5

Deep inelastic scattering and the parton model

In this chapter we will consider the notion of *partons*, in the way Feynman introduced them. The parton model (PM) corresponds to a very clever application of the concepts behind the method of virtual quanta, which we described in Chapter 2. The theoretical reasons why the PM provides a relevant description of the hadronic constituents are, however, very complicated and this chapter only contains a first introduction.

The road to the PM goes through experiment. Over many years physicists have performed in various contexts a type of experiment which can be traced back to Rutherford. They have used a charged particle to extract information on the charge and mass structure of smaller and smaller constituents of matter. Rutherford made use of α-radiation on nuclear targets and very quickly made two essential observations.

He and his assistant were able to detect the scattering of the α-particles by direct observation of the flashes that they produced on a screen. They found, firstly, that *most of the beam particles simply continued through the target as if it was empty of matter*. But, secondly, *every now and then they found quite an appreciable deviation.*

It was Rutherford's genius that not only he did take his observations seriously but also used them to provide a description of the atom. We are going to consider his result, together with the necessary corrections due to relativity, spin and the internal structure of the target.

He explained the source of the α-particle deviations by a classical mechanics calculation of the orbits of charged particles in a Coulomb field and he attributed this Coulomb field to a precise charge value placed inside a very tiny region indeed, i.e. an atomic *nucleus*. He was pretty lucky, however, that his classical mechanics calculation agreed with the quantum mechanical results.

This is by no means trivial. In principle Nature could have chosen to use something other than an inverse square law for the force between electrically charged particles (although this would have been difficult

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

to accommodate with many other phenomena, among them ourselves!). Then Rutherford would have obtained a result which subsequent quantum mechanical corrections would have made obsolete: he did not at that time know anything about quantum mechanics and his beautiful atomic model would have been irrelevant.

The Rutherford scattering cross section is also at the basis of high $p_\perp$-scattering among hadronic constituents. Therefore the results will occur again in connection with deep inelastic scattering in the linked dipole chain model, in section 20.7, when we consider the hadronic wave function in a Feynman diagrammatic description of perturbative QCD. The (color-)charged constituents (the 'partons') will be sensitive to the strong Coulomb fields between them (such fields are inherent properties of any gauge field theory). In particular, when we use small wavelength probes, Heisenberg's indeterminacy principle implies that the observable partons must have large energy-momenta, i.e. their interactions will correspond to large momentum transfers.

After Rutherford, when more energetic beams of charged particles became available, experiments were performed on nuclear targets directly. A great amount of information was extracted about the charges inside (or actually mostly on the surface of) the nucleus. Still later, people were able to study scattering from the simplest nucleon, i.e. the proton itself and for a long time there was a general understanding that the proton was a complex charged object but that the charge seemed to be smeared out in a continuous way. It was necessary, in order to describe the reaction of a proton to an electromagnetic field pulse, to introduce a *form factor*. Such a form factor corresponds classically to an extended charge distribution.

When I was a young student, my teacher Källén referred to the next possible observational tool, the Stanford linear accelerator (SLAC), as the 'Monster'. It was understood from the beginning that the Monster might provide beams sufficiently high in energy to smash the proton but there were few people around who believed that this would lead to a new concept of constituents. The young Bjorken was around, however, and based upon theoretical investigations in current algebra he predicted that *one should find a 'scaling' cross section.*

Physicists have always used dimensional analysis to derive results of the kind usually referred to as 'back-of-an-envelope' calculations. Thus when one considers a particular dynamical situation there are always dimensional parameters. The typical space size may in a quantum mechanical description of a particle either be the Compton wavelength $1/m$, the Bohr radius $1/m\alpha$ or the 'classical charge radius' α/m (which occurs in the Thompson cross section for long-wavelength radiation scattering on a charged particle) with m the particle mass and $\alpha \simeq 1/137$ the fine structure constant. Based upon such quantities it is in general easy to find

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

the possible size of an effect, besides some plain (usually combinatorial) numbers such as $3! = 6$ and factors like $2/3$ (from spin) or (multiple) 2π's. (Note that π is almost a dimensional number because high-energy physicists generally obtain it either from the conversion of Planck's constant $h \to \hbar = h/2\pi$ – the π's in the conversion of the volume factors to cross sections are generally of that kind – or from integrals over the azimuthal angle.) We will use such considerations repeatedly in this book.

For the proton it was already known that there was a scale involved in connection with the form factor. This length scale corresponds to the extension of the proton charge distribution and it is of the same order as the inverse proton mass. Bjorken's statement can be rephrased to mean that *there should be no new length scales deeper inside the proton.*

The process, which is called deep inelastic scattering (DIS), will be discussed further within the Lund model in Chapter 20 and within the conventional QCD scenario in Chapter 19. It contains three dimensional numbers: the squared momentum transfer to the proton from the impinging electron, conventionally called $-Q^2$; the squared mass of the final-state (smashed) system, conventionally called W^2; and then the squared mass of the original system, i.e. the squared proton mass m_p^2.

The reason why Källén and his contemporaries called the machine the Monster was the fact that it would produce beams such that $m_p^2 \ll Q^2$ and/or W^2. Bjorken's suggestion was that the cross section should depend (besides a trivial Q^2-dependence) only on the ratio Q^2/W^2 of the two larger dimensional numbers. This turned out to be essentially correct.

According to Dick Taylor, who was present at the time, Feynman used to come over to SLAC to learn about the experimental results. One day he presented the experimentalists with the PM as an explanation for the scaling phenomena. Since Feynman's proposal there have been few high-energy theorists who have not produced some kind of work on the PM at some time in their career. We who have worked on the Lund model were very late arrivals on the scene.

In order to exhibit the PM we will provide a brief description of Rutherford's classical mechanics calculation and then show how to obtain the same result in a potential scattering model in quantum mechanics. This discussion is relevant to lepton-hadron scattering when the hadron can be considered as very heavy, i.e. its mass is much larger than any parameter with energy dimension in the problem. We will after that turn to the question of scattering on a composite system and introduce the idea of a *form factor*. This will lead to the Rosenbluth formula, which describes elastic scattering within the most general framework possible in a Lorentz-covariant and parity-invariant setting.

We will finally consider inelastic scattering, in which the incident lepton

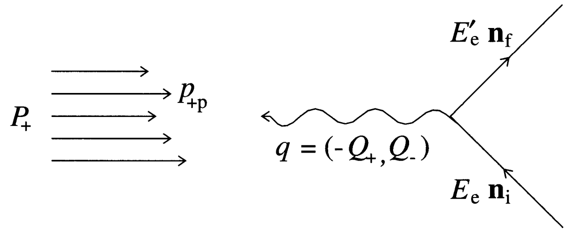

Fig. 5.1. The inelastic scattering of an electron from the field quanta of a hadron with notation described in the text.

produces field pulses, i.e. momentum transfers, which are so large that the initial hadron disintegrates. We will start with an excursion into lightcone physics and in particular indicate some of the steps that led Bjorken to suggest scaling cross sections.

Finally, we will use the results to exhibit the PM. We will show how parton flux factors arise and, in particular, the importance of spin and the other quantum numbers of the quark-partons for the resulting description.

5.1 The parton model: Feynman's proposal

Feynman used the results of the method of virtual quanta (MVQ), cf. Chapter 2, in an ingenious way. He assumed that the interaction ability of a hadron with respect to an electromagnetic field pulse is defined by a set of quanta which he called partons. Partons are at this stage operationally defined by the single property that *they are able to scatter elastically with an electron by absorbing a radiation quantum.*

In order to give a precise description we will assume that an accelerator provides us with electrons, of high energy E_i, coming in along a well-defined direction $\mathbf{n}_i$. We also assume that such an electron is scattered in the field of the hadron so that afterwards we observe it to have energy $E_f < E_i$ moving outwards in a direction $\mathbf{n}_f$ described by the angle θ (i.e. $\mathbf{n}_i \cdot \mathbf{n}_f = \cos\theta$, see Fig. 5.1).

From this situation we conclude that the electron has been exposed to a four-momentum transfer, conventionally called q:

$$q \equiv (q_0, -l\mathbf{n}) = (E_i - E_f,\ p_i\mathbf{n}_i - p_f\mathbf{n}_f) \tag{5.1}$$

As we have seen in Chapter 2 this four-vector must be spacelike, i.e. q^2 must be negative, $q^2 = -Q^2$, in order that the incoming and outgoing electrons stay on the mass shell $E_i^2 - p_i^2 = E_f^2 - p_f^2 = m_e^2$.

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

The momentum transfer corresponds (for large values of Q^2) to a very highly collimated electromagnetic field pulse with a space-time size of the order of the wavelength, $1/\sqrt{Q^2}$. We will use lightcone components along the vector $\mathbf{n}$ in Eq. (5.1) to describe this field pulse and so define positive $Q_\pm$ with $Q_+Q_- = Q^2$ (note the definition of q in Eq. (5.1))

$$-Q_+ = q_0 - l, \quad Q_- = q_0 + l \tag{5.2}$$

In Fig. 5.1 the hadron comes in as a cloud of (massless) partons together having a large positive-lightcone component P_+. The interaction between the radiative pulse described by q and one of the partons with a positive-lightcone component p_{+p} corresponds to an absorption of this radiation quantum. In order to stay on the mass shell the parton will have to reverse direction so that after the collision it will have a negative-lightcone component p_{-p}. Note that, as the parton is massless and is assumed to move along the direction $\pm\mathbf{n}$, it will before and after have a single nonvanishing lightcone component in this picture.

From energy-momentum conservation we conclude that all the kinematical properties of the interaction are fixed by

$$p_{\pm p} = Q_\pm \tag{5.3}$$

There are two observable (large) Lorentz invariants, i.e. $Q^2 = Q_+Q_-$ and $2Pq \simeq Q_-P_+$. We have neglected the hadronic mass and we note that in this approximation the final-state mass square of the smashed hadron has increased to $W^2 = (P+q)^2 \simeq 2Pq - Q^2$. Because the cross section depends only upon the ratio of these Lorentz invariants it must therefore depend only upon the fraction of the energy-momentum of the hadron, which is carried by the scattered parton (the index refers to Bjorken)

$$x_B = \frac{-q^2}{2Pq} = \frac{Q_+}{P_+} = \frac{p_{+p}}{P_+} \tag{5.4}$$

This sole dependence upon x_B can be understood as follows: *the interaction depends only upon the number of partons with that particular value of the fractional energy-momentum.* Thus the hadron has been reduced to a flux of partons with respect to the interaction, just as in the MVQ a charged particle is described by the flux of photons.

This assumption of Feynman about the interaction between the field pulse and the constituents implies the possibility of an experimental study of the flux of the partons, i.e. to decide upon the detailed structure of the hadron under study. It is then only necessary to consider the electron before and after the interaction. The probability of finding a large momentum transfer is directly related to the amount of suitable absorbers, i.e. partons, in the hadron.

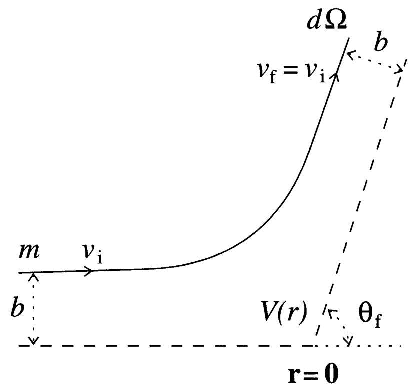

Fig. 5.2. Particles moving in a central force field are deflected in a definite direction characterised by the solid angle $d\Omega$.

Large values of the fraction x_B correspond to the partons which carry a large part of the total energy-momentum of the hadron. Therefore they should be major constituents of the hadronic wave function. For smaller values of x_B Feynman suggested that there should be a bremsstrahlung spectrum like the one we found for the photons in a moving Coulomb field according to the MVQ,

$$\sim dx_B/x_B \tag{5.5}$$

This is usually referred to as 'Feynman's wee parton spectrum'.

5.2 Rutherford's formula from classical mechanics

A detailed derivation of the Rutherford formula is given in Goldstein's book and we will only provide a brief description. In classical mechanics everything is completely determined by the force law and the initial conditions on the particle(s) involved. Consequently there is always a definite orbit along which every particle moves in space-time and a corresponding trajectory in phase space.

We assume that a particle with mass m is approaching the force centre in a field described by a potential $V(r)$, see Fig. 5.2, which vanishes as $r \to \infty$. Thus the force is spherically symmetric, $\mathbf{F} = -\left[dV(r)/dr\right]\mathbf{e}_r$ where $\mathbf{e}_r$ is a unit vector pointing radially outwards. We also assume that the particle has velocity v_i far from the centre, impact parameter b_i and orientation along some azimuthal angle ϕ_i.

This means that we can define an incident flux $Id\phi_i b_i db_i$ of such particles. All these particles will move along the same orbit and after

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

the encounter will end up moving outwards in a definite direction, which we will characterise in terms of a solid angle $d\Omega_f = \sin\theta_f d\theta_f d\phi_f \equiv d\Omega$.

RI The orbital angular momentum, $\mathbf{L}$, is conserved and therefore the particles will move in a plane perpendicular to $\mathbf{L}$. This means that the angles ϕ_i and ϕ_f coincide. The size of $|\mathbf{L}| = L$ is from the initial conditions $L = mv_i b_i$. Further the energy is conserved, cf. RIV below. Therefore the initial speed is equal to the final one and thus the same is true for the impact parameters, $b_i = b_f \equiv b$.

RII The cross section for the scattering of these particles is the fraction of particles scattered into the solid angle $d\Omega_f$ per unit time, divided by the incoming flux. It is then obtained by equating the outgoing and the ingoing fluxes:

$$\frac{d\sigma}{d\Omega_f} I\, d\Omega_f = -I b_i db_i d\phi_i \tag{5.6}$$

The minus sign is introduced because the larger the value of b the smaller the force and therefore the smaller the scattering. From this equation we conclude that

$$\frac{d\sigma}{d\Omega} = \frac{-b_i}{\sin\theta_f} \frac{db_i}{d\theta_f} \tag{5.7}$$

Therefore we must calculate the relationship between the impact parameter and the scattering angle.

RIII In order to calculate this orbit relation we use cylindrical coordinates $r(t)$, $\theta(t)$, so that the velocity is $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ (with dots indicating time derivatives). We obtain for the Lagrangian

$$\mathscr{L} = T - V(r) \text{ with } T = m\mathbf{v}^2/2 = m(\dot{r}^2 + (r\dot{\theta})^2)/2 \tag{5.8}$$

As $\mathscr{L}$ is independent of the angle θ the corresponding angular momentum component is conserved:

$$p_\theta = \frac{d\mathscr{L}}{d\dot{\theta}} = mr^2\dot{\theta} \equiv L \tag{5.9}$$

This can be used to reorganise the time dependence of $r(t)$ and $\theta(t)$ and from this we obtain an equation for the orbit $r = r(\theta)$:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta} = -\frac{L}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{5.10}$$

Using $u = 1/r$ and $u' = du/d\theta$ we can then write the kinetic energy term T in Eq. (5.8) as

$$T = \frac{L^2}{2m}(u')^2 + \frac{L^2u^2}{2m} \tag{5.11}$$

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

For an attractive Coulomb force between a charge $-e$ and a charge Ze we obtain for the potential term:

$$V = V(r) = -\frac{Ze^2}{4\pi r} = -Z\alpha u \tag{5.12}$$

(where we have introduced the fine structure constant α).

RIV As the total energy is conserved and expressible in terms of T and V we obtain

$$\begin{aligned} E \equiv \frac{mv_i^2}{2} &= T + V \\ &= \frac{L^2}{2m}[(u')^2 + u^2] - Z\alpha u = \frac{L^2}{2m}[(u')^2 + (u - u_0)^2 - u_0^2] \end{aligned} \tag{5.13}$$

where u_0, the displacement of u, is given by

$$u_0 = \frac{Z\alpha m}{L^2} \tag{5.14}$$

Equation (5.13) is equivalent to the harmonic oscillator relationship discussed in Chapter 3 and we can immediately write down the solution:

$$u \equiv \frac{1}{r} = u_0(1 + \epsilon \cos\theta) \tag{5.15}$$

This is the equation for a hyperbola since ϵ, the eccentricity, is larger than 1:

$$\epsilon = \sqrt{1 + \left(\frac{2Eb}{Z\alpha}\right)^2} \tag{5.16}$$

RV There are then two values of θ for which $r \to \infty$; these are given by $\cos\theta = -1/\epsilon$ and the angle between these directions is evidently $\pi - \theta_f$ (see Fig. 5.2). A little algebra then leads to the result that

$$b_i \equiv b = \frac{Z\alpha}{2E}\cot\left(\frac{\theta_f}{2}\right) \tag{5.17}$$

The final result for the Rutherford cross section is from Eq. (5.6)

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z^2\alpha^2}{16E^2}\right)\frac{1}{\sin^4(\theta_f/2)} \tag{5.18}$$

We will meet the same expression when we do the calculations using quantum mechanics. The energy E in Eq. (5.18) is given by the nonrelativistic kinetic energy $(mv_i^2)/2$.

The formula is singular for small scattering angles because the small-angle region corresponds to large impact parameters $b = b_i$ according to Eq. (5.17). The particles come in far from the force centre and are consequently deflected very little. The Coulomb force *per se* has infinite range but it is evident that any charge centre in real life will be screened by other charges (e.g. by its own electrons if it is an atomic nucleus).

In order to get an estimate of the cross section for a screened situation we will assume that the impact parameter is equal to w times the corresponding Bohr radius, i.e.

$$b_i = \frac{w}{Zm\alpha} \equiv wr_B \tag{5.19}$$

Then from Eq. (5.16) the parameter $\epsilon = \sqrt{1 + (wE/E_0)^2}$ with E_0 equal to the corresponding Rydberg energy:

$$E_0 = \frac{m(Z\alpha)^2}{2} \tag{5.20}$$

We now consider a fixed energy E much larger than E_0. This means that the velocity v_i will be much greater than $Z\alpha \simeq Z/(137)$, where we have introduced the well-known value for the fine structure constant in QED. This leaves, at least for small Z-values, a region where we may neglect relativistic corrections and still fulfil the requirement. We then obtain $\theta_f \sim 2E_0/(wE)$.

If we exchange the angular variation for one with respect to the parameter w we obtain a smooth behaviour,

$$d\sigma \simeq 2wdw\pi r_B^2 \tag{5.21}$$

and the cross section is independent of the energy E as long as $wE \gg E_0$.

Note that the cross section only depends upon the square of the charge combination $Z\alpha$. Therefore we obtain the same formula if the two charges have the same sign, i.e. if the attractive Coulomb potential in Eq. (5.12) is exchanged for a repulsive one: $-Z\alpha \to Z\alpha$. The displacement u_0 will in that case change sign, however. This means that the force centre will no longer be the internal focus of the hyperbola but instead the external one. Or, in other words, while the particle will go around the force centre for an attractive force it will go in an outside hyperbola if the force is repulsive. But the scattering angles are the same!

5.3 Rutherford's formula in relativistic quantum mechanics

1 *The calculation of the cross section*

We will in this section again consider the scattering of a charged particle from a Coulomb potential. This is a preliminary for treating the scattering of two charged particles. We will again meet Rutherford's result although

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

this time in terms of the square of the Fourier transform of the potential. We use the transition operator $\mathcal{T} = \int dx j_\mu A^\mu$ and assume that the external potential A^μ depends only upon the space coordinates. At the end we shall specialise to the ordinary Coulomb shape $A_\mu = -\delta_{\mu,0} Ze/(4\pi r)$, which was used in the previous section.

The transition matrix element between an incoming electron (energy-momentum k) and an outgoing one (energy-momentum k') is

$$\langle k'| \mathcal{T} |k\rangle = \frac{e}{2V\sqrt{k_0 k_0'}} \int dx \{\mathbf{k}'|jA(\mathbf{x})|\mathbf{k}\} \exp\left[-ix(k-k')\right] \tag{5.22}$$

We have here introduced the reduced matrix elements of the current operator, which we discussed in Chapter 4.

Time integration produces an energy-conserving δ-distribution and space integration leads to the Fourier transform of the vector potential:

$$\langle k'| \mathcal{T} |k\rangle = \frac{2\pi e}{2V\sqrt{k_0 k_0'}} \{\mathbf{k}'|j\mathscr{A}(\mathbf{q})|\mathbf{k}\}\delta(k_0 - k_0') \tag{5.23}$$

with $q = k' - k$. Momentum is not conserved in this case, because the infinitely heavy potential takes up the recoil. To calculate the cross section we use the techniques described in Chapter 3:

$$\begin{aligned} d\sigma &= \left(\frac{w}{\delta t}\right)\left(\frac{V}{v}\right)\left(\frac{Vd^3k'}{(2\pi)^3}\right) \\ &= \frac{e^2}{(2\pi)^2}\left(\frac{1}{4|\mathbf{k}|k_0'}\right)\int d^3k'\delta(k_0-k_0')|\{\mathbf{k}'|j\mathscr{A}(\mathbf{q})|\mathbf{k}\}|^2 \\ &= d\Omega'\frac{\alpha}{4\pi}|\{\mathbf{k}'|j\mathscr{A}(\mathbf{q})|\mathbf{k})\}|^2 \end{aligned} \tag{5.24}$$

The first factor in the first line is the transition probability per unit time, the second the (inverse) flux of incoming particles with $v = |\mathbf{k}|/k_0$ and the third the number of final states. In the second line we have rewritten the whole expression and in the third gone over from the integration variable $|\mathbf{k}'|$ to k_0' and performed the integral by means of the δ-distribution.

We may now make use of the analysis presented in Chapter 4 for the reduced matrix element combination summed over the spin states:

$$\begin{aligned} &\sum_{spins}\{\mathbf{k}'|j_\mu|\mathbf{k}\}\{\mathbf{k}|j_\nu|\mathbf{k}')\} = (T_1+T_2)_{\mu\nu}, \\ &T_{1\mu\nu} = g_{\mu\nu}q^2 - q_\mu q_\nu, T_{2\mu\nu} = (k+k')_\mu(k+k')_\nu \end{aligned} \tag{5.25}$$

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

2 The Mott cross section and the form factors

Gathering the different factors and assuming that the four-vector potential only has a time component, $A_0 = V(r)$, we obtain the cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha}{2\pi}\left[\mathbf{k}^2(1+\cos\theta)+2m^2\right]|\mathscr{V}(\mathbf{q})|^2 \tag{5.26}$$

where θ is the scattering angle. (We have neglected a few steps, leaving it to the reader to obtain this result.) There are two terms multiplying the squared Fourier transform of the potential. Depending upon whether the lepton rest mass m or the momentum $|\mathbf{k}|$ dominates we obtain a nonrelativistic or an extreme relativistic approximation.

For the Coulomb potential of a point particle with charge Ze we obtain

$$\mathscr{V}(\mathbf{q}) = -\frac{Ze}{4\pi}\int \frac{d^3x}{|\mathbf{x}|}\exp(-i\mathbf{x}\cdot\mathbf{q}) = \frac{Ze}{q^2} \tag{5.27}$$

The simplest way to see this is to use the coordinate-space differential equation for the Coulomb potential,

$$\Delta V(\mathbf{x}) = Ze\delta(\mathbf{x}) \tag{5.28}$$

and perform the Fourier transform, thereby changing the Laplacian Δ to $-\mathbf{q}^2 \equiv q^2$ (Note that $\Delta\exp[i\mathbf{q}\cdot\mathbf{x}] = -\mathbf{q}^2\exp[i\mathbf{q}\cdot\mathbf{x}]$).

It is at this point that Rutherford was lucky in his classical mechanics approach. The squared Fourier transform of the Coulomb potential evidently contains an inverse power of the squared momentum transfer $(q^2)^2 = (-|\mathbf{q}|^2)^2 = 4k^2(1-\cos\theta)^2 = 16k^4\sin^4(\theta/2)$ (where k is the cms conserved momentum of the particles), which is just what Rutherford obtained from his calculation of the variation of the impact parameter with angle. This relation between the Fourier transform of the potential and the variation of the impact parameter is only true for a Coulomb potential.

This leads to the so-called Mott cross section in the limit where we may neglect the electron mass:

$$\frac{d\sigma}{d\Omega}_{Mott} = \left(\frac{Z^2\alpha^2}{4E^2}\right)\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \tag{5.29}$$

There is a factor $4\cos^2(\theta/2)$ as compared to the Rutherford formula. If we go back to Rutherford's derivation we find that it is based upon nonrelativistic kinematics. The projectile mass is assumed to be much larger than its kinetic energy. This means according to Eq. (5.26) that $\mathbf{k}^2 \ll m^2$ and we obtain in this limit

$$\frac{d\sigma}{d\Omega} = \frac{\alpha}{\pi}(m^2)|\mathscr{V}(\mathbf{q})|^2 = \left(\frac{Z^2\alpha^2}{16(\mathbf{k}^2/2m)^2}\right)\frac{1}{\sin^4(\theta/2)} \tag{5.30}$$

which is Rutherford's result (with $E = E_{kin} = |\mathbf{k}|^2/2m$).

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

If the electron encounters not a point charge but a charge distribution $Zef(\mathbf{x})$ then on the right-hand side of Eq. (5.28) the exchange $Ze\delta(\mathbf{x}) \to Zef(\mathbf{x})$ should be made; this evidently means that in place of Eq. (5.27) we will have

$$\mathscr{V}(\mathbf{q}) = -\frac{Ze}{\mathbf{q}^2}\tilde{f}(\mathbf{q}), \quad \tilde{f}(\mathbf{q}) = \int d^3x f(\mathbf{x}) \exp(-i\mathbf{x} \cdot \mathbf{q}) \tag{5.31}$$

The normalisation condition $\int d^3xf = 1$ corresponds to $\tilde{f}(|\mathbf{q}|^2 = 0) = 1$. We conclude that with the introduction of the charge distribution f the Mott (or Rutherford) cross section is changed as follows:

$$\frac{d\sigma}{d\Omega}_{Mott} \to \frac{d\sigma}{d\Omega}_{Mott} |\tilde{f}(\mathbf{q})|^2 \tag{5.32}$$

Provided that the momentum transfer $\sqrt{|q^2|}$ is smaller than the inverse of any length scale in the charge distribution, or in other words provided that the wavelength of the electromagnetic pulse cannot resolve the target structures, then we have the same pointlike cross section. For larger momentum transfers the scattering experiment can be used to measure (the Fourier transform of) the charge distribution. The function $\tilde{f}$ is known as a form factor.

5.4 The target recoil and the general elastic cross section for the scattering of spin 1/2 particles

The form factor introduced at the end of the last section is too simple to describe scattering from a baryon target. Firstly, one cannot consider baryons as merely charge distributions. They also have magnetic moments and an electromagnetic pulse will influence that aspect of the baryon structure, too. Secondly, they are not infinitely heavy and so we must include also the recoil of the target, i.e. we must introduce not only energy but also momentum conservation in the scattering.

We have already, in Chapter 3 on field theory, considered a simplified model for this scattering situation, the scalar $g{:}\psi^2{:}\phi$-model. From the results in Eqs. (3.104)–(3.110) we now generalise the situation to two different ψ-particles, ψ_e indexed 1, 3, and ψ_B, indexed 2, 4, with 1, 2 the incoming pair (Fig. 5.3). We have in mind particles such as electrons and baryons and as they are both spin 1/2 particles the interaction is governed by the four-vector currents $j_e \propto {:}\psi^*\psi{:}$ and j_B likewise expressed in terms of Dirac spinors.

This means that the coupling constant factor $4g^2$ in Eq. (3.110) should be replaced by e^2 (this is plain combinatorics). Further the factor B in Eq. (3.110) contains three pole terms. Due to the fact that the lepton

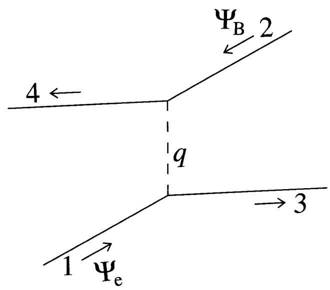

Fig. 5.3. The elastic scattering of an incoming electron (index 1) from a baryon (2) to the final state (3, 4) with the exchange of a virtual photon.

and baryon cannot mutually annihilate or be exchanged there is in the present situation only one of the factors left, the momentum-transfer pole $1/(k_1 - k_3)^2 = 1/(k_2 - k_4)^2$ (with $M_\phi = 0$ for the photon).

With these modifications we can use the result in Eq. (3.110):

$$d\sigma = \frac{e^2}{2(2\pi)^2\sqrt{\lambda(s, M_e^2, M_B^2)}}|B|^2 \times \prod_{j_f=3}^{4} dk_{j_f}\delta^+(k_{j_f}^2 - M_{j_f}^2)\delta(k_1 + k_2 - k_3 - k_4) \tag{5.33}$$

this time with B expressed in terms of the reduced matrix elements

$$B = \sum_{spins} \left(\frac{\{k_3e|j^\mu|k_1e\}\{k_4p|j^\mu|k_2p\}}{(k_2 - k_4)^2} \right)$$

Comparing with the result in Eq. (5.24) we find that the Fourier transform $\mathscr{A}^\mu$ of the four-vector potential A^μ has been replaced as follows:

$$\mathscr{A}^\mu \to \frac{ie}{2V\sqrt{E_2E_4}} \frac{\{k_4p|j^\mu|k_2p\}}{(k_2 - k_4)^2} \tag{5.34}$$

This is exactly in accordance with our physical intuition that we should now obtain the four-vector potential A^μ from the baryon current, j_B^μ:

$$A^\mu(x) \to \int dx' D_0(x - x') \langle k_4 | j_B^\mu(x') | k_2 \rangle + \text{g.t.} \tag{5.35}$$

where g.t. again stands for gauge-dependent terms of no interest because of the coupling to the conserved (electron) current. The result in Eq. (5.35) is, as easily seen, equal to the energy-momentum space result in Eq. (5.34) and corresponds to a solution of Maxwell's equations for the

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

vector potential in terms of the (baryon) current j_B^μ (note that the Green's function $\Box D_0(x) \propto \delta(x)$ is chosen as the Feynman propagator).

If we sum over the final-state spins and average over the initial ones for both the baryon and the electron we obtain two tensors (cf. Eq. (4.36)), one for the electron (which we have already written out in Eq. (5.25)) and one for the baryon, similarly with the two parts:

$$\begin{aligned}
T_B &= T_{1B} + T_{2B}, \\
T_{1B\mu\nu} &= g_{\mu\nu} q^2 - q_\mu q_\nu, \\
T_{2B\mu\nu} &= 4\left[k_2 - q(qk_2)/q^2\right]_\mu \left[k_2 - q(qk_2)/q^2\right]_\nu
\end{aligned} \tag{5.36}$$

For T_{2B} we have used the form explained in connection with Eq. (4.45).

Multiplying the electron and baryon tensors together we obtain the cross section. It is at this point useful to write it in an invariant form because we will need this later. To that end we introduce the two invariants corresponding to the energy and the scattering angle, the cms squared energy s and the squared momentum transfer $q^2 = -Q^2$:

$$\begin{aligned}
\hat{s} &\equiv s - M^2 \simeq 2k_1k_2 \simeq 2k_3k_4, \\
Q^2 &= -q^2 = 2k_1k_3 = 2(k_2k_4 - M^2)
\end{aligned} \tag{5.37}$$

Here we shall neglect the lepton mass and write $M \equiv M_B$. Note that in this case $\sqrt{\lambda} \simeq \hat{s}$. We obtain (note the factor $(1/2)^2$ from the initial spin-averaging)

$$\begin{aligned}
\frac{d\sigma}{dQ^2} &= \frac{4\alpha^2 De}{\hat{s}Q^4} I, \quad De = Q^4 - 2(\hat{s} + M^2)Q^2 + 2\hat{s}^2 \\
I &= \int dk_3 dk_4 \delta(k_3^2)\delta(k_4^2)\delta(k_1 + k_2 - k_3 - k_4)\delta((k_1 - k_3)^2 + Q^2)
\end{aligned} \tag{5.38}$$

We have here used the same trick as before, introducing a derivative of a function by means of a δ-distribution, this time in Q^2.

The integral I is more complicated (because of the δ-distribution in Q^2) than the phase-space integrals we have encountered before. To calculate it we introduce the vector $P = k_1 + k_2$, the total energy-momentum in the cms where $P = (W, \mathbf{0})$ and we place the vector k_1 along the 3-axis. For simplicity we shall calculate the integral in detail for the case when we can neglect the mass M, although we will at the end introduce it into the result. We obtain

$$\begin{aligned}
I &= \int dk_3 \delta(\mathbf{k}_3^2 - E_3^2)\delta(W^2 - 2WE_3)\delta(-2E_1E_3(1 - \cos\theta) + Q^2) \\
&= \frac{\pi}{2W^2} \to \frac{\pi}{2\hat{s}}
\end{aligned} \tag{5.39}$$

We have here performed the E_3-integral by means of the second δ, the $|\mathbf{k}_3|$-integral by the first δ and then the $d\Omega(= d\theta \sin\theta d\phi)$-integrals by

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

means of the last δ. The last line contains the generalisation to the case when $M \neq 0$.

In this way we arrive at the following result for the invariant Rutherford scattering cross section:

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2[Q^4 - 2(\hat{s} + M^2)Q^2 + 2\hat{s}^2]}{(Q^2)^2\hat{s}^2} \tag{5.40}$$

(the factor 2π corresponds to the fact that in a spin-averaged cross section there is no dependence upon the azimuthal angle).

We have obtained the cross section for the process $e_1 + p_2 \to e_3 + p_4$ by the use of the spin sums over the current matrix element in (5.33). According to crossing symmetry (mentioned after Eq. (4.37)) we may from this result easily obtain the result for the process $e_1 + \bar{e}_3 \to \bar{p}_2 + p_4$, i.e. the annihilation of the pair $e_1\bar{e}_3$ into $\bar{p}_2, p_4$ by the exchanges $p_3 \to -p_3$ and $p_2 \to -p_2$ in the matrix element. At the same time we note that the (squared) cms energy is in this situation $(p_1 + p_3)^2 \simeq 2p_1p_3$ while the momentum transfer variable is $Q^2 = -(p_1 - p_2)^2 \simeq 2p_1p_2$, i.e. we obtain the relevant cross section with $s \simeq \hat{s} \leftrightarrow Q^2$ (neglecting the masses). We will later only need the result for the case when all the particles are massless and we obtain after some straightforward calculations the (spin- and azimuthal angle-averaged) annihilation cross section

$$\frac{d\sigma_A}{dQ^2} = \frac{2\pi\alpha^2(s^2 - 2sQ^2 + 2Q^4)}{s^4} \tag{5.41}$$

5.5 The extension to non-pointlike baryons, form factors

Written in this form it is easy to evaluate the above cross section in any Lorentz frame. Conventionally we use the laboratory (lab) frame, in which the baryon is initially at rest.

In the lab frame the electron energies before and after the interaction, E and E' respectively, are different and in particular fulfil the relations

$$\begin{aligned} \frac{E'}{E} &= \frac{1}{1 + (E/M)(1 - \cos\theta)} \\ s &= M^2 + 2ME \\ Q^2 &= 2EE'(1 - \cos\theta) \end{aligned} \tag{5.42}$$

We shall leave the reader to prove these and also to show that

$$\left| \frac{dQ^2}{\sin\theta d\theta} \right| = 2E'^2 \tag{5.43}$$

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

Using these relations we obtain by straightforward means the cross section in the lab frame from Eq. (5.40):

$$\frac{d\sigma}{d\Omega_{lab}} = \frac{d\sigma}{d\Omega_{Mott}} \frac{E'}{E} \left[1 + \frac{Q^2}{2M_B^2} \tan^2\left(\frac{\theta}{2}\right)\right] \tag{5.44}$$

There are two new factors: the electron energy is not the same before and afterwards in the lab system; as a Dirac particle, the baryon also has a magnetic moment.

We will not go into detail with respect to the electric and magnetic interaction properties of a Dirac particle. Just as there are different electric and magnetic fields in different Lorentz frames, these properties are also frame dependent. It is useful to remember, however, that if we multiply in the factor $\cos^2(\theta/2)$ from the numerator in the Mott cross section then the factor inside the brackets in Eq. (5.44) becomes

$$\cos^2\left(\frac{\theta}{2}\right) + \frac{Q^2}{2M_B^2} \sin^2\left(\frac{\theta}{2}\right) \tag{5.45}$$

Here we have two obviously independent terms stemming from the parts $2\hat{s}^2 - 2(\hat{s} + M^2)Q^2$ and Q^4 of the factor De in Eq. (5.40).

We have up to now treated the baryon as a point Dirac particle; however, according to experiment it is not. It turns out that there are two independent form factors, just as we saw in Eq. (5.36) that the squared current leads to two independent tensors, T_{1B} and T_{2B}. These form factors can be introduced in different ways. The most symmetrical version involves the so-called electric and magnetic form factors G_E and G_M.

These play the roles of electric and magnetic couplings in the Breit frame, [84]. But their main importance is that they can be shown to be invariants, i.e. to depend only upon Q^2, and that they occur in a simple way. The bracketted terms in Eq. (5.44) are then exchanged as follows:

$$1 + \frac{Q^2}{2M_B^2} \tan^2\left(\frac{\theta}{2}\right) \to \frac{G_E^2 + (Q^2/4M_B^2)G_M^2}{1 + (Q^2/4M_B^2)} + \frac{Q^2}{2M_B^2} \tan^2\left(\frac{\theta}{2}\right) G_M^2 \tag{5.46}$$

With this exchange in Eq. (5.44) we obtain the general elastic cross section formula for lepton-baryon scattering when parity is conserved. It is called the *Rosenbluth formula* and has been thoroughly investigated experimentally. One finds that both the electric and the magnetic form factors behave in the same way:

$$G_E \propto G_M \propto [1 + Q^2/(M_0)^2]^{-1}, \quad M_0 \simeq 0.71\,\text{GeV} \tag{5.47}$$

In the early days of investigation of the proton and neutron this result lead to many speculations. Actually, the finding that the form factors were pole-dominated even led to the prediction that there should be

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

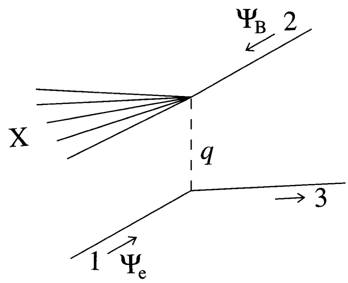

Fig. 5.4. The inelastic scattering of an electron from a baryon with one-photon exchange (a single electromagnetic pulse q) leading to a final state in which the baryon has fragmented into a complex system X.

(resonance) particles, [97], with the quantum numbers needed for the form factors, i.e. spin 1 particles. The fact that the ρ- and ω-particles fulfil these requirements and also have masses close to M_0 created particular attention. It is, however, not possible to prove from first principles that the elastic form factors should be analytic functions of $q^2 = -Q^2$ in the same way as we proved via the Källén-Lehman representation that the propagator should be analytic; within the Källén-Lehman formalism developed in Chapter 4 it would be natural to obtain a pole from an intermediate state.

Depending upon temperament and taste one may consider Eq. (5.47) as either a surprising finding or a reason for building a model. Such a model, *the vector dominance model* for the evaluation of matrix elements containing operators with the quantum numbers of the electromagnetic currents, [60], has been extensively used but is outside the scope of this book.

5.6 The inelastic scattering of electrons on baryons; lightcone physics

We will now consider the seemingly much more complex situation when the electromagnetic pulse q from the electron towards the baryon is such that the baryon breaks up into many final-state fragments (see Fig. 5.4).

The way in which we have introduced the elastic cross section makes it, however, rather easy to extend the formalism to the inelastic case, at least if we are only going to observe the electron before and after the interaction. According to Eqs. (5.34) and (5.35) the baryon is observable only through its current. For the case at hand, with a final state $\langle X|$ for the baryon containing all kinds of fragment particles, we need only to

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

make the exchange

$$\langle k_4 |\, j^\mu_B(0)\, | k_2 \rangle \to \langle X |\, j^\mu_B(0)\, | k_2 \rangle \tag{5.48}$$

We then obtain the same cross section but the baryon current parts are then described by (after averaging over the initial baryon spin)

$$\begin{aligned} W^{\mu\nu} &= \frac{VE_2}{4\pi M} \sum_X (2\pi)^4 \delta(q + k_2 - k_X) \langle k_2 |\, j^\mu_B(0)\, | X \rangle \langle X |\, j^\nu_B(0)\, | k_2 \rangle \\ &= \frac{VE_2}{4\pi M} \int dx \exp(iqx) \sum_X \langle k_2 |\, j^\mu_B(x)\, | X \rangle \langle X |\, j^\nu_B(0)\, | k_2 \rangle \\ &= \frac{VE_p}{4\pi M} \int dx \exp(iqx) \langle p |\, j^\mu_B(x) j^\nu_B(0)\, | p \rangle \end{aligned} \tag{5.49}$$

where in the last line, we have gone over to the conventional notation $p = (E_p, \mathbf{p})$ instead of the earlier k_2. In the second line we have re-defined the δ-distribution as a Fourier transform using $\langle p |\, j^\mu_B(x)\, | X \rangle = \langle p |\, j^\mu_B(0)\, | X \rangle \exp[ix(p - k_X)]$. In the third line we have used the completeness relation $\sum_X | X \rangle \langle X | = 1$ to arrange the result into a two-current matrix element in the initial (spin-averaged) baryon state $| p \rangle$.

We evidently need the the factor $2VE_p$ to cancel a volume factor and to obtain the invariant combination $E_p |\mathbf{k}_1| \to s - M^2$. The same factors are also needed in Eq. (5.49) to make the tensor W into an invariant according to our conventions. The momentum transfer four-vector q is defined in terms of the initial- and final-state (observable) lepton energy-momenta: $q = k_1 - k_3$. Finally, the factor $2M$ is introduced for conventional reasons.

It is useful at this point to note that

$$\begin{aligned} &\int dx \exp(iqx) \langle p |\, j^\mu_B(0) j^\nu_B(x)\, | p \rangle \\ &= \sum_{X'} (2\pi)^4 \delta(q + k_{X'} - p) \langle p |\, j^\mu_B(0)\, | X' \rangle \langle X' |\, j^\nu_B(0)\, | p \rangle = 0 \end{aligned} \tag{5.50}$$

because in this case the masses of the states X' must be smaller than the baryon mass and there are no such states containing a baryon (the electromagnetic interactions conserve baryon number). To see this we note that the mass M_X of a state X occurring in Eq. (5.49) must fulfil

$$M^2 \le M_X^2 = (p + q)^2 = M^2 - Q^2 + \nu \Rightarrow \nu \ge Q^2 \tag{5.51}$$

where $\nu = 2pq$ (note that different authors use somewhat different definitions of ν). Therefore the mass of X' in Eq. (5.50) must fulfil $M_{X'}^2 = (p - q)^2 \le M^2 - 2Q^2 < M^2$.

We may use this fact to rewrite the tensor W in terms of a commutator matrix element:

$$W^{\mu\nu} = \frac{VE_p}{4\pi M} \int dx \exp(iqx) \langle p |\, [j^\mu_B(x), j^\nu_B(0)]\, | p \rangle \tag{5.52}$$

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

In Chapter 3 we argued that due to causality the commutator of a local-field operator at two different points vanishes if the points are spacelike with respect to each other. This means that the integral in Eq. (5.52) is actually not over all space-time but only over the lightcones and their interior, i.e. $x^2 \geq 0$. It turns out to be reasonable to make the case that only the lightcone itself plays a role in the limit $\nu \to \infty$ with $x_B = Q^2/\nu$ nonvanishing. (Note that Eq. (5.51) implies the limit $x_B \leq 1$).

We will present a few steps in connection with such an argument (which is basically the scaling argument presented by Bjorken). We firstly choose to make use of the baryon rest frame in which $q = (q_0, \mathbf{0}_\perp, -|\mathbf{q}|)$ and note that in this frame

$$\nu = 2Mq_0 \quad \Rightarrow \quad |\mathbf{q}| = \sqrt{\frac{\nu^2}{4M^2} + Q^2} \simeq \frac{\nu}{2M} + Mx_B \tag{5.53}$$

so that the lightcone components of q along the 3-axis are approximately

$$q_- = q_0 + |\mathbf{q}| \simeq \frac{\nu}{M}, \quad q_+ = q_0 - |\mathbf{q}| \simeq -Mx_B \tag{5.54}$$

Then we consider a simplified model of the causal tensor function $W^{\mu\nu}$ in Eq. (5.52):

$$W(\nu, x_B) = \int dx \exp(iqx) F(x^2, px) \tag{5.55}$$

where $F = 0$ if $x^2 < 0$. (Note that there are only three possible invariants that the integrand F for a scalar W can depend upon, x^2, px, p^2, and that the third of these is a constant, $p^2 = M^2$.)

The argument in the oscillating exponent is then $iqx = i(q_- x_+ + q_+ x_-)/2 \simeq i(x_+ \nu/M - x_- M x_B)/2$. According to the theory of the Fourier transform the function W can then only obtain significant contributions from the integration regions $x_+ \leq M/\nu$ and $x_- \leq 1/Mx_B$. For the limit $\nu \to \infty$ this evidently means the region $0 \leq x^2 = x_+ x_- - \mathbf{x}_\perp^2 \leq 1/(x_B \nu) - \mathbf{x}_\perp^2$. Therefore the inverse of $Q^2 = x_B \nu$ limits the transverse area inside which the integral obtains significant contributions and we are then led towards the lightcone itself when $Q^2 \to \infty$.

There are several pitfalls in this argument and it only works for sufficiently well-behaved functions F in the integrand. If F is of that kind we may continue the argument a little further and assume that the main contribution to such an F constructed from scalar currents,

$$F = \frac{VE_p}{2\pi} \langle p| \, [j(x), j(0)] \, |p\rangle \tag{5.56}$$

will be a singularity along the lightcone, similar to the one obtained in

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

Eq. (3.77) for the ordinary commutator, multiplied by a function $f(px)$:

$$F = \frac{i\epsilon(x)}{2\pi}\delta(x^2)\left[\int da\exp(iapx)\tilde{f}(a) + \cdots\right] \tag{5.57}$$

The dots indicate less singular terms and we have written f in terms of its Fourier transform, $\tilde{f}$. If this is introduced into Eq. (5.55) we obtain, neglecting the terms indicated by ellipses and using the Fourier transform occurring in Eq. (3.77),

$$\begin{aligned} W &= 2\pi\int da\tilde{f}(a)\epsilon(q+ap)\delta((q+ap)^2) \\ &\simeq \frac{2\pi}{v}\int da\tilde{f}(a)\delta(a-x_B) = \frac{2\pi}{v}\tilde{f}(x_B) \end{aligned} \tag{5.58}$$

This is apart from the dimensional factor v^{-1} a result which only depends upon the Bjorken scaling variable x_B through the Fourier transform of f.

It is of particular interest to note that the scaling variable x_B in this way occurs as the inverse Fourier transform variable (the 'canonical coordinate') of the quantity px, which intuitively describes the variations of the matrix elements along the lightcone $x^2 = 0$. The result stems from the assumption that the (scalar) current commutator behaves as the free-field commutator in Eq. (3.77). The argument can, however, be generalised to include more complex situations where the lightcone singularity contains derivatives of δ-distributions. The main point throughout is that no new scale is involved. The lightcone *per se* is evidently the same everywhere.

5.7 The parton model revisited

We have seen in the previous section how to make use of some simple causality arguments, and some perhaps optimistic limits, to obtain the scaling behaviour of the inelastic cross section. In this section we will arrive at the same result by an analysis of the cross section we obtain from the inelastic scattering situation. We obtain, by introducing the tensor W into Eq. (5.33),

$$d\sigma = \frac{e^2}{2(2\pi)^2(s-M^2)t^2}dk_3\delta^+(k_3^2)2MT_{\mu\nu}W^{\mu\nu} \tag{5.59}$$

The tensor W can be constructed from the two T-tensors we have used before. In conventional notation we write (with two scalar form factors W_j):

$$\begin{aligned} W_{\mu\nu} = e^2\Bigg[&W_1\left(-g_{\mu\nu} + \frac{q_\mu q_\mu}{q^2}\right) \\ &+ W_2\frac{1}{M^2}\left(p_\mu - \frac{pq}{q^2}q_\mu\right)\left(p_\nu - \frac{pq}{q^2}q_\nu\right)\Bigg] \end{aligned} \tag{5.60}$$

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

which leads to

$$T_{\mu\nu}W^{\mu\nu} = e^2\left\{2Q^2W_1 + W_2\left[\frac{\hat{s}(\hat{s}-\nu)}{M^2} - Q^2\right\}\right) \quad (5.61)$$

using $\nu = 2pq$. Next, we introduce instead of the vector k_3 the two invariants Q^2, ν by means of the usual trick:

$$dQ^2d\nu\int dk_3\delta^+(k_3^2)\delta(Q^2 - 2k_1k_3)\delta(\nu - 2p(k_1 - k_3)) = \frac{\pi}{2\hat{s}}dQ^2d\nu \quad (5.62)$$

The cross section is then given by

$$d\sigma = \frac{2\pi\alpha^2 dQ^2 d\nu}{\hat{s}^2Q^4}\left[2MQ^2W_1 + \frac{W_2}{M}(\hat{s}^2 - \hat{s}\nu - M^2Q^2)\right] \quad (5.63)$$

According to Feynman's suggestion this cross section should be expressible in terms of a flux factor $\sum_j e_j^2 f_j(x)dx$ of partons, all massless and scattering like point particles (with squared charge $e_j^2e^2$) from the electron. Their cross section should then be given by the invariant cross section in Eq. (5.40), so that

$$d\sigma = dQ^2\frac{2\pi\alpha^2(Q^4 - 2\hat{s}Q^2 + 2\hat{s}^2)}{Q^4\hat{s}^2}d\nu\delta(\nu - Q^2) \quad (5.64)$$

If everywhere we replace the parton energy-momentum p by xp, this implies the following changes:

$$Q^2 = -q^2 \to Q^2, \quad \hat{s} = 2pk_1 \to x\hat{s}, \quad \nu = 2pq \to x\nu \quad (5.65)$$

We then obtain, by comparing coeffients,

$$\frac{W_2}{M} = \int 2\sum_j e_j^2 f_j(x)xdx\delta(x\nu - Q^2) \quad \Leftrightarrow \quad \frac{\nu W_2}{2M} = \sum_j e_j^2 f_j(x_B)x_B \quad (5.66)$$

$$2MW_1 = \int\sum_j e_j^2 f_j(x)Q^2dx/x\delta(x\nu - Q^2) \quad \Leftrightarrow \quad 2MW_1 = \sum_j e_j^2 f_j(x_B)$$

where $x_B = Q^2/\nu$ is the Bjorken scaling variable. In this way we have been able to give a precise relationship between Feynman's parton flux factors and the inelastic form factors W_1 and W_2.

We note that the fact that we have assumed the partons to be spin 1/2 particles provides a very precise relationship between the two structure functions W_1 and W_2, i.e.

$$\frac{\nu W_2}{2M} = x2MW_1 \equiv xf_B(x) \quad (5.67)$$

using the subscript B to denote a baryon target. There will be different parton flux factors (or, as they are called, *structure functions*) for a proton (p) and a neutron (n) as we will see below.

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

If we had assumed that the partons were spin 0 particles the corresponding analysis would have led to the result that W_2 still has the same scaling shape but now $W_1 = 0$. Needless to say the original SLAC experiments proved conclusively that W_j fulfils the scaling laws in Eq. (5.66) and therefore that the partons involved were, just as the quarks should be, spin 1/2 particles.

It is instructive to compare these results with the case of elastic scattering and the corresponding form factors from the Rosenbluth formula (Eqs. (5.44), (5.46). If we put $G_E = G_M = G$ then we obtain the correspondences

$$W_1 \to \frac{Q^2}{4M^2} G^2(Q^2)\delta(v - Q^2), \quad W_2 \to G^2(Q^2)\delta(v - Q^2) \tag{5.68}$$

In this case there is another scale, $M_0 \simeq 0.71$ GeV, from the dipole formula for the baryon elastic form factors. Therefore it is impossible to rearrange these expressions into a scaling form.

We note, however that for $x_B = 1$ we go from the inelastic to the elastic contribution. In real-life experiments it is not actually a δ-peak, although it does stand out by several orders of magnitude (depending upon the way one plots it) from the inelastic background. In the neighborhood $x_B \simeq 1$ there are also contributions from several nucleon resonances and it is interesting that the inelastic cross section as described above takes over in a very smooth way. If we take an average over these resonances then we smoothly go over to the general inelastic cross section (the Drell-Yan-West relations, [54]). This means that the nucleon splits up into partons as smoothly as possible.

5.8 The partons as quarks

We will mention, just for completeness, a few properties of the structure functions for baryons when the partons are identified as quarks, in accordance with Gell-Mann's and Neeman's suggestion. For more extensive discussions we refer to [77].

With the wild proliferation of new particles, found in high-energy interactions at the end of the 1950s and in the 1960s, it quickly became clear that all these quantities could not be fundamental quanta. Therefore several different classification schemes were suggested, all of them building upon some idea of a basic symmetry in the interactions. The one which was successful, the SU(3)-group classification, contains besides the singlet, octet and higher representations a triplet also (corresponding to the spin 1/2 representation in SU(2)). This triplet (which we from now on will call 3_f, f for *flavor*) contains three 'building blocks', the u, d and s quarks (q-particles, or q's). Together with the corresponding antitriplet,

$\overline{3}_f$, containing the antiquarks, the $\bar{u}$, $\bar{d}$ and $\bar{s}$ ($\bar{q}$-particles, or $\bar{q}$'s), they can be used to build up all known higher representations of the hadrons without charm and bottom flavors. The quarks must have a set of internal quantum numbers in order to be useful.

Q1 *The quark electric charges* are $+2e/3$ for the u and $-e/3$ for the d and s, with e the fundamental electric charge. As it is the square of the charges which occurs in the cross sections (the square of the matrix elements) the u will couple four times as strongly as the d and s to electromagnetic interactions.

This means that the effective flux factors for electromagnetic interactions contain a different weighting between the quark species so that the observed flux must be proportional to

$$f(x) = \tfrac{4}{9}\left[u(x) + \bar{u}(x)\right] + \tfrac{1}{9}\left[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)\right] \tag{5.69}$$

when electromagnetic probes are used. We use the notation $x_B = x$ and the quark names for the distributions.

Q2 *The pairs* u, d *and* $\bar{u}, \bar{d}$ *each form an isospin* $1/2$ *doublet*. The s and $\bar{s}$ do not carry isospin but instead *strangeness* and *antistrangeness*). This means that the SU(3) flavor-group contains fundamental building blocks both in abstract isospin space (in both directions, u 'up' and d 'down') and along the strangeness direction.

The strong interaction conserves these quantum numbers so that the total isospin I and the strangeness content is conserved; further they do not care about the directions in isospin space. This means that states with the same I but different I_3 (i.e. different steps in the u- or d-directions) react in the same way to the strong interaction.

In particular the proton, p, and the neutron, n, form an isospin doublet with $I = 1/2$; they contain uud and ddu respectively. Therefore a knowledge of the u-content (u_p) of the p is equivalent to a knowledge of the d-content (d_n) of the n. The same goes for $d_p = u_n$.

Q3 *SU(3)-symmetry of the 'ocean'*. One usually assumes (for lack of evidence to the contrary) that there are two particular kinds of parton distributions, for *valence constituents* and for *'ocean' q- and* $\bar{q}$*-particles*. Thus $u_p = u_{pv} + u_{po}$, i.e. the sum of the valence and the ocean contributions and a similar relation holds for d_p.

Further one often assumes that *all the ocean parts are equal*, i.e. $u_{po} = \bar{u}_p = d_{po} = \bar{d}_p = s_p = \bar{s}_p \equiv o$ (note that for a baryon all the antiquarks then belong to the ocean).

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press

Then we can rewrite the relations for the effective structure functions of the p, n, f_p and f_n, and their difference, as

$$\begin{aligned} f_p(x) &= \tfrac{1}{9}(4u_{pV} + d_{pV}) + \tfrac{4}{3}O(x) \\ f_n(x) &= \tfrac{1}{9}(4d_{pV} + u_{pV}) + \tfrac{4}{3}O(x) \\ f_p(x) - f_n(x) &= \tfrac{1}{3}(u_{pV} - d_{pV}) \end{aligned} \tag{5.70}$$

Q4 *The q- and $\overline{q}$-partons carry spin* $1/2$, as we have shown above.

Taken together this means that (if property Q3 is fulfilled) that there are three different structure functions for the quarks in the baryons. There is also the gluon structure function $g(x)$, which is often taken as closely related to the ocean quark properties.

The experimental results provide both a direct measurement of some combinations of the structure functions and also constraints on all of them. We will end by pointing out that the original SLAC experiments had already given constraints on the behaviour of $g(x)$. It is evident that the following integral will contain all the momentum carried by q and $\bar{q}$:

$$\int_0^1 x dx (u + \bar{u} + d + \bar{d} + s + \bar{s}) = I \tag{5.71}$$

From their measurements on protons and neutrons the experimentalists were able to determine that

$$\begin{aligned} \int_0^1 x dx f_p(x) &\simeq \tfrac{4}{9}I_u + \tfrac{1}{9}I_d \simeq 0.18 \\ \int_0^1 x dx f_n(x) &\simeq \tfrac{1}{9}I_u + \tfrac{4}{9}I_d \simeq 0.12 \end{aligned} \tag{5.72}$$

with the approximation that one neglects the strange and antistrange contributions.

From here we conclude that the fraction of the proton's energy-momentum carried by the u and $\bar{u}$, I_u, and the fraction carried by the d and $\bar{d}$, I_d, are approximately 0.36 and 0.18, respectively. Therefore in this approximation I for the proton is 0.54. This means that about 50% of the proton momentum is carried by the field or, as we will in general say, the field quanta, the *gluons*.

https://doi.org/10.1017/9781009401296.005 Published online by Cambridge University Press