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REGULAR CATEGORIES AND REGULAR FUNCTORS

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Introduction. Let $\mathscr{X}$ be a category with nice factorization-properties. If a functor $G:\mathscr{A} \to \mathscr{X}$ which has a left-adjoint behaves nice with respect to factorizations then it can be shown quite easily that G behaves well in many other respects, especially that it lifts nice properties from $\mathscr{X}$ into $\mathscr{A}$.

1. Regular categories.

1.1 *Definitions.* Let $\mathscr{X}$ be a category.

(a) A morphism $e:X \to Y$ of $\mathscr{X}$ is called a *regular epimorphism* if and only if (e, Y) is the coequalizer of some pair (r, s) of $\mathscr{X}$-morphisms.

(b) A pair (X, f) is called a *source* in $\mathscr{X}$ provided X is an object of $\mathscr{X}$ and $f = (f_i)_{i \in I}$ is a family of $\mathscr{X}$-morphisms $f_i:X \to X_i$ with common domain X. The index-class I may be a proper class. Also, it may be empty.

(c) A source (X, m) is called a mono-source, provided whenever

$$Y \underset{s}{\overset{r}{\rightrightarrows}} X$$

is a pair of morphisms such that $m_i \cdot r = m_i \cdot s$ for all i then $r = s$.

(d) A factorization

$$X \xrightarrow{f_i} X_i = X \xrightarrow{e} Y \xrightarrow{m_i} X_i$$

is called a *regular factorization* of the source (X, f) if and only if e is a regular epimorphism and (Y, m) is a mono-source.

(e) $\mathscr{X}$ is called *regular* if and only if every source in $\mathscr{X}$ has a regular factorization. (The concept of a regular category, due to M. Barr, differs from our concept.)

1.2 *Examples and remarks.* (a) Most decent categories are regular. Any category of algebras defined by (not necessary finite) operations and equations is regular, e.g. the categories *Set* of sets, *Grp* of groups, *R-Mod* of left-R-moduls, *Rng* of rings, *C*-Alg* of C^*-algebras, *C Lat* of complete lattices, *C Boo Alg* of complete Boolean algebras, *Comp Haus* of compact Hausdorff spaces. Any category of algebras defined by implications between identities is regular, e.g. the categories of torsion free Abelian groups and of zerodimensional, compact Hausdorff spaces. The categories *Field* of fields, *Top* of topological spaces,

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Pos of partially ordered sets, and *Rel* of relations are regular. Any partially ordered class, considered as a category, is regular. Any full (regular epi)-reflective subcategory of a regular category is regular.

(b) Regular categories need not be decent. A simple example of a regular category which is not even concretizable is the following:

Objects: A_α, B, C_α, T where α runs through all ordinals.

Morphisms: All identities, for each object X exactly one morphism $t_x: X \to T$ and the following:

hom $(A_\alpha, B) = \{f_\alpha, g_\alpha\}$ with $f_\alpha \neq g_\alpha$
hom $(B, C_\alpha) = \{h_\alpha\}$
hom $(A_\alpha, C_\beta) = \{f_{\alpha\beta}, g_{\alpha\beta}\}$ with $f_{\alpha\alpha} = g_{\alpha\alpha}$ and $f_{\alpha\beta} \neq g_{\alpha\beta}$ for $\alpha \neq \beta$.

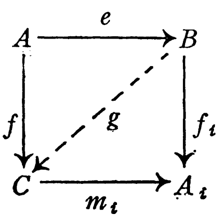
Composition: $h_\beta \circ f_\alpha = f_{\alpha\beta}$, $h_\beta \circ g_\alpha = g_{\alpha\beta}$.

The resulting category is obviously regular. It is not "concretizable" since B has a proper class of pairwise non-equivalent regular quotients $B \to C_\alpha$, and any concrete category is regular-cowellpowered.

(c) The regular factorizations can, if they exist, usually be constructed directly. In fact, the concept of "factorizations" seems to be a basic concept in category theory. For those who prefer to apply some machinery there are theorems like the following: Any category $\mathscr{X}$ which is complete, wellpowered, cowellpowered, and such that the class of regular epimorphisms in $\mathscr{X}$ is closed under composition (or under pullbacks) is regular.

(d) The category *Cat* of small categories is not regular. In fact, the obvious functor F from $\cdot \to \cdot$ into the category with precisely two morphisms 1 and f and the property $f^2 = f$ has no regular factorization. (F is the composite of two regular epimorphisms but not regular itself.)

1.3 Proposition. *If $e: A \to B$ is a regular epimorphism, $f: A \to C$ an arbitrary morphism, (B, f) a source, and (C, m) a mono-source such that $m_i \cdot f = f_i \cdot e$ for each i, then there exists a unique morphism $g: B \to C$ such that for each i the following diagram commutes.*



1.4 Corollaries. (a) *Regular factorizations are, if they exist, essentially unique.*

(b) *In a regular category the class of regular epimorphisms is closed under composition.*

(c) *In a regular category, if $g \cdot f$ is a regular epimorphism then so is g.*

1.5 Proposition. *Every regular category has coequalizers.*

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Proof. For any pair

$$A \overset{r}{\underset{s}{\rightrightarrows}} B$$

of morphisms let (B, f) be the source consisting of all those morphisms f_i for which $f_i \cdot r = f_i \cdot s$. If

$$B \xrightarrow{f_i} B_i = B \xrightarrow{e} C \xrightarrow{m_i} B_i$$

is the regular factorization of (B, f) then (e, C) is the coequalizer of (r, s).

2. Regular functors.

2.1 *Definition.* Let $\mathscr{X}$ be a regular category. A functor $G:\mathscr{A} \to \mathscr{X}$ is called *regular* if and only if it satisfies the following conditions:

(1) G has a left-adjoint;

(2) G *creates regular factorizations*, i.e. if (A, f) is a source in $\mathscr{A}$ and

$$GA \xrightarrow{Gf_i} GA_i = GA \xrightarrow{e} X \xrightarrow{m_i} GA_i$$

is a regular factorization of (GA, Gf) in $\mathscr{X}$ then there exists a factorization

$$A \xrightarrow{f_i} A_i = A \xrightarrow{\bar{e}} \bar{X} \xrightarrow{\overline{m_i}} A_i$$

of (A, f) in $\mathscr{A}$ which is uniquely determined by the property "$G\bar{e} = e$ and $G\bar{m}_i = Gm_i$ for each i" and which, in addition, is a regular factorization of (A, f).

2.2 *Convention.* In this section we will suppose that: $\mathscr{X}$ is a regular category; $G:\mathscr{A} \to \mathscr{X}$ is a regular functor with left-adjoint F, front-adjunctions $\eta_X:X \to GFX$ and back-adjunctions $\epsilon_A:FGA \to A$.

2.3 PROPOSITION. *$\mathscr{A}$ is regular. G preserves and reflects regular factorizations. G preserves and reflects regular epimorphisms, monomorphisms, and isomorphisms.*

2.4 PROPOSITION. *G is transportable, i.e. for any $\mathscr{A}$-object A and $\mathscr{X}$-isomorphism $f:GA \to X$ there exists a unique $\mathscr{A}$-isomorphism $g:A \to B$ such that $Gg = f$.*

2.5 PROPOSITION. *G is faithful.*

Proof. Let

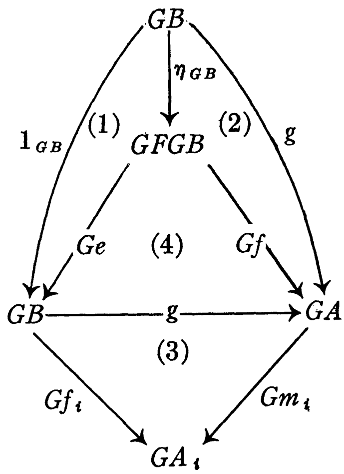
$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

be a pair of $\mathscr{A}$-morphisms with $Gf = Gg$. Then $G\epsilon_A \cdot \eta_{GA} = 1_{GA}$ implies that (a) $G\epsilon_A$ is a retraction (= split epimorphism), hence a regular epimorphism; hence ϵ_A is a regular epimorphism, (b) $G(f \cdot \epsilon_A) \cdot \eta_{GA} = Gf \cdot 1_{GA} = Gg \cdot 1_{GA} = G(g \cdot \epsilon_A) \cdot \eta_{GA}$ which implies $f \cdot \epsilon_A = g \cdot \epsilon_A$. Now (a) and (b) imply $f = g$.

2.6 PROPOSITION. *Every regular epimorphism $e:A \to B$ in $\mathscr{A}$ is G-final, i.e. for any A-morphism $f:A \to C$ and any $\mathscr{X}$-morphism $g:GB \to GC$ with $Gf = g \cdot Ge$ there exists a unique $\mathscr{A}$-morphism $h:B \to C$ with $Gh = g$.*

2.7 PROPOSITION. *Every mono-source (A, m) in $\mathscr{A}$ is G-initial, i.e. for any source (B, f) in $\mathscr{A}$ and any $\mathscr{X}$-morphism $g:GB \to GA$ with $Gf_i = Gm_i \cdot g$ for each i there exists a unique $\mathscr{A}$-morphism $h:B \to A$ with $Gh = g$.*

Proof. There exist unique A-morphisms $e:FGB \to B$ and $f:FGB \to A$ such that the triangles (1), (2), (and (3)) in the following diagram commute:



Since η_{GB} is a front-adjunction and (GA, Gm) is a mono-source this implies that (4) commutes. Hence Proposition 2.6 can be applied.

2.8 LEMMA. *G reflects limits, i.e. if $D:\mathscr{I} \to \mathscr{A}$ is a diagram, $f = (f_i:A \to D_i)_{i \in I}$, and (A, f) is a source in $\mathscr{A}$ such that (GA, Gf) is a limit of $G \cdot D$ then (A, f) is a limit of D.*

Proof. Since G is faithful (A, f) is a lower bound of D. If (B, g) is an arbitrary lower bound of D then (GB, Gg) is a lower bound of $G \cdot D$. Consequently there exists an $\mathscr{X}$-morphism $g:GB \to GA$ such that $Gg_i = Gf_i \cdot g$ for each i. As a limit (GA, Gf) is a mono-source. Therefore (A, f) is a mono-source and Proposition 2.7 implies that there exists an $\mathscr{A}$-morphism $h:B \to A$ with $Gh = g$ Hence $g_i = f_i \cdot h$ for each i. Uniqueness follows from the fact that (A, f) is a mono-source.

2.9 THEOREM. *G creates limits, i.e. if $D:\mathscr{I} \to \mathscr{A}$ is a diagram and (X, f) is a*

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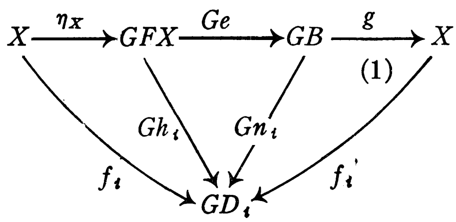
limit of $G \cdot D$ then there exists a source (A, g) with $g_i: A \to D_i$ in $\mathscr{A}$ which is uniquely determined by the property "$GA = X$ and $Gg_i = f_i$ for all i" and which, in addition, is a limit of D.

Proof. For each i there exists a unique $\mathscr{A}$-morphism $h_i: FX \to D_i$ with $f_i = Gh_i \cdot \eta_X$. If

$$FX \xrightarrow{h_i} D_i = FX \xrightarrow{e} B \xrightarrow{n_i} D_i$$

is the regular factorization of (FX, h) then (GB, Gn) is a lower bound of GD. Hence there exists a unique $\mathscr{X}$-morphism $g: GB \to X$ such that for each i triangle (1) in the following diagram commutes:

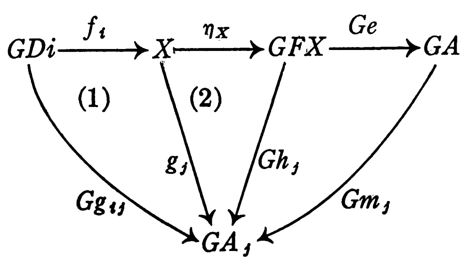


g is a retraction, since $g \cdot Ge \cdot \eta_X = 1_X$. g is a monomorphism since $Gn_i = f_i \cdot g$ and (GB, Gn) is a mono-source. Therefore g is an isomorphism and the remaining part of the proof follows immediately from Proposition 2.4 and Lemma 2.8.

2.10 COROLLARY. *If $\mathscr{X}$ is complete then $\mathscr{A}$ is complete.*

2.11 THEOREM. *G detects colimits, i.e. if $D: \mathscr{I} \to \mathscr{A}$ is a diagram and $G \cdot D$ has a colimit then so has D.*

Proof. Let $((f_i)_{i \in I}, X)$ be a colimit of $G \cdot D$, and let $(g_{ij}, A_j)_{j \in J}$ be the family of all upper bounds of D. Then for each $j \in J$ there exist unique morphisms $g_j: X \to GA_j$ and $h_j: FX \to A_j$ such that for each i the triangles (1) and (2) in the following diagram commute:



If

$$FX \xrightarrow{h_j} A_j = FX \xrightarrow{e} A \xrightarrow{m_j} A_j$$

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is the regular factorization of the source $(FX, (h_j)_{j\in J})$ then Proposition 2.7 implies that for any i there exists a unique $\mathscr{A}$-morphism $k_i:D_i \to A$ with $Gk_i = Ge \cdot \eta_X \cdot f_i$. It follows immediately that $((k_i)_{i\in I}, A)$ is a colimit of D.

2.12 *Definition.* (a) An $\mathscr{X}$-morphism $g:X \to GA$ *G-generates* A if and only if the following conditions hold (1) for any pair

$$A \underset{s}{\overset{r}{\rightrightarrows}} B$$

of $\mathscr{A}$-morphisms $Gr \cdot g = Gs \cdot g$ implies $r = s$; (2) whenever $g = Gm \cdot f$ for some $\mathscr{A}$-monomorphism $m:B \to A$ then m is an isomorphism.

(b) Let $(A_i)_I$ be a family of $\mathscr{A}$-objects and $(X, (f_i:X \to GA_i)_I)$ be a source in $\mathscr{X}$. A factorization

$$X \xrightarrow{f_i} GA_i = X \xrightarrow{g} GA \xrightarrow{Gm_i} GA_i$$

is called a *G-regular factorization of* $(X, (f_i, A_i)_I)$ if and only if g G-generates A and $(A, (m_i:A \to A_i)_I)$ is a mono-source.

2.13 PROPOSITION. *For any family* $(A_i)_I$ *of* $\mathscr{A}$*-objects and any source* $(X, (f_i:X \to GA_i)_I)$ *there exists an essentially unique G-regular factorization of* $(X, (f_i, A_i)_I)$.

Proof. For each i there exists a unique $\mathscr{A}$-morphism $g_i:FX \to A_i$ with $f_i = Gg_i \cdot \eta_X$. If

$$FX \xrightarrow{g_i} A_i = FX \xrightarrow{e} B \xrightarrow{m_i} A_i$$

is a regular factorization of the source $(FX, (g_i)_I)$ then

$$X \xrightarrow{f_i} GA_i = X \xrightarrow{Ge \cdot \eta_X} GB \xrightarrow{Gm_i} GA:$$

is a regular factorization of $(X, (f_i, A_i)_I)$. Essential uniqueness follows from the next proposition.

2.14 *Remark.* As a consequence of 2.13 a functor $G:\mathscr{A} \to \mathscr{X}$ is regular if and only if G creates regular factorizations and G-regular factorizations exist.

2.15 PROPOSITION. *If* $g:X \to GA$ *G-generates* A, $f:X \to GB$ *is an* $\mathscr{X}$*-morphism*, $(A, (f_i:A \to A_i)_I)$ *is a source, and* $(B, (m_i:B \to A_i)_I)$ *is a mono-source such that* $Gm_i \cdot f = Gf_i \cdot g$ *for each* i *then there exists a unique* $\mathscr{A}$*-morphism*

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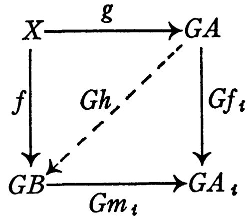
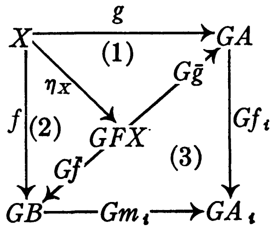
$h: A \to B$ such that for each i the following diagram commutes:



Proof. There exist unique morphisms $\bar{g}: FX \to A$ and $\bar{f}: FX \to B$ such that the triangles (1) and (2) and hence also (3) in the following diagram commute:



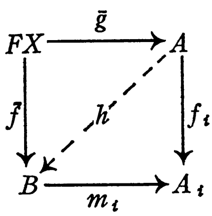
If

$$FX \xrightarrow{\bar{g}} A = FX \xrightarrow{e} B \xrightarrow{m} A$$

is the regular factorization of $\bar{g}$ then

$$X \xrightarrow{g} GA = X \xrightarrow{Ge \cdot \eta_X} GB \xrightarrow{Gm} GA.$$

Since g G-generates A this implies that m is an isomorphism. Hence $\bar{g}$ is a regular epimorphism, and Proposition 1.3 implies that there exists a morphism $h: A \to B$ such that the diagram



commutes for each i.

Therefore h is the required morphism. Uniqueness is clear.

2.16 Theorem. *If*

$$\mathscr{B} \xrightarrow{H} \mathscr{X} = \mathscr{B} \xrightarrow{K} \mathscr{A} \xrightarrow{G} \mathscr{X}$$

then H is regular if and only if K is regular.

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Proof. If K is regular then H is obviously regular too. If H is regular then $\mathscr{B}$ is regular and K preserves and reflects regular factorizations. Therefore K creates regular factorizations. Hence it remains to show that K has a left-adjoint, i.e. that for any $\mathscr{A}$-object A there exists a K-universal map. Let $(f_i, B_i)_I$ be the class of all pairs with $\mathscr{B}$-objects B_i and $\mathscr{A}$-morphisms $f_i:A \to KB_i$, and let

$$GA \xrightarrow{Gf_i} HB_i = GA \xrightarrow{g} HB \xrightarrow{Hm_i} HB_i$$

be the H-regular factorization of $(GA, (Gf_i, B_i)_I)$. Since

$$GA \xrightarrow{Gf_i} GKB_i = GA \xrightarrow{g} GKB \xrightarrow{GKm_i} GKB_i$$

and $(KB, (Km_i)_I)$ is a mono-source Proposition 2.7 implies that there exists a unique $\mathscr{A}$-morphism $f:A \to KB$ with $Gf = g$. It follows immediately that $f:A \to KB$ is K-universal.

3. Characterizations of regular functors. In this section we suppose that $\mathscr{X}$ is a regular category, and that $G:\mathscr{A} \to \mathscr{X}$ is a functor with left-adjoint F, front-adjunctions $\eta_X:X \to GFX$, and back-adjunctions $\epsilon_A:FGA \to A$.

3.1 THEOREM. *The following conditions are equivalent:*

(1) *G is regular;*

(2) *$\mathscr{A}$ is regular, G is transportable and preserves and reflects regular factorizations;*

(3) *$\mathscr{A}$ has coequalizers, G is transportable and preserves and reflects regular epimorphisms.*

Proof. Obviously (1) implies (2), and (2) implies (3). It remains to show that (3) implies (1). As in Proposition 2.5 it can be shown that G is faithful (any functor with left-adjoint and epimorphic back-adjunctions is faithful), and hence reflects mono-sources. Let (A, f) be a source in $\mathscr{A}$, and let

$$GA \xrightarrow{Gf_i} GA_i = GA \xrightarrow{e} E \xrightarrow{m_i} GA_i$$

be a regular factorization of (GA, Gf). Then (e, E) is the coequalizer of some pair

$$Y \underset{r_2}{\overset{r_1}{\rightrightarrows}} GA$$

of $\mathscr{X}$-morphisms. For $j = 1, 2$ there exists a unique $\mathscr{A}$-morphism FY

$$FY \xrightarrow{s_j} A$$

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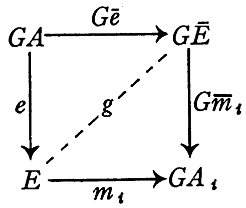
with $r_j = Gs_j \cdot \eta_Y$. Let $(\bar{e}, \bar{E})$ be a coequalizer of (s_1, s_2). Then there exists a unique $\mathscr{X}$-morphism $f:E \to G\bar{E}$ such that $G\bar{e} = f \cdot e$, and for each i there exists a unique $\mathscr{A}$-morphism $\bar{m}_i:\bar{E} \to A_i$ such that $f_i = \bar{m}_i \cdot \bar{e}$. Since $G\bar{e}$ is a regular epimorphism, (E, m) is a mono-source, and $G\bar{m}_i \cdot G\bar{e} = Gf_i = m_i \cdot e$ for each i, Proposition 1.3 implies that there exists a unique $\mathscr{X}$-morphism $g:G\bar{E} \to E$ such that for each i the following diagram commutes:



Hence $f \cdot g = 1_{G\bar{E}}$ and $g \cdot f = 1_E$. Therefore g is an isomorphism. Since G is transportable, this implies that the regular factorization of (GA, Gf) can be lifted uniquely. Since G reflects mono-sources, the lifted factorization is regular.

3.2 *Remark.* The requirement in condition (3) of Theorem 3.1 that $\mathscr{A}$ has coequalizers is not superfluous as the following example demonstrates. Let $\mathscr{A}$ be the full subcategory of the category *SGrp* of semigroups whose objects are precisely those semigroups whose underlying set is either empty or contains at least two elements, and let $G:\mathscr{A} \to Set$ be the forgetful functor. Then the following hold:

(1) G has a left-adjoint, is transportable, reflects and preserves regular epimorphisms.

(2) G is not regular, $\mathscr{A}$ is not regular, has no terminal object, does not have equalizers or coequalizers.

3.3 THEOREM. *If G is the embedding of a reflective full sub-category $\mathscr{A}$ of $\mathscr{X}$ into $\mathscr{X}$, then the following conditions are equivalent:*

(1) *G is regular;*

(2) *$\mathscr{A}$ contains with any source its regular factorization in $\mathscr{X}$;*

(3) *$\mathscr{A}$ contains with any morphism its regular factorization in $\mathscr{X}$;*

(4) *$\mathscr{A}$ contains each object which is simultaneously a subobject of some $\mathscr{X}$-object and a regular quotient of some $\mathscr{X}$-object;*

(5) *$\mathscr{A}$ is isomorphism-closed and G preserves regular epimorphisms.*

Proof. Obviously (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

(4) $\Rightarrow$ (5). Let $f:A \to B$ be a regular epimorphism in $\mathscr{A}$. If

$$A \xrightarrow{f} B = A \xrightarrow{e} X \xrightarrow{m} B$$

is a regular factorization of (A, f) in $\mathscr{X}$, then (4) implies that X and hence e and m belong to $\mathscr{A}$. Since f is a regular epimorphism in $\mathscr{A}$ and m is a mono-

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morphism in $\mathscr{X}$ and hence in $\mathscr{A}$, m must be an isomorphism in $\mathscr{A}$ and hence in $\mathscr{X}$. Consequently f is a regular epimorphism in $\mathscr{X}$.

(5) $\Rightarrow$ (1) follows immediately from Theorem 3.1 since each embedding of a full, reflective subcategory reflects regular epimorphisms.

3.4 *Remarks.* (a) the embedding $G:\mathscr{A} \to \mathscr{X}$ of an isomorphism-closed, full, bireflective subcategory $\mathscr{A}$ of $\mathscr{X}$ into $\mathscr{X}$ need not be regular. In fact, if $\mathscr{X}$ = *Top*, $\mathscr{A}$ = *Reg* is the full subcategory of $\mathscr{X}$ whose objects are precisely the regular spaces (regular does not imply T_1), and $G:\mathscr{A} \to \mathscr{X}$ is the embedding, then the following hold:

(1) $\mathscr{A}$ is an isomorphism-closed, full, bireflective subcategory of X;

(2) G does not preserve regular epimorphisms and hence is not regular;

(b) If the embedding $G:\mathscr{A} \to \mathscr{X}$ of a full, reflective subcategory $\mathscr{A}$ of $\mathscr{X}$ into $\mathscr{X}$ is regular, then $\mathscr{A}$ need neither be closed under subobjects nor under regular quotients. The embedding functor G: *Comp Haus* $\to$ *Top* provides an example.

3.5 PROPOSITION. *If $\mathscr{X}$ is a partially ordered class, then the following conditions are equivalent:*

(1) *G is regular;*

(2) *G is an embedding of a full, reflective subcategory of $\mathscr{X}$ into $\mathscr{X}$.*

4. Related concepts. This section contains a short discussion of the relation between "regular functors" and "primitive" respectively "quasiprimitive categories of algebras" on one hand, and "monadic" (= "tripleable" = "varietal") functors on the other hand—the latter being quite intriguing. We do not give proofs. These can be easily obtained by using the characterizations of primitive and quasiprimitive categories of algebras given by W. Felscher [**1**], the characterizations of varietal categories over *Set* given by F. E. J. Linton [**4**; **5**; **6**] or some results of J. R. Isbell [**2**; **3**].

4.1 THEOREM. *For any functor $G:\mathscr{A} \to Set$ the following conditions are equivalent:*

(1) *G is regular;*

(2) *there exist a quasiprimitive category $\mathscr{B}$ of algebras with forgetful functor $U:\mathscr{B} \to Set$ and an isomorphism $K:\mathscr{A} \to \mathscr{B}$ with $G = U \cdot K$.*

Note. A fork

$$X \underset{s}{\overset{r}{\rightrightarrows}} Y \overset{t}{\to} Z$$

is called a *congruence-fork* if and only if (t, Z) is the coequalizer of (r, s), and (r, s) is the congruence-relation (= kernel pair) of t.

4.2 THEOREM. *For any functor $G:\mathscr{A} \to Set$ which has a left adjoint the following conditions are equivalent:*

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(1) *G is monadic;*
(2) *G is regular and reflects congruence-relations (= kernel-pairs);*
(3) *$\mathscr{A}$ has congruence-relations and whenever*

$$B \overset{p}{\underset{q}{\rightrightarrows}} A$$

is a pair of $\mathscr{A}$-morphisms and $f:GA \to X$ is a function such that

$$GB \overset{Gp}{\underset{Gq}{\rightrightarrows}} GA \xrightarrow{f} X$$

is a congruence-fork, then there exists an $\mathscr{A}$-morphism $g:A \to C$, uniquely determined by the property $Gg = f$, and, in addition,

$$B \overset{p}{\underset{q}{\rightrightarrows}} A \xrightarrow{g} C$$

is a congruence fork in $\mathscr{A}$;
(4) *G creates congruence-forks, i.e. if*

$$X \overset{r}{\underset{s}{\rightrightarrows}} GA \xrightarrow{t} Y$$

is a congruence fork, then the following conditions hold:
(a) *If (r, s) can be lifted to $\mathscr{A}$ then t can be lifted uniquely to $\mathscr{A}$,*
(b) *if t can be lifted to $\mathscr{A}$ then (r, s) can be lifted uniquely to $\mathscr{A}$.*
And in each case the resulting fork in $\mathscr{A}$ is a congruence-fork.

4.3 *Examples.* The forgetful functors from the categories of torsion free Abelian groups resp. of zerodimensional compact Hausdorff-spaces into *Set* are regular but not monadic.

4.4 *Remark.* (a) As the above results 4.2 and 4.3 show, the concept of monadic functors is for base-category $\mathscr{X} = Set$ properly stronger than the concept of regular functors. The picture changes completely if we replace *Set* by other base-categories. In fact, even for nice base-categories $\mathscr{X}$ a monadic functor may fail severely to have all the nice properties of regular functors, exhibited in § 2.

(b) It can be easily shown that any regular functor $G:\mathscr{A} \to \mathscr{X}$ can be written as a composition of two monadic functors. In fact, if T is the monad (= triple) induced by G, $\mathscr{A}^T$ is the category of T-algebras, $G^T:\mathscr{A}^T \to \mathscr{X}$ is the forgetful functor and $K:A \to A^T$ the so called comparison-functor, then $G = G^T \cdot K$ and K as well as G^T are monadic. Moreover K is an embedding of a full, reflective subcategory of $\mathscr{A}^T$ which is closed under subobjects. But, vice versa, the composition of two monadic functors may fail badly to be regular even for the base-category $\mathscr{X} = Set$.

4.5 *Examples.* (a) If X is a partially ordered class, then $G:\mathscr{A} \to \mathscr{X}$ is monadic if and only if G is regular.

(b) Any regular, full embedding is monadic. In fact, any embedding $G:\mathscr{A} \to \mathscr{X}$ of a full, isomorphism-closed, reflective subcategory $\mathscr{A}$ of $\mathscr{X}$ into $\mathscr{X}$ is monadic. It may fail to be regular, as the embedding $Reg \to Top$ shows (cf. 3.4 (a)).

(c) Let $G: Cat \to Set$ be the forgetful functor from the category Cat of small categories into Set which associates with every small category its set of morphisms. Then G is a composite of two monadic functors (see below) but fails badly to be regular:

(1) Cat is not regular. The obvious functor F from $\cdot \to \cdot$ into the category with precisely two morphisms 1 and f and the property $f^2 = f$ has no regular factorization. In fact F is a composite of two regular epimorphisms, but is itself not regular.

(2) G neither preserves nor reflects regular epimorphisms.

(3) Regular epimorphisms in Cat need not be G-final.

Let $Grph$ be the category of graphs, i.e. of algebras (X, d, c) where X is a set and d and c are unary operations satisfying the equations $d^2 = c \cdot d = d$ and $c^2 = d \cdot c = c$. Let $U: Grph \to Set$ be the forgetful functor, and let $K: Cat \to Grph$ be the obvious functor which associates with any small category C the Graph (X, d, c) where X is the set of C-morphisms, and d, respectively c, are the domain-, respectively codomain- function of C (objects are supposed to be identified with their corresponding identities). Then $G = U \cdot K$, and U as well as K are monadic. $Grph$ is a nice category, but the monadic functor $K: Cat \to Grph$ fails badly to be regular. In fact it is subject to the same mishaps (1), (2), and (3) above as G.

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