{f^{nl}(x): n = 1, 2, \ldots\}$$

converges to ξ_i; it is readily seen that $\{X_1, X_2, \ldots, X_l\}$ is a cover of X, as desired. We now define $\kappa_i: X \to [0, 1]$ by setting $\kappa_i[X_i] = 1$ and $\kappa_i[X \sim X_i] = 0$. Clearly then, κ_i are continuous and $\kappa_j(f(x)) = \kappa_i(x)$ whenever $f(\xi_i) = \xi_j$. Define now h by setting

$$h(x) = ((\kappa_1(x), \kappa_2(x), \ldots, \kappa_k(x)), (\varphi_1(x), \ldots, \varphi_n(x), \ldots)),$$

where the $\varphi_n: X \to [0, 1]$ are defined with respect to members of a base for $X \sim \bigcap_1^\infty f^n[X]$, as in the proof of Theorem 1. It is a straightforward matter to verify that h is as desired.

The proof of the following theorem in which the κ_is together with the ψ_is of the proof of Theorem 2 are combined to yield the desired mapping should now be obvious and is therefore omitted.

THEOREM 4. *Let f, X, λ, and p be as in Theorem 3 and suppose, in addition, that f is one-to-one. Then a homeomorphism h of X into $E^k \times l_2$ exists such that hfh^{-1} is the restriction to $h[X]$ of the transformation which assigns to $((x_1, x_2, \ldots, x_k), y)$ the element $((x_{p(1)}, x_{p(2)}, \ldots, x_{p(k)}), \lambda y)$.*

REFERENCES

1. M. Edelstein, *A short proof of a theorem by L. Janos*, Proc. Amer. Math. Soc. *20* (1969), 509–510.

2. ——— *On the representation of contractive homeomorphisms as transformations in Hilbert space* (to appear).

3. L. Janos, *A converse of Banach's contraction theorem*, Proc. Amer. Math. Soc. *18* (1967), 287–289.

4. ——— *Linearization of a contractive homeomorphism*, Can. J. Math. *20* (1968), 1387–1390.

Dalhousie University,
Halifax, Nova Scotia