Canad. Math. Bull. Vol. **14** (4), 1971

OSCILLATION ON FINITE OR INFINITE INTERVALS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS([1])

BY
D. WILLETT

1. **Introduction.** Recently, Ronveaux [11] has shown how to use a combination of a Riccati transformation and a homographic transformation to estimate both from below and above the distance between a zero and the succeeding or preceding extremum (zero of y') of solutions of

$$y''+p(t)y = 0. \tag{1.1}$$

In this paper, we show how such transformations can be used to derive an equation from which the distance between successive zeros of a solution y of (1.1) can be estimated directly.

More precisely, we consider the equation

$$[r(t)y']'+q(t)y = 0, \tag{1.2}$$

with $r \in C^1$, $q \in C$, $r > 0$, $q \geq 0$. Suppose that $y(t)$ is a positive solution of (1.2) on (a, b) with $y(a)=y(b)=0$. We derive sequences b_n and c_n, which are functions of r, q, and a, such that

$$b_n \downarrow b \quad \text{and} \quad c_n \uparrow b, \quad \text{as } n \uparrow \infty.$$

The numbers b_n and c_n are defined in terms of the solutions of transcendental equations. For example, the number c_0 relative to equation (1.1) with $a=0$ satisfies

$$\int_0^{c_0} t(c_0-t)p_+(t)\,dt = c_0, \tag{1.3}$$

where $p_+(t)=\max\{p(t), 0\}$. Condition (1.3) is originally due to Hartman and Wintner [6].

The problem of determining b goes back to at least Lyapunov [8] and de la Vallée Poussin [12]. Considerable work (cf. the bibliography and the references therein) has been carried out over the years.

In §4, we obtain similar *convergent* sequences for the problem involving the distance between a zero and adjacent extremes. This problem is one order less difficult than the problem of distance between zeros, the meaning of which will become clear in §4.

Received by the editors September 16, 1970 and, in revised form, November 24, 1970.

([1]) Research supported, in part, by National Research Council of Canada Grant A3053 to the Univ. of Alberta.

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press

Finally, we derive some new necessary and sufficient criteria for disconjugacy of (1.2) on a finite or infinite interval $[a, b)$. Actually, in this regard, Theorems 1 through 3 are further results of the general method derived in Willett [14]. For a general survey of the oscillation results for (1.2), see (Willett [13], [14]).

2. **Disconjugacy and the distance between zeros.** Throughout the remainder of this paper, we consider equation (1.2) under the assumptions $q \in C[a, b)$, $r \in C^1[a, b)$, $r > 0$, and with

$$R(t) = \int_a^t r^{-1}(s)\,ds, \qquad M(t) = \int_a^t R^2(s)q(s)\,ds,$$

$$P(t) = \int_a^t R(s)[R(t)-R(s)]q(s)\,ds,$$

$$P_0(t) = M(t)P(t)R^{-1}(t) = M(t)\int_a^t M(s)r^{-1}(s)R^{-2}(s)\,ds,$$

$$P_1(t) = \int_a^t P(s)[P(s)-R(s)]q(s)\,ds,$$

$$P_n(t) = \int_a^t R^2(s)M^{-2}(s)P_{n-1}(s)\left[P_{n-1}(s)+2\sum_{k=0}^{n-2} P_k(s)\right]q(s)\,ds, \quad n = 2, \ldots,$$

$$Q_0(t) = P_0(t)-\int_a^t R(s)P(s)q(s)\,ds = \int_a^t M^2(s)r^{-1}(s)R^{-2}(s)\,ds,$$

$$Q_1(t) = \int_a^t P(s)\left[P(s)-R(s)M^{-1}(t)\int_a^t R(\tau)P(\tau)q(\tau)\,d\tau\right]q(s)\,ds,$$

$$Q_n(t) = \int_a^t R^2(s)M^{-2}(s)Q_{n-1}(s)\left[Q_{n-1}(s)+2\sum_{k=0}^{n-2} Q_k(s)\right]q(s)\,ds, \quad n = 2, \ldots.$$

From the definition of $M(t)$, we note that $M(t)=0$ for $t>a$ can occur only if $q(t)=0$. Hence, we define $q(t)M^{-1}(t)=0$ when $t>a$ and $M(t)=0$, and assume q is not identically zero on $[a, b)$. We also assume that all integrals are improper integrals at a. Since

$$\lim_{t\to a+}\frac{M(t)}{R^3(t)} = \lim_{t\to a+}\frac{R^2(t)q(t)r(t)}{3R^2(t)} = \frac{q(a)r(a)}{3}, \tag{2.0}$$

it is a simple matter to show that all the above functions are well-defined continuous functions on $[a, b)$.

THEOREM 1. *Assume $q \ge 0$ and $a < b \le \infty$. Equation (1.2) is disconjugate on $[a, b)$, if and only if $\sum_{k=0}^{\infty} Q_k(t)$ converges for $a \le t < b$ and*

$$\sum_{k=0}^{\infty} Q_k(t) < M(t) \text{ or } M(t) = 0, \quad a < t < b. \tag{2.1}$$

COROLLARY 1. *Assume $q \ge 0$ and $a < b < \infty$. Equation (1.2) has a positive solution y on (a, b) such that*

$$y(a) = 0 = y(b),$$

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press

if and only if

$$\sum_{k=0}^{\infty} Q_k(b) = M(b). \tag{2.2}$$

Thus, if b_n is such that $M(b_n)>0$ and

$$\sum_{k=0}^{n} Q_k(b_n) \geq M(b_n), \tag{2.3}$$

then

$$b_n > b;$$

and if equality occurs in (2.3) for $n \geq N$, then

$$b_n \downarrow b, \quad \text{as } n \uparrow \infty.$$

THEOREM 2. *Assume $q \geq 0$ and $a<b\leq\infty$. Equation (1.2) is disconjugate on $[a, b)$, if and only if $\sum_{k=0}^{\infty} P_k(t)$ converges for $a\leq t<b$ and*

$$\sum_{k=0}^{\infty} P_k(t) < M(t) \text{ or } M(t) = 0, \quad a < t < b. \tag{2.4}$$

COROLLARY 2. *Assume $q\geq 0$ and $a<b<\infty$. Equation (1.2) has a positive solution y on (a, b) such that*

$$y(a) = 0 = y(b),$$

if and only if

$$\sum_{k=0}^{\infty} P_k(b) = M(b). \tag{2.5}$$

Thus, if c_n, $n=0, 1, \ldots$, is such that

$$\sum_{k=0}^{n} P_k(c_n) \leq M(c_n), \tag{2.6}$$

then

$$c_n < b$$

and (1.2) is disconjugate on $[a, c_n]$. Furthermore, if equality occurs in (2.6) for $n\geq N$, then

$$c_n \uparrow b, \quad \text{as } n \uparrow \infty.$$

COROLLARY 3. *Assume $a<b\leq\infty$ and $f \in C[a, b)$. Let $q=f_+=f \vee 0$ and assume M, P, and P_n are defined as above. If (2.6) holds, then the equation*

$$(ry')'+fy = 0 \tag{2.7}$$

is disconjugate on $[a, c_n]$.

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press

Proof of Theorem 1. Assume that (1.2) is disconjugate on $[a, b)$. Thus, any solution of (1.2) satisfying initial conditions $y(a)=0$, $y'(a)>0$ is positive on (a, b). For one such solution, let

$$z = [1-r(t)R(t)y'(t)y^{-1}(t)]R(t), \quad a < t < b. \tag{2.8}$$

L'Hôpital's Rule implies

$$\lim_{t\to a+} \frac{R(t)}{y(t)} = \frac{1}{r(a)y'(a)};$$

hence,

$$\lim_{t\to a+} \frac{z(t)}{R(t)} = 0. \tag{2.9}$$

Furthermore, (1.2) and (2.8) imply that z is a $C^1(a, b)$-solution of the Riccati equation

$$z' = q(t)R^2(t)+r^{-1}(t)R^{-2}(t)z^2, \quad a < t < b. \tag{2.10}$$

At this point, we assume without loss of generality that $q(t)$ is not identically zero in any right neighborhood of a, for if $q(t)=0$ for $a\le t\le \bar{a}$ with $\bar{a}$ maximal, then $z(t)=0$ for $a\le t\le \bar{a}$ and we replace a by $\bar{a}$ in the following analysis. However, we do not replace a by $\bar{a}$ in the definition of R. Thus, (2.10) and $z(a)=0$ imply

$$z(t) > 0 \text{ for } a < t < b. \tag{2.11}$$

Next, let

$$w = [1-M(t)z^{-1}(t)]M(t), \quad a < t < b. \tag{2.12}$$

Clearly

$$0 < w(t) < M(t), \quad a < t < b, \tag{2.13}$$

and so $w(a+)=0$. Furthermore, (2.10) implies

$$w' = \frac{M^2(t)}{r(t)R^2(t)}+\frac{R^2(t)}{M^2(t)}q(t)w^2, \quad a < t < b. \tag{2.14}$$

Hence,

$$w(t) = Q_0(t)+\int_a^t R^2(s)M^{-2}(s)q(s)w^2(s)\,ds, \quad a < t < b. \tag{2.15}$$

A procedure for solving integral equations of the type (2.15) is presented in Willett [14]. It goes as follows. Let

$$\begin{cases} w_0(t) = Q_0(t), \\ w_n(t) = Q_0(t)+\int_a^t R^2(s)M^{-2}(s)q(s)w_{n-1}^2(s)\,ds, n = 1,\dots. \end{cases} \tag{2.16}$$

By induction, it is not difficult to show that

$$w_n - w_{n-1} = Q_n;$$

hence, we have

$$w_n(t) = \sum_{k=0}^{n} Q_k(t) \le w(t) < M(t), \quad a < t < b. \tag{2.17}$$

Since (2.16) implies $w_n \ge w_{n-1}$, it follows that $\sum_{k=0}^{\infty} Q_k(t)$ converges for $a<t<b$. Finally, (2.17) implies (2.1).

Now, suppose that (2.1) holds, and let

$$w(t) = \sum_{k=0}^{\infty} Q_k(t), \quad a \le t < b. \tag{2.18}$$

We are still assuming without loss of generality that q is not identical to zero in any neighborhood of a, that is $M(t) > 0$ for $t > a$. Let

$$w_n = \sum_{k=0}^{n} Q_k$$

so that (2.16) holds. Since $w_n(t) < M(t)$ for $a<t<b$ and all n, the Lebesgue dominated convergence theorem implies that $w(t)$ is a solution of (2.15). Since $w(t) = \sum_{n=0}^{\infty} Q_n(t) < M(t)$ for $a<t<b$ by assumption, the function

$$f(t) = M^2(t)/[M(t) - w(t)]R^2(t)r(t) \tag{2.19}$$

is positive on (a, b). Fix τ, $a<\tau<b$, and define

$$y(t) = R(t) \exp\left(-\int_{\tau}^{t} f(s)\, ds\right), \quad a < t < b. \tag{2.20}$$

(Although it is not needed, one can actually show that $f(t) \to 0$, as $t \to a+$; hence, $\tau = a$ is also correct.) The fact that w is a solution of (2.14) implies that y is a solution of (1.2). Since y is positive on (a, b), Sturm theory implies that (1.2) is disconjugate on $[a, b)$. We note that in case $q(t)=0$ for $a \le t \le \bar{a}$ and $\bar{a}$ is maximal, then (2.20) still gives a positive solution of (1.2) provided one defines $f(t)=0$ for $a \le t \le \bar{a}$.

Proof of Theorem 2. The main difference in the proofs of Theorems 1 and 2 is in the choice of initial function $w_0(t)$. Consider the general situation. Let

$$\begin{cases} w_0(t) = M(t) \displaystyle\int_a^t \frac{M(s)}{r(s)R^2(s)}\, ds = P_0(t), \\ w_n(t) = Q_0(t) + \displaystyle\int_a^t R^2(s) M^{-2}(s) q(s) w_{n-1}^2(s)\, ds, \quad n = 1, \ldots. \end{cases} \tag{2.21}$$

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press

Here, we again assume without loss of generality that $M(t)>0$ for $t>a$. Then,

$$w_0'(t) = \frac{M^2(t)}{r(t)R^2(t)}+\frac{R^2(t)}{M(t)}q(t)w_0(t), \quad a < t < b,$$

and $\exists$ a maximum b_0, $a<b_0\le b$, such that

$$w_0(t) < M(t), \quad a < t < b_0.$$

Since

$$w_1'(t) = \frac{M^2(t)}{r(t)R^2(t)} + \frac{R^2(t)}{M^2(t)}q(t)w_0^2(t), \quad a < t < b,$$

and $w_0(a)=0=w_1(a)$, it is clear that

$$w_1(t) \le w_0(t), \quad a \le t \le b_0.$$

Thus, $\exists$ maximum b_1, $b_0\le b_1\le b$, such that

$$w_1(t) < M(t), \quad a < t < b_1.$$

In general, $\exists$ a sequence (w_n, b_n) such that

$$0 \le w_n(t) \le w_{n-1}(t), \quad a \le t < b_{n-1},$$

$$w_n(t) < M(t), \quad a < t < b_n,$$

$$b_{n-1} \le b_n \le b.$$

Hence, $\exists w_*(t)\ge 0$ and $b_*\le b$ such that

$$w_n \downarrow w_* \quad \text{and} \quad b_n \uparrow b_*, \quad \text{as } n\uparrow\infty.$$

Suppose now that (2.4) holds. If $b_n<b$ for all $n=0, 1, \ldots$, then $w_*(t)<M(t)$ for $a<t<b_*$ and

$$\lim_{t\to b_*}[M(t)-w_*(t)] = 0. \tag{2.22}$$

But (2.21) implies

$$w_*(t) = \lim_{n\to\infty}\sum_{k=0}^{n} P_k(t),$$

so that (2.4) implies $M(b_*)-w_*(b_*)>0$ if $b_*<b$. Thus, $b_*=b$.

Letting $n\to\infty$ in (2.21), we conclude by the Monotone convergence theorem that w_* is a solution of (2.15). Since

$$w_*(t) \le w_0(t) = M(t)\int_a^t r^{-1}(s)R^{-2}(s)M(s)\,ds, \quad a < t < b_0,$$

it is also true that

$$\lim_{t\to a+} w_*(t)/M(t) = 0.$$

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press

As in the proof of Theorem 1, it now follows that

$$y_*(t) = R(t) \exp \left[-\int_a^t \frac{M^2(s)\, ds}{(M(s)-w_*(s))R^2(s)r(s)} \right]$$

is a positive solution of (1.2) on (a, b). Hence, (1.2) is disconjugate on $[a, b)$.

Now, suppose that (1.2) is disconjugate on $[a, b)$. Then, as in the first part of the proof of Theorem 1, $\exists$ a solution $w \in C^1(a, b)$ of (2.14) satisfying (2.13). On the other hand, the sequence w_n defined in (2.21) converges from above to the solution $w_* \in C^1(a, b^*)$. Hence, w and w_* are solutions of the same differential equation on (a, b^*) and satisfy the same initial condition, and

$$0 < w(t) \le w_*(t) < M(t), \quad a < t < b^*. \tag{2.23}$$

We will prove that

$$w(t) = w_*(t), \quad a < t < b^*,$$

and hence,

$$0 = \lim_{t \to b^*_-} [M(t) - w_*(t)] = \lim_{t \to b^*_-} [M(t) - w(t)],$$

which implies $b^* = b$ by (2.13). Thus, the conclusion of the theorem would follow from (2.23).

Let

$$\Delta(t) = w_*(t) - w(t), \quad a \le t < b^*.$$

Since w and w_* are both solutions of (2.15) on $[a, b^*)$,

$$\begin{aligned} 0 \le \Delta(t) &\le \int_a^t M'(s)M^{-2}(s)[w_*(s)+w(s)]\, \Delta(s)\, ds \\ &\le 2\int_a^t M'(s)M^{-2}(s)w_*(s)\, \Delta(s)\, ds, \quad a < t < b^*. \end{aligned}$$

This is a form of the well-known Gronwall inequality. It implies $\Delta(t) \le 0$, $a < t < b^*$, provided $M'M^{-2}w_*$ is integrable on $[a, t]$ for all t such that $a < t < b^*$. But in the neighborhood of a, which is the only place there is a problem, $w_* \le w_0$. Hence,

$$\frac{M'(s)}{M^2(s)} w_*(s) \le \frac{M'(s)}{M(s)} \int_a^s \frac{M(\tau)\, d\tau}{r(\tau)R^2(\tau)} \le \frac{M'(s)}{M^{2/3}(s)} \int_a^s \left[\frac{M(\tau)}{R^3(\tau)}\right]^{2/3} \frac{d\tau}{r(\tau)},$$

which is integrable by (2.0).

Corollaries 1 and 2 are direct consequences of Theorems 1 and 2, respectively. Corollary 3 follows from Sturm theory, since (2.7) is disconjugate on $[a, b]$ if

$$(r(t)y')' + f_+ y = 0$$

is disconjugate on $[a, b)$.

Theorem 2 and its corollaries depend essentially upon the sequence w_n defined in (2.21). We note here that there is a feasible alternate way of choosing the sequence w_n. Let

$$(2.24)\qquad \begin{cases} v_0(t) = w_0(t), \\ v_n(t) = Q_0(t)+\int_a^t R^2(s)M^{-2}(s)q(s)v_{n-1}(s)v_n(s)\,ds, \quad n = 1,\ldots, . \end{cases}$$

Since the integral equation involving v_n is linear in v_n, it can be solved explicitly. So (2.24) determines a unique function v_n for each value of n. It can be shown that the sequence v_n converges monotonically to a solution v_* of (2.15), and (1.2) is disconjugate on $[a, b)$, if and only if

$$v_*(t) < M(t), \quad a < t < b.$$

Thus, $v_n(c_n) \le M(c_n)$ implies $c_n < b$ with $c_n \uparrow b$, as $n \uparrow \infty$, in the case $v_n(c_n) = M(c_n)$ for all $n \ge N$.

The advantage of using w_n instead of v_n is computational. Solution of (2.24) for v_n will show that the formula defining v_n contains exponential functions. (2.21) shows that this is not the case for w_n. On the other hand, the sequence v_n will in general converge faster than the sequence w_n.

3. **Application.** As a special case of the results obtained in the previous section, consider

$$(3.1)\qquad y''+p(t)y = 0, \quad a \le t \le b, \quad p(t) \ge 0,$$

and let

$$H_0(t) = \int_a^t (s-a)(t-s)p(s)\,ds/(b-a),$$

$$H_1(t) = \int_a^t (s-a)p(s)H_0(s)\,ds \Big/ \int_a^t (s-a)^2p(s)\,ds,$$

$$H_2(t) = \int_a^t p(s)H_0^2(s)\,ds \Big/ \int_a^t (s-a)^2p(s)\,ds.$$

COROLLARY 4. *Equation (3.1) is disconjugate on $[a, b]$ if either of the following conditions hold*:

(i) $H_0(b) \le 1$

(ii) $H_0(b)-(b-a)H_1(b)+(b-a)^2H_2(b) \le 1$;

and is conjugate on $[a, b]$ if either of the following conditions hold:

(iii) $H_0(b)-(b-a)H_1(b) \ge 1$

(iv) $H_0(b)-(b-a)H_1(b)+(b-a)^2H_2(b)-(b-a)^2H_1^2(b) \ge 1$.

The conditions (i)–(iv) correspond respectively to (2.6) with $n=0, 1$ and (2.3) with $n=0, 1$. For the trivial equation $y''+y=0$, conditions (ii) and (iv) imply

$$2.604 < \pi < 3.366.$$

A nontrivial equation to which Corollary 4 can be applied is

$$y''+\lambda(\sin t)y = 0, \quad 0 \le t \le \pi. \tag{3.2}$$

Equation (3.2) is disconjugate on $[0, \pi]$ if

$$\lambda \le 1.056,$$

and is conjugate on $[0, \pi]$ if

$$\lambda \ge 1.242.$$

For the equation

$$y''+g(t)y'+f(t)y = 0, \tag{3.3}$$

which is equivalent to (1.1) with

$$r(t) = \exp\left(\int_a^t g(s)\,ds\right), \quad q(t) = f(t)\exp\left(\int_a^t g(s)\right) ds,$$

the condition (2.6) with $n=0$ and $f_+(t)=\max(0, f(t))$ is

$$\int_a^b \left(\int_a^s r^{-1}(\tau)\,d\tau\right)\left(\int_s^b r^{-1}(\tau)\,d\tau\right) r(s)f_+(s)\,ds \le \int_a^b r(\tau)\,d\tau. \tag{3.4}$$

This result includes the result of Hartman and Wintner [7], which states that (3.3) is disconjugate on $[a, b]$ if

$$\int_a^b (s-a)(b-s)f_+(s)\,ds+\max\left\{\int_a^b s\,|g(s)|\,ds, \int_a^b (b-s)\,|g(s)|\,ds\right\} \le b-a.$$

4. **Disconjugacy and the distance between zeros and focal points.** The basis for the development in §2 involved the relationship between equation (2.14) and equation (1.2). Information about (1.2) was obtained by analyzing (2.14). A similar development with respect to the problem of locating the first zero of $y'(t)$ is possible using equation (2.10) in place of (2.14). We list the pertinent results in this section and outline the proofs.

Let

$$M_0(t) = M(t) = \int_a^t R^2(s)q(s)\,ds,$$

$$M_1(t) = \int_a^t r^{-1}(s)R^{-2}(s)M^2(s)\,ds,$$

$$M_n(t) = \int_a^t r^{-1}(s)R^{-2}(s)M_{n-1}(s)\left[M_{n-1}(s)+2\sum_{k=0}^{n-2} M_k(s)\right] ds, \quad n = 2,\ldots,$$

$$N_0(t) = R(t)\int_a^t R(s)q(s)\,ds,$$

$$N_1(t) = \int_a^t r^{-1}(s)R^{-2}(s)N_0(s)[N_0(s)-R(s)]\,ds,$$

$$N_n(t) = \int_a^t r^{-1}(s)R^{-2}(s)N_{n-1}(s)\left[N_{n-1}(s)+2\sum_{k=0}^{n-2} N_k(s)\right] ds, \quad n = 2,\ldots.$$

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press

THEOREM 3. *Assume $a<b\leq\infty$ and $M\geq 0$. Equation (1.2) is disconjugate on $[a, b)$, if and only if $\sum_{k=0}^{\infty} M_k(t)$ converges for $a\leq t<b$.*

THEOREM 4. *Assume $a<b\leq\infty$, $M\geq 0$, and (1.2) is disconjugate on $[a, b)$. Let y be any nontrivial solution of (1.2) such that $y(a)=0$. Then, $\exists c\in(a, b)$ such that $y'(c)=0$, if and only if*

$$\sum_{k=0}^{\infty} M_k(c) = R(c). \tag{4.1}$$

COROLLARY 4. *Assume that y is a solution of (1.2) such that $y(a)=0$, $y'(c)=0$, and $y'(t)>0$ for $a<t<c$. If*

$$\sum_{k=0}^{n} M_k(b_n) \geq R(b_n), \tag{4.2}$$

then

$$b_n > c.$$

Furthermore, if equality occurs in (4.2) for $n\geq N$, then

$$b_n \downarrow c, \quad \text{as } n\uparrow\infty.$$

THEOREM 5. *Assume $a<b\leq\infty$, $M\geq 0$, and (1.2) is disconjugate on $[a, b)$. Let y be any nontrivial solution of (1.2) such that $y(a)=0$. Then, $\exists c\in(a, b)$ such that $y'(c)=0$ and $y'(t)\neq 0$ for $a<t<c$, if and only if $\sum_{k=0}^{\infty} N_k(t)$ converges for $a<t<c$ and*

$$\sum_{k=0}^{\infty} N_k(c) = R(c). \tag{4.3}$$

COROLLARY 5. *Assume that y is a solution of (1.2) such that $y(a)=0$, $y'(c)=0$, and $y'(t)>0$ for $a<t<c$. If*

$$\sum_{k=0}^{n} N_k(c_n) \leq R(c_n), \tag{4.4}$$

then,

$$c > c_n.$$

Furthermore, if equality occurs in (4.4) for $n\geq N$, then

$$c_n \uparrow c, \quad \text{as } n\uparrow\infty.$$

When $n=0$ in Corollaries 4 and 5, the bounds

$$c_0 < c < b_0,$$

where

$$\int_a^{b_0} R^2(s)q(s)\,ds \geq R(b_0) \quad \text{and} \quad \int_a^{c_0} R(s)q(s)\,ds \leq 1$$

are obtained. These estimates have previously been obtained by Ronveaux [11].

Proof of Theorem 3 (outline). The proof of Theorem 3 is similar to the proof of Theorem 1. The proof depends primarily upon the fact that the equation

$$z' = qR^2 + z^2/rR^2 \tag{4.5}$$

has a unique solution in $C(a, b)$ satisfying

$$\lim_{t\to a^+} \frac{z(t)}{R(t)} = 0, \tag{4.6}$$

if and only if, equation (1.2) is disconjugate on $[a, b)$. The unique solution of (4.5)–(4.6) can be obtained by iterating in either the manner described in the proof of Theorem 1 or in the proof of Theorem 2. In the former case,

$$z(t) = \sum_{k=0}^{\infty} M_k(t), \tag{4.7}$$

which holds for $a<t<b$. In the latter case,

$$z(t) = \sum_{k=0}^{\infty} N_k(t), \tag{4.8}$$

which holds for all t such that $z(t)<R(t)$.

We will now prove that (4.5)–(4.6) has at most one solution. Suppose it has two solutions z_1 and z_2 on an interval $[a, b)$. Then

$$\Delta(t) = |z_1(t)-z_2(t)| \le \int_a^t \frac{|z_1(s)+z_2(s)|}{r(s)R^2(s)} \Delta(s)\, ds.$$

Once again, the Gronwall inequality implies $\Delta(t)=0$ if

$$f(s) = r^{-1}(s)R^{-2}(s)\, |z_1(s)+z_2(s)|$$

is integrable on $[a, t]$ for all $a<t<b$. Clearly, we need to show only that f is integrable in some right neighborhood of a. For any solution z of (4.5), we obtain

$$\left(\frac{z}{R}\right)' = qR + \frac{z}{rR^2}\left(\frac{z}{R}-1\right).$$

Hence, if z also satisfies (4.6), then $\exists \delta=\delta_z>0$ such that

$$[z(t)R^{-1}(t)]' \le q(t)R(t), \quad a < t < a+\delta.$$

So

$$z(t) \le R(t)\int_a^t R(s)q(s)\, ds, \quad a < t < a+\delta.$$

Hence, for the solutions z_1 and z_2 and for

$$\delta = \min(\delta_{z_1}, \delta_{z_2}),$$

we obtain

$$f(t) \le 2r^{-1}(t)R^{-1}(t)\int_a^t R(s)q(s)\,ds$$
$$\le 2r^{-1}(t)\int_a^t q(s)\,ds, \quad a \le t < a+\delta,$$

and so $f(t)$ is integrable on $[a, a+\delta]$.

Proof of Theorem 4. Since (1.2) is disconjugate, the solution y does not vanish on (a, b). Hence, the function z defined by (2.8) is $C^1(a, b)$ and satisfies (4.5)–(4.6). (2.8) also implies that $y'(c)=0$, if and only if $z(c)=R(c)$. The conclusion of the theorem follows from the fact that (4.5)–(4.6) has the unique solution (4.7) on (a, b).

Proof of Theorem 5. The proof is the same as the proof of Theorem 4 with the substitution of (4.8) for (4.7).

References

1. D. Banks, *Bounds for eigenvalues and generalized convexity*, Pacific J. Math. **13** (1963), 1031–1052.

2. A. Elbert, *On the solutions of the differential equation* $y''+q(x)y=0$, *where* $[q(x)]^{\gamma}$ *is concave, II*, Studia Sci. Math. Hung. **4** (1969), 257–266.

3. A. M. Fink, *On the zeros of* $y''+py=0$ *with linear, convex, and concave p*, J. Math. Pures Appl. **46** (1967), 1–10.

4. A. M. Fink and D. F. St. Mary, *On an inequality of Nehari*, Proc. Amer. Math. Soc. **21** (1969), 640–642.

5. A. Galbraith, *On the zeros of solutions of ordinary differential equations of the second order*, Proc. Amer. Math. Soc. **17** (1966), 333–337.

6. P. Hartman and A. Wintner, *On an oscillation criterion of Lyapunov*, Amer. J. Math. **73** (1951), 885–890.

7. ——— and ———, *On an oscillation criterion of de la Vallée Poussin*, Quart. Appl. Math. **13** (1955), 330–332.

8. A. Lyapunov, *Sur une série relative à la théorie des équations différentielles linéaires à coefficient périodiques*, C.R. Acad. Sci. Paris, **123** (1896), 1248–1252.

9. Z. Opial, *Sur les intégrales oscillantes de l'équation différentielle* $u''+f(t)u=0$, Ann. Polon. Math. **4** (1958), 308–313.

10. ———, *Sur une inégalité de C. de la Vallée Poussin dans la théorie de l'équation différentielle linéaire du second ordre*, Ann. Polon. Math. **6** (1959–60), 87–91.

11. A. Ronveaux, *Equations différentielles du second ordre: distances entre zéro et extremum des solutions*, Ann. Soc. Sci. Bruxelles Sér. I, **84** (1970), 5–20.

12. C. de la Vallée Poussin, *Sur l'équation différentielle linéaire du second ordre*, J. Math. Pures Appl. **8** (1929), 125–144.

13. D. Willett, *Classification of second order linear differential equations with respect to oscillation*, Advances in Math. **3** (1969), 594–623.

14. ———, *A necessary and sufficient condition for the oscillation of some linear second order differential equations*, Rocky Mt. Math. J. **1** (1970), 357–365.

University of Alberta,
Edmonton, Alberta

University of Utah,
Salt Lake City, Utah

https://doi.org/10.4153/CMB-1971-096-8 Published online by Cambridge University Press