DISTRIBUTIONS ASSOCIATED WITH THE FACTORS OF WILKS' Λ IN DISCRIMINANT ANALYSIS

A. M. KSHIRSAGAR

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1. Introduction

Exact tests, based on factorization of Wilks' Λ were derived by Williams (1952, 1955) and Bartlett (1951), for testing the goodness of fit of a single hypothetical discriminant function, in the case of several groups. An analytical derivation of the distributions associated with these tests was given by the author (1964 a), after expressing the test statistics in canonical forms. Williams (1961) and later Radcliffe (1966) extended these factorizations of Wilks' Λ to cover the case of s ($s > 1$) hypothetical discriminant functions. Radcliffe, in his paper, states that an analytical derivation of the distributions, as well as of the independence of the factors of Λ is desirable. This is done in the present paper, by expressing the test criteria in simpler forms and using matrix transformations in the multivariate Beta distribution (Kshirsager 1961). This is a straightforward extension of the author's paper (1964 a), which dealt with only one hypothetical discriminant function.

The notation in this paper is the same as that of Radcliffe's (1966) and is slightly different from that of the author's earlier paper (1964 a).

2. Factorisation of Wilks' Λ and some preliminary results

Let there be two vectors

$$\boldsymbol{x}' = [x_1, x_2, \dots x_p]$$

and

$$\boldsymbol{y}' = [y_1, y_2, \dots y_q]$$

with $p \leqq q$ and let the matrix of the corrected sum of squares and sum of products (s.s. and s.p.) of these $p+q$ variables be

$$(2.1) \qquad \begin{array}{cc} \left[\begin{array}{c|c} \boldsymbol{C}_{xx} & \boldsymbol{C}_{xy} \\ \hline \boldsymbol{C}_{yx} & \boldsymbol{C}_{yy} \end{array}\right] & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p \quad & \quad q \end{array} & \end{array}$$

based on n degree of freedom (d.f.). If we consider the regression of $\boldsymbol{x}$ on $\boldsymbol{y}$, we shall get the following partitioning of the "total" s.s. and s.p. matrix

Source	d.f.	Matrix of s.s. and s.p.
Regression	q	$\boldsymbol{C}_{xy} \cdot \boldsymbol{C}_{yy}^{-1} \cdot \boldsymbol{C}_{yx} = \boldsymbol{A}$
Residual	$n-q$	$\boldsymbol{C}_{xx} - \boldsymbol{C}_{xy} \cdot \boldsymbol{C}_{yy}^{-1} \cdot \boldsymbol{C}_{yx} = \boldsymbol{C}_{xx \cdot y} = \boldsymbol{B}-\boldsymbol{A}$
Total	n	$\boldsymbol{C}_{xx} = \boldsymbol{B}$

If we have $q+1$ multivariate populations and if we carry out a multivariate analysis of varience on the p characters $\boldsymbol{x}$, we shall get a similar partitioning of the total matrix $\boldsymbol{B}$ into "Between populations" (d.f.q.) and "Within Populations" (d.f.n-q). We can call these qd.f. here as corresponding to a set of dummy variables $\boldsymbol{y}$ and then the "Between Populations" matrix $\boldsymbol{A}$ is the regression of $\boldsymbol{x}$ on these dummy variables (Bartlett, 1951). The means of these $q+1$ populations will lie in only an s-dimensional space ($s < p$), if and only if the canonical correlations $\rho_1, \rho_2, \ldots, \rho_p$ between $\boldsymbol{x}$ and $\boldsymbol{y}$ are such that $\rho_1 > \rho_2 > \ldots > \rho_s > 0$ and $\rho_{s+1} = \rho_{s+2} = \ldots = \rho_p = 0$.

The canonical variables corresponding to $\rho_1, \ldots, \rho_s$ are then the s discriminant functions adequate for discriminating between the $q+1$ groups. Given a set of s discriminants $\boldsymbol{\Gamma}'\boldsymbol{x}$ where $\boldsymbol{\Gamma}'$ is an (sxp) matrix of rank s, one will be interested in testing, whether these assigned functions are adequate for discriminating between the $q+1$ populations or not. Williams (1961) and Radcliffe (1966) obtained the following test criteria for this purpose. Wilk's Λ is $|\boldsymbol{B}-\boldsymbol{A}|/|\boldsymbol{B}|$. The Λ criterion based on $\boldsymbol{\Gamma}'\boldsymbol{x}$ alone is

$$\Lambda_0 = |\boldsymbol{\Gamma}'(\boldsymbol{B}-\boldsymbol{A})\boldsymbol{\Gamma}|/|\boldsymbol{\Gamma}'\boldsymbol{B}\boldsymbol{\Gamma}|. \tag{2.2}$$

The residual likelihood ratio criterion is, therefore,

$$\Lambda^* = \Lambda/\Lambda_0 \tag{2.3}$$

and this is factorized as

$$\Lambda^* = \Lambda'\Lambda'' \tag{2.4}$$

or

$$\Lambda^* = \Lambda^{\mathrm{V}}\Lambda^{\mathrm{VI}} \tag{2.5}$$

where

$$\Lambda' = |\boldsymbol{\Gamma}'\boldsymbol{A}\boldsymbol{B}^{-1}(\boldsymbol{B}-\boldsymbol{A})\Gamma|/\Lambda_0|\boldsymbol{\Gamma}'\boldsymbol{A}\boldsymbol{\Gamma}|, \tag{2.6}$$

and

$$\Lambda^{\mathrm{V}} = \Lambda|\boldsymbol{\Gamma}'\boldsymbol{A}\boldsymbol{\Gamma}+\boldsymbol{\Gamma}'\boldsymbol{A}(\boldsymbol{B}-\boldsymbol{A})^{-1}\boldsymbol{A}\boldsymbol{\Gamma}|/|\boldsymbol{\Gamma}'\boldsymbol{A}|. \tag{2.7}$$

Λ' is called the direction factor and Λ'' the partial coplanarity factor of Λ, because Λ' deals with the adequacy of the directions of the s assigned functions, while Λ'' is more concerned with whether s linear functions are sufficient at all, for discrimination. In the alternative factorization (2.5), Λ^{V} is the coplanarity factor and Λ^{VI} is the partial direction factor.

We now proceed to deiive the distribution of Λ', Λ'', Λ^{V} and Λ^{VI} analytically and show that Λ' is independently distributed of Λ'' and so also Λ^{V} of Λ^{VI}. The distributions are derived under the null hypothesis viz.

The canonical correlations $\rho_{s+1}, \ldots, \rho_p$ are all zero and that $\boldsymbol{\Gamma}'\boldsymbol{x}$ are the canonical variables corresponding to the non-null correlations $\rho_1, \rho_2, \ldots, \rho_s$. There is no loss of generality in assuming

$$\boldsymbol{\Gamma}'\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix}$$

because, if it is not so, we can always transform from $x_1, \ldots x_p$ to $x_1^*, \ldots x_p^*$ and $y_1, \ldots y_q$ to $y_1^* \cdots y_q^*$ where $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are the canonical variables and then suppress the stars. Therefore, we can assume that this is already done, so that $\boldsymbol{x}$ and $\boldsymbol{y}$ are the canonical variables in the population. Consequently,

$$\boldsymbol{\Gamma}' = \begin{array}{c} \left[\begin{array}{c:c} \boldsymbol{I} & \boldsymbol{0} \end{array}\right] s \\ \scriptstyle s \quad p-s \end{array} \tag{2.8}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{0}$ is the null matrix. It is well-known that the distribution of $\boldsymbol{A}$ is a non-central Wishart distribution (see for example Kshirsagar **1964** b) and that of $\boldsymbol{B}-\boldsymbol{A}$ is the central Wishart distribution, independent of $\boldsymbol{A}$. The non-central Wishart distribution is given by James (1964). Make the transformations

$$\boldsymbol{B}-\boldsymbol{A} = \boldsymbol{C}\boldsymbol{L}\boldsymbol{C}' \tag{2.9}$$

$$\boldsymbol{B} = \boldsymbol{C}\boldsymbol{C}' \tag{2.10}$$

$$\boldsymbol{L} = \boldsymbol{T}\boldsymbol{T}' \tag{2.11}$$

where $\boldsymbol{C}$ and $\boldsymbol{T}$ are lower triangular matrices of order p. It can then be easily seen that the distribution of $\boldsymbol{T}$ is

$$\text{const. } h(\boldsymbol{T}_1; \rho_1, \ldots, \rho_s) \prod_{i=1}^{p} t_{ii}^{n-q-i} |\boldsymbol{I}-\boldsymbol{T}\boldsymbol{T}'|^{(q-p-1)/2} d\boldsymbol{T} \tag{2.12}$$

where $h(\boldsymbol{T}_1; \rho_1, \ldots, \rho_s$ is a function of $\boldsymbol{T}_1$ and the non-null ρ's only, $\boldsymbol{T}_1$ being the $s \times s$ submatiix of

$$\boldsymbol{T} = \begin{array}{c} \left[\begin{array}{c:c} \boldsymbol{T}_1 & \boldsymbol{0} \\ \hdashline \boldsymbol{T}_3 & \boldsymbol{T}_4 \end{array}\right] \begin{array}{l} s \\ p-s \end{array} \\ \scriptstyle s \quad\quad p-s \end{array} \tag{2.13}$$

An explicit form of h is not necessary at all; it is sufficient to note that it involves T_1 only. When $s = 1$, an explicit form of h is given by the author

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(1961). For the sake of brevity, we shall denote $h(\boldsymbol{T}_1; \rho_1, \rho_2, \ldots, \rho_s)$ by $h(\boldsymbol{T}_1, \rho)$ only. From (2.9) to (2.11)

$$\Lambda = |\boldsymbol{L}| = \prod_{i=1}^{p} t_{ii}^2 \tag{2.14}$$

and when all ρ's are zero, it is known that the distribution of t_{ii}^2 $(i = 1, 2, \ldots, p)$ is

$$\text{const } (t_{ii}^2)^{((n-q-i+1)/2)-1}(1-t_{ii}^2)^{(q/2)-1}d(t_{ii}^2), \tag{2.15}$$

and that of $\boldsymbol{L}$ is

$$\text{const } |\boldsymbol{L}|^{(n-q-p-1)/2}|\boldsymbol{I}-\boldsymbol{L}|^{(q-p-1)/2}dL \tag{2.16}$$

(see Kshirsagar 1961, 1964 b). Λ is then distributed as the product of the p independent Beta variables t_{ii}^2 given by (2.15). This distribution of Λ is known as the $\Lambda(n, p, q)$ distribution. $\Lambda(n, p, q)$ and $\Lambda(n, q, p)$ are the same. We therefore have the following two lemmas:

LEMMA 1: *If the distribution of a $p \times p$ lower triangular matrix $\boldsymbol{T}$ is*

$$\text{const} \prod_{1}^{p} t_{ii}^{n-q-1}|\boldsymbol{I}-\boldsymbol{T}\boldsymbol{T}'|^{(q-p-1)/2}d\boldsymbol{T},$$

the distribution of $\prod_1^p t_{ii}^2 = |\boldsymbol{T}\boldsymbol{T}'|$ is the $\Lambda(n, p, q)$ distribution.

LEMMA 2: *If the distribution of a $p \times p$ matrix $\boldsymbol{L}$ is* (2.16), *the distribution of $|\boldsymbol{L}|$ is the $\Lambda(n, p, q)$ distribution.*

If $\boldsymbol{Z}$ is an $n \times p$ matrix of independent standard normal variables, $\boldsymbol{S} = \boldsymbol{Z}'\boldsymbol{Z}$ has the Wishart distribution. This leads to

LEMMA 3:

$$\int_{\boldsymbol{Z}'\boldsymbol{Z}=\boldsymbol{S}} d\boldsymbol{Z} = \text{const } |\boldsymbol{S}|^{(n-p-1)/2}d\boldsymbol{S}$$

where $\int_{\boldsymbol{Z}'\boldsymbol{Z}=\boldsymbol{S}}$ *means, transforming from $\boldsymbol{Z}$ to $\boldsymbol{S}$ and some other variables and integrating out these other variables.*

We need the following results also:

Result 1: $|\boldsymbol{I}-\boldsymbol{P}\boldsymbol{Q}| = |\boldsymbol{I}-\boldsymbol{Q}\boldsymbol{P}|$.

Result 2: $(\boldsymbol{I}-\boldsymbol{P}\boldsymbol{Q})^{-1} = \boldsymbol{I}+\boldsymbol{P}(\boldsymbol{I}-\boldsymbol{Q}\boldsymbol{P})^{-1}\boldsymbol{Q}$.

Result 3: $\left[\begin{array}{c|c} \boldsymbol{P} & \boldsymbol{Q} \\ \hline \boldsymbol{Q}' & \boldsymbol{S} \end{array}\right]^{-1} = \left[\begin{array}{c|c} \boldsymbol{P}^{-1}+\boldsymbol{P}^{-1}\boldsymbol{Q}(\boldsymbol{S}-\boldsymbol{Q}'\boldsymbol{P}^{-1}\boldsymbol{Q})^{-1}\boldsymbol{Q}'\boldsymbol{P}^{-1} & \cdots \\ \hline \cdots & \cdots \end{array}\right]$.

Result 4: $\left[\begin{array}{c|c} \boldsymbol{P} & \boldsymbol{Q} \\ \hline \boldsymbol{Q}' & \boldsymbol{S} \end{array}\right] = |\boldsymbol{P}||\boldsymbol{S}-\boldsymbol{Q}'\boldsymbol{P}^{-1}\boldsymbol{Q}|$.

3. Simpler expressions for Λ', Λ'', Λ^{V} and Λ^{VI}

Using (2.9), (2.10), (2.11) and (2.8), it can be seen by a little algebra, that

$$\Lambda = \prod_{1}^{p} t_{ii}^2, \quad \Lambda_0 = \prod_{1}^{s} t_{ii}^2, \quad \Lambda^* = \prod_{s+1}^{p} t_{ii}^2 = |\boldsymbol{T}_4 \boldsymbol{T}_4'|,$$

and

$$\Lambda' = \frac{|\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1-\boldsymbol{T}_3'\boldsymbol{T}_3|}{|\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1|} = |\boldsymbol{I}-\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-1}\boldsymbol{T}_3'|, \tag{3.1}$$

$$\Lambda^{\mathrm{V}} = |\boldsymbol{T}_4\boldsymbol{T}_4'+\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-1}\boldsymbol{T}_3'|, \tag{3.2}$$

so that

$$\Lambda'' = \prod_{i=s+1}^{p} t_{ii}^2/\Lambda', \tag{3.3}$$

and

$$\Lambda^{\mathrm{VI}} = |\boldsymbol{T}_4\boldsymbol{T}_4'|/\Lambda^{\mathrm{V}} \tag{3.4}$$

We have repeatedly used results 1 to 4 of the previous section in obtaining the above expressions.

4. Distribution of Λ' and Λ''

From (2.13),

$$\begin{aligned} |\boldsymbol{I}-\boldsymbol{T}\boldsymbol{T}'| = |\boldsymbol{I}-\boldsymbol{T}'\boldsymbol{T}| &= \left|\begin{array}{c:c} \boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1-\boldsymbol{T}_3'\boldsymbol{T}_3 & -\boldsymbol{T}_3'\boldsymbol{T}_4 \\ \hdashline -\boldsymbol{T}_4'\boldsymbol{T}_3 & \boldsymbol{I}-\boldsymbol{T}_4'\boldsymbol{T}_4 \end{array}\right| \\ &= |\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1-\boldsymbol{T}_3'\boldsymbol{T}_3||\boldsymbol{I}-\boldsymbol{T}_4'\{\boldsymbol{I}+\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1-\boldsymbol{T}_3'\boldsymbol{T}_3)^{-1}\boldsymbol{T}_3'\}\boldsymbol{T}_4|. \end{aligned} \tag{4.1}$$

We can always express the $(p-s)\times(p-s)$ matrix

$$\{\boldsymbol{I}+\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1-\boldsymbol{T}_3'\boldsymbol{T}_3)^{-1}\boldsymbol{T}_3'\}^{-1} \tag{4.2}$$

as $\boldsymbol{D}\boldsymbol{D}'$, where

$$\boldsymbol{D} = \begin{bmatrix} d_{s+1,\,s+1} & 0 & \cdots & 0 \\ d_{s+2,\,s+1} & d_{s+2,\,s+2} & \cdots & 0 \\ \cdots & & & \\ d_{p,\,s+1} & d_{p,\,s+2} & \cdots & d_{pp} \end{bmatrix} \tag{4.3}$$

is a lower triangular matrix. Hence from (4.1)

$$\begin{aligned} |\boldsymbol{I}-\boldsymbol{T}\boldsymbol{T}'| &= |\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1-\boldsymbol{T}_3'\boldsymbol{T}_3||\boldsymbol{I}-\boldsymbol{T}_4'(\boldsymbol{D}\boldsymbol{D}')^{-1}\boldsymbol{T}_4| \\ &= |\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1||\boldsymbol{I}-\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-1}\boldsymbol{T}_3'||\boldsymbol{I}-\boldsymbol{T}_4'(\boldsymbol{D}\boldsymbol{D}')^{-1}\boldsymbol{T}_4| \\ &= |\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1||\boldsymbol{I}-\boldsymbol{M}\boldsymbol{M}'||\boldsymbol{I}-\boldsymbol{E}\boldsymbol{E}'| \end{aligned} \tag{4.4}$$

where

(4.5) $$M' = (I - T_1' T_1)^{-\frac{1}{2}} T_3'$$

and

(4.6) $$E = \begin{bmatrix} e_{s+1,\, s+1} & 0 & \cdots & 0 \\ \cdots & & & \\ \cdots & & & \\ e_{p,\, s+1} & e_{p,\, s+2} & \cdots & e_{pp} \end{bmatrix} = D^{-1} T_4.$$

From (4.6)

(4.7) $$e_{ii} = t_{ii}/d_{ii}; \qquad (i = s+1, s+2, \ldots, p)$$

and from (4.2)

$$\begin{aligned} |DD'| = \prod_{i=s+1}^{p} d_{ii}^2 &= |I + T_3(I - T_1'T_1 - T_3'T_3)^{-1} T_3'|^{-1} \\ &= |I - T_1'T_1 - T_3'T_3||I - T_1'T_1|^{-1} \\ &= |I - T_3(I - T_1'T_1)^{-1} T_3'| \end{aligned}$$

But from (3.1), this is Λ' and this is also

(4.8) $$= |I - MM'|,$$

from (4.5).

Observe that, from (4.6) and (4.8)

(4.9) $$|EE'|| = \prod_{s+1}^{p} e_{ii}^2 = \frac{|T_4 T_4'|}{|DD'|} = \frac{\Lambda^*}{\Lambda'} = \Lambda''.$$

From (2.12), (2.13) and (4.4), the distribution of T_1, T_3 and T_4 is

(4.10) $$\begin{aligned} &\text{const}\; h(T_1, \rho) \prod_{i=1}^{s} t_{ii}^{n-q-i} \prod_{i=s+1}^{p} t_{ii}^{n-q-i} |I - T_1 T_1'|^{(q-p-1)/2} \\ &\qquad \cdot |I - MM'|^{(q-p-1)/2} |I - EE'|^{(q-p-1)/2} dT_1 dT_3 dT_4. \end{aligned}$$

Transform from T_4 to E by (4.6) and from T_3 to M by (4.5). The Jacobians of these transformations are respectively, $\prod_{i=s+1}^{p} d_{ii}^{i-s}$ and $|I - T_1 T_1'|^{(p-s)/2}$ (see Deemer and Olkin, 1951). Thus, using (4.7) and (4.8), the distribution of T_1, M and E is

$$\begin{aligned} &\text{const}\; h(T_1, \rho) \prod_{i=1}^{s} t_{ii}^{n-q-i} |I - T_1 T_1'|^{(q-s-1)/2} |I - M'M|^{(n-p-s-1)/2} \\ &\qquad\qquad \cdot \prod_{i=s+1}^{p} e_{ii}^{n-q-i} |I - EE'|^{(q-p-1)/2} dT_1 dM dE. \end{aligned}$$

Now use lemma 3 to obtain the distribution of $M'M$, from that of M above. This results in proving that T_1, $M'M$ and E are independently distributed, the distribution of $M'M$ being

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$$\text{const } |\boldsymbol{M}'\boldsymbol{M}|^{((p-s)-s-1)/2}|\boldsymbol{I}-\boldsymbol{M}'\boldsymbol{M}|^{((n-p)-s-1)/2}d(\boldsymbol{M}'\boldsymbol{M}) \tag{4.11}$$

and of $\boldsymbol{E}$ being,

$$\text{const } \prod_{j=1}^{p-s} e_{j+s,\, j+s}^{(n-2s)-(q-s)-j}|\boldsymbol{I}-\boldsymbol{E}\boldsymbol{E}'|^{((q-s)-(p-s)-1)/2}d\boldsymbol{E}. \tag{4.12}$$

Now apply lemma 2 to (4.11) and lemma 1 to (4.12). This shows that

$$|\boldsymbol{I}-\boldsymbol{M}'\boldsymbol{M}| = \Lambda'$$

has the $\Lambda(n-s,\ s,\ p-s)$ distribution, and

$$|\boldsymbol{E}\boldsymbol{E}'| = \Lambda'' \text{ has the } \Lambda(n-2s,\ p-s,\ q-s)$$

distribution and that both are independent as E and M are independent.

7. Distributions of Λ^{V} and Λ^{VI}

From (2.13),

$$\begin{aligned} |\boldsymbol{I}-\boldsymbol{T}\boldsymbol{T}'| &= \left|\begin{array}{c|c} \boldsymbol{I}-\boldsymbol{T}_1\boldsymbol{T}_1' & -\boldsymbol{T}_1\boldsymbol{T}_3' \\ \hline -\boldsymbol{T}_1\boldsymbol{T}_3' & \boldsymbol{I}-\boldsymbol{T}_3\boldsymbol{T}_3'-\boldsymbol{T}_4\boldsymbol{T}_4' \end{array}\right| \\ &= |\boldsymbol{I}-\boldsymbol{T}_1\boldsymbol{T}_1'||\boldsymbol{I}-\boldsymbol{T}_3\boldsymbol{T}_3'-\boldsymbol{T}_4\boldsymbol{T}_4'-\boldsymbol{T}_3\boldsymbol{T}_1'(\boldsymbol{I}-\boldsymbol{T}_1\boldsymbol{T}_1')^{-1}\boldsymbol{T}_1\boldsymbol{T}_3'| \\ &= |\boldsymbol{I}-\boldsymbol{T}_1\boldsymbol{T}_1'|\boldsymbol{I}-\boldsymbol{G}\boldsymbol{G}'| \end{aligned} \tag{5.1}$$

where

$$\boldsymbol{G} = \begin{bmatrix} g_{s+1,\, s+1} & 0 & \cdots & 0 \\ \cdots & & & \\ \cdots & & & \\ g_{p,\, s+1} & g_{p,\, s+2} & \cdots & g_{pp} \end{bmatrix} \tag{5.2}$$

is a lower triangular matrix such that

$$\begin{aligned} \boldsymbol{G}\boldsymbol{G}' &= \boldsymbol{T}_3\boldsymbol{T}_3'+\boldsymbol{T}_4\boldsymbol{T}_4'+\boldsymbol{T}_3\boldsymbol{T}_1'(\boldsymbol{I}-\boldsymbol{T}_1\boldsymbol{T}_1')^{-1}\boldsymbol{T}_1\boldsymbol{T}_3' \\ &= \boldsymbol{T}_4\boldsymbol{T}_4'+\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-1}\boldsymbol{T}_3', \end{aligned} \tag{5.3}$$

by using result 2 in sect. 2. Thus by (3.2),

$$|\boldsymbol{G}\boldsymbol{G}'| = \Lambda^{\mathrm{V}} \tag{5.4}$$

Also note that,

$$\begin{aligned} |\boldsymbol{T}_4\boldsymbol{T}_4'| &= |\boldsymbol{G}\boldsymbol{G}'-\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-1}\boldsymbol{T}_3'| \\ &= |\boldsymbol{G}\boldsymbol{G}'||\boldsymbol{I}-(\boldsymbol{G}\boldsymbol{G}'|^{-1}\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-1}\boldsymbol{T}_3'| \\ &= |\boldsymbol{G}\boldsymbol{G}'||\boldsymbol{I}-\boldsymbol{H}'\boldsymbol{H}| \end{aligned} \tag{5.5}$$

where

$$\boldsymbol{H} = (\boldsymbol{G}\boldsymbol{G}')^{-\frac{1}{2}}\boldsymbol{T}_3(\boldsymbol{I}-\boldsymbol{T}_1'\boldsymbol{T}_1)^{-\frac{1}{2}}. \tag{5.6}$$

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Using (5.1) in (2.12) the distribution of T_1, T_3 and T_4 is

$$\text{const } h(T, \rho) \prod_{i=1}^{s} t_{ii}^{n-q-i} \prod_{i=s+1}^{p} t_{ii}^{n-q-i} |I - T_1 T_1'|^{(q-p-1)/2} \cdot |I - GG'|^{(q-p-1)/2} dT_1 dT_3 dT_4. \tag{5.7}$$

Transform from T_4 to G by (5.3), the Jacobian of the transformation, denoted by $J(T_1 : G)$ is

$$J(T_4 : T_4 T_4') J(T_4 T_4' : GG') J(GG' : G)$$

and equals to

$$\frac{J(GG' : G)}{J(T_4 T_4' : T_4)}$$

as $J(T_4 T_4' : GG')$ is unity, by (5.3). Using Deemer and Olkin (1951), the required Jacobian is, therefore

$$\prod_{i=s+1}^{p} \left(\frac{g_{ii}}{t_{ii}}\right)^{p+1-i}. \tag{5.8}$$

Now use this Jacobian and (5.5) to obtain the distribution of T_1, T_3 and G in the form

$$\text{const } h(T_1, \rho) \prod_{i=1}^{s} t_{ii}^{n-q-i} |I - T_1 T_1'|^{(q-p-1)/2} \prod_{i=s+1}^{p} g_{ii}^{p+1-i} \cdot |I - GG'|^{(q-p-1)/2} |I - H'H|^{(n-q-p-1)/2} \cdot |GG'|^{(n-q-p-1)/2} dT_1 dT_3 dG. \tag{5.9}$$

Now transform from T_3 to H by (5.6). The Jacobian is (Deemer and Olkin, 1951)

$$|GG'|^{s/2} |I - T_1' T_1|^{(p-s)/2}$$

The distribution of T_1, H and G is therefore,

$$\text{const } h(T_1, \rho) \prod_{i=1}^{s} t_{ii}^{n-q-i} |I - T_1 T_1'|^{(q-s-1)/2} \prod_{i=s+1}^{p} g_{ii}^{n-q+s-i} \cdot |I - GG'|^{(q-p-1)/2} |I - H'H|^{(n-q-p-1)/2} dT_1 dH dG. \tag{5.10}$$

Finally apply lemma 3 to obtain the distribution of $H'H$ from that of H. This shows that T_1, G and $H'H$ are independent; the distribution of G is

$$\text{const} \prod_{j=1}^{p-s} g_{j+s,\, j+s}^{(n-s)-(q-s)-j} |I - GG'|^{((q-s)-(p-s)-1)/2} dG, \tag{5.11}$$

and that of $H'H$ is

$$\text{const } |H'H|^{((p-s)-s-1)/2} |I - H'H|^{[[(n-q)-(p-s)]-s-1]/2} d(H'H). \tag{5.12}$$

Apply Lemma 1 to the distribution of $\boldsymbol{G}$ and lemma 2 to that of $\boldsymbol{H}'\boldsymbol{H}$ to observe that $|\boldsymbol{G}\boldsymbol{G}'| = \Lambda^{\mathrm{V}}$ has the $\Lambda(n-s, q-x, p-s)$ distribution and by (5.5) $|\boldsymbol{I}-\boldsymbol{H}'\boldsymbol{H}| = |\boldsymbol{T}_4\boldsymbol{T}_4'|/|\boldsymbol{G}\boldsymbol{G}'| = \Lambda^{\mathrm{VI}}$ has the $\Lambda(n-q, s, p-s)$ distribution.

Further they are independent.

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Southern Methodist University
Dallas, Texas, U.S.A.