MODELLING THE EFFECT OF EARTH TIDES IN THE LUNAR ORBITAL MOTION

J. Derral Mulholland and Odile Calame
Centre d'Etudes et de Recherches Géodynamiques et Astronomiques, Grasse, France

ABSTRACT. The effect of tidal friction in the lunar orbit is one of the classic examples of *ad hoc* modelling of an unknown physical phenomenon. The two basic calculational approaches in current use are developed in some detail, and numerical tests and comparisons are presented. Although attention is normally concentrated on the acceleration in orbital longitude produced as a result of terrestrial dissipation, it is shown that the variation in Earth-Moon distance due to Earth tides is far from negligible. A significant, though minor, fraction of this variation is supplied by the radial component of the acceleration, which would exist even in the absence of tidal friction. In fact, this represents almost all the difference in the predictions of the two models.

1. INTRODUCTION

Over approximately the past fifty years, the most classic example in the lunar orbit motion of purely empirical modelling of an unknown or non-understood physical phenomenon has been what is supposed to be the effect of tidal friction in the Earth. Logical physical arguments (e.g. Darwin 1898, Munk & Macdonald 1960) lead inescapably to the conclusion that the existence of tidal friction in the Earth insures the production of a secular acceleration in the lunar orbit, as a means of conserving the total angular momentum of the Earth-Moon system. Until fairly recently, this phenomenon was treated in a completely arbitrary manner; since there was no geophysical theory with which to predict the magnitude, *any* unexplainable secular acceleration in the observed lunar longitude was simply assumed to be due to the tides. The absence of a theory made it impossible to contradict this point of view.

Although the tidal friction now has a competitor for the production of a secular acceleration (e.g. Van Flandern 1981), the logical situation has not much changed today. Equivalent physical models have been proposed, but there is still no theory to support them; estimates of the magnitude of the tidal friction effect are and must still be purely empirical. Nonetheless, all serious numerical integrations of the lunar

O. Calame (ed.), High-Precision Earth Rotation and Earth-Moon Dynamics, 199–206.

orbital motion have included some provision for introducing the effect of the tides. There have been two basic ways to do this, one being purely *ad hoc*, the other with a certain amount of geophysical window-dressing. Mulholland (1980) has designated these two procedures as the arithmetic and analogue methods, respectively. We are not aware of any explicit and detailed development of these two methods in the open literature, and our own derivations appear to differ slightly from those used elsewhere. It thus seems appropriate to present them here, as well as the results of some numerical studies.

2. THE ARITHMETIC MODEL

The arithmetic method consists simply of inserting an *ad hoc* acceleration into the equation of motion in such a way as to mimic the average secular acceleration in longitude W determined from observation. Suppose, following Oesterwinter & Cohen (1972), that the acceleration in longitude is produced by a force of constant magnitude, acting always in the osculating orbital plane of the Moon and normal to the lunar radius vector. Bearing in mind that no physical justification is given for this assumption, we can generate the desired direction vector as the vector product of the angular momentum vector $\vec{h}$ with the Earth-Moon vector $\vec{r}$. For the magnitude of this acceleration, Gauss's form of the planetary equations (e.g. Brouwer & Clemence 1961, p. 301) gives

$$\tfrac{1}{2}\, n\, \mathrm{d}a/\mathrm{d}t \tag{1}$$

where a is the semi-major axis, n the anomalistic mean motion, or the first time derivative of the mean anomaly, and t the time. Using Kepler's third law, and supposing that the unmodelled acceleration in longitude is identical with the unmodelled acceleration in mean anomaly, then the cartesian acceleration required to produce an average acceleration in the lunar mean longitude L of

$$\Delta\, \mathrm{d}^2L/\mathrm{d}t^2 = W \tag{2}$$

is given by

$$\Delta\, \mathrm{d}^2\vec{r}/\mathrm{d}t^2 = \frac{1}{3}\, a\, W\, (\vec{h}/h) \times (\vec{r}/r) \tag{3}$$

This method contains no physics, and it ignores the periodic effects caused by the 5% variation in lunar distance, but it sometimes serves.

3. THE ANALOGUE MODEL

3.1 The Tide-Raising Potential

The analogue method has a lot of physics in it, or at least what appears to be physics. We begin with the potential function felt by a test point inside the orbit of the Moon, due to the gravitational

https://doi.org/10.1017/S0252921100002608 Published online by Cambridge University Press

attractions of Earth and Moon,

$$V = \frac{GE}{r_1} + \frac{GM}{r} \sum_{i=2}^{\infty} (r_1/r)^i \, P_i(\cos S) \tag{4}$$

where G is the Universal gravitational constant, E and M the masses of Earth and Moon respectively, r the geocentric distance of the Moon, r_1 the geocentric distance of the test point, P_i the Legendre polynomial of degree i , and S the geocentric angle subtended by the Moon and the test point. If the test point were a zero-mass free body, this would be the restricted problem of three bodies. Suppose, however, that the test point is fixed to the surface of Earth, which we suppose for the moment to be perfectly rigid. The first term, now constant, is superfluous and can be discarded. That which remains is the "tide-raising potential at the Earth's surface", due to the action of the Moon:

$$U = \frac{GM}{r} \sum_{i=2}^{\infty} (R/r)^i \, P_i(\cos S) \tag{5}$$

where we have replaced r_1 with the (constant) radius R of the Earth. Following the usual procedure, we will replace the angle S with its equivalent in spherical coordinates. For the sake of conceptual simlicity, the reader may suppose for the moment that the angles λ and ϕ are the terrestrial longitude and latitude, although the form is invariant with the spherical system chosen; once again the unsubscripted and subscript 1 variables refer to the Moon and the test point, respectively. It can be readily verified that the potential (5) may then be written

$$U = (GM/r) \sum_{i=2}^{\infty} (R/r)^i \sum_{j=0}^{i} (2-\delta_{0j})\,[(i-j)!/(i+j)!] \cdot P_{ij}(\sin\phi)\, P_{ij}(\sin\phi_1)\, \cos j(\lambda-\lambda_1) \tag{6}$$

where δ_{0j} is the Kronecker delta, and the P_{ij} are the associated Legendre functions.

3.2 The Lunar Earth Tide and its Gravitational Influence on an External Free Body

We now have a tide-raising potential of the form

$$U = \sum_i U_i \tag{7}$$

a spherical harmonic expansion in the coordinates of the Moon and the surface point. The Earth, however, is not perfectly rigid and will therefore distort under the action of this potential. The deformation

https://doi.org/10.1017/S0252921100002608 Published online by Cambridge University Press

of the body of the Earth will be accompanied by a corresponding deformation of its gravitational potential. At the surface, this will be

$$U_s = \sum_i k_i U_i \tag{8}$$

which is essentially a definition of the Love numbers k_i. The extra potential that will perturb any free body in the vicinity is, according to a theorem by Dirichlet,

$$U = \sum_i (R/r_2)^{i+1} k_i U_i \tag{9}$$

where r_2 is the geocentric distance of the perturbed body.

Let us now return to the question of the coordinate system in which eqn. (6) is written. The relation has the same form no matter what set of spherical coordinates is used, so we should choose the most convenient. A qualitative description of the tidal distortion is that Earth is stretched out with approximate rotational symmetry about the extensional axis, which is approximately coincident with the line joining the centers of Earth and Moon. Suppose, instead of geographic coordinates (longitude and latitude), we choose to measure λ and ϕ in and normal to the lunar orbit plane. With this frame, it is clear that we can now justify simplifying the calculation by considering only the bimodal tesseral deformation, i.e. the term $i=j=2$. The potential then reduces to

$$U = (9/12)\ k_2 (GM/r^3)(R^5/r_2^3)\ \cos^2\phi\ \cos^2\phi_1\ \cos 2(\lambda-\lambda_1) \tag{10}$$

To this point, the development is valid even for a perfectly elastic Earth, one in which there are no frictional losses. Qualitatively, the existence of tidal friction produces a time lag T in the response of the Earth to the tidal perturbation from the Moon. Since Earth's rotation rate ω is different from the lunar mean motion, this implies that the tidal bulge will be displaced from the Earth-Moon direction by the tidal lag angle

$$\delta = (\omega - n)\ T \tag{11}$$

[We adopt the sign convention of Kaula (1968), rather than that of Yoder et al. (1978)]. Another way of saying this is that the deformation of the Earth at time $t = t^* + T$ is caused by the lunar position at time t^*, and

$$\lambda_1 = \lambda + \delta \tag{12}$$

If we further recognize that both the Moon and the bulge will lie in or very near the lunar orbit plane, we may set $\phi=\phi_1=0$, and expression (10) reduces to

$$U = (3/4)\ k_2 \cos 2\delta\ [GM/r^3(t^*)][R^5/r_2^3(t)] \tag{13}$$

https://doi.org/10.1017/S0252921100002608 Published online by Cambridge University Press

Taking the gradient with respect to the coordinates of the perturbed body, we finally obtain for the inertial acceleration imposed on any exterior free body at time t by the semi-diurnal lunar Earth tide

$$d^2\vec{\rho}_2/dt^2 = -\,3\,qk_2\,\cos 2\delta\,[GM/r^3(t^*)]\,[R/r_2(t)]^5\,\vec{r}_2(t) \tag{14}$$

where, for reasons that will be clarified later, we have introduced the constant q=3/4.

3.3 The Tidal Acceleration of the Moon

The lunar tide produced on the Earth perturbs the Moon's own orbit. That it is the Moon itself that generates this tide is irrelevant to its action on the orbit. In evaluating eqn. (14), however, the requirement to evaluate the lunar position at two different times is inconvenient. This is circumvented by recognizing that the inferred value of T is about ten minutes, during which time the Moon moves only about 1/6 of its diameter. One may then use the linearized relation

$$\vec{r}(t) = \vec{r}(t^*) - \delta\,\vec{r}(t^*) \times \vec{k} \tag{15}$$

to eliminate t from the left-hand side, which also breaks the vector into explicit radial and transverse components, $\vec{k}$ being the unit z-vector. Finally, recognizing that the magnitude of the acceleration is extremely small, and that previous simplifications have already been at a more questionable level, we suppose that the acceleration at time t^* is negligibly different from that at time t. The results in the final expression for the cartesian acceleration

$$d^2\vec{r}/dt^2 = -\,3qk_2\,\cos 2\delta\,GM(1+M/E)(R^5/r^8)(\vec{r} - \delta\,\vec{r} \times \vec{k}) \tag{16}$$

where the factor $(1+M/E)$ accomplishes the translation from barycenter to geocenter.

3.4 The Constant q

We have introduced the constant q in an attempt to minimize the confusion for those who compare the above result with eqn. (5) of Williams et al. (1978). Accounting for the sign convention on δ, the only difference between us is the factor $q\cos 2\delta$ given above. While no details are given of the other derivation, it seems evident that they have used the approximation $\cos 2\delta=1$. Independent derivations by several different people associated with our work have invariably produced the result q=3/4, rather than the value unity implied by our JPL colleagues. Surely, we will soon discover where the problem lies. Nonetheless, it is important to point out that this in no way affects the major conclusions in the numerical discussion that follows. Indeed, one may legitimately take the operational point of view that the analysis of observations provides an experimental value of $q\delta$ instead of δ.

4. NUMERICAL COMPARISONS AND CONCLUSIONS

Comparing eqns. (3) and (16), one notes three differences: a) the directions of the transverse components are nearly the same; b) the analogue method introduces a radial component that is not dependent on tidal friction, only on the elastic deformation; and c) its scalar coefficient has a periodic variation. Both methods have empirical parameters to be determined from observation, insuring that the average values of the coefficients will be the same. In principle, the radial term will look like an extra mass, affecting the mean motion. In principle, the variable coefficient will produce larger short-periodic variations. How important are these differences in numerical application?

This question can, and should, be answered definitively by analytical means. It was a natural thing, however, to perform some numerical tests while converting our integration program from the arithmetic to an analogue formulation. We will discuss three specific cases here: 1/ the arithmetic method with W=-26 arcsec/century^2; 2/ the analogue method with k_2=0.30 and δq=-2.55 degrees, chosen to produce the same effect in longitude; and 3/ the elastic analogue case with k_2=0.30 and zero lag angle. Figure 1 shows the first of these cases, while Figure 2 gives the third, both over a 400-day interval. We were surprised to find that a graph of the differences between cases 2 and 3, overlaid on Figure 1, shows only barely discernible variations from those curves. The same situation obtains when the differences between cases 2 and 1 are overlain on Figure 2. Both of these discoveries point to the same conclusion: the only difference in practice between the arithmetic and analogue methods is due to the radial, non-frictional, term. It is a result of the Earth's non-rigidity, but not of dissipation processes. The greater rigor of derivation in the analogue approach appears to buy no advantage whatsoever in modelling the acceleration in longitude. Stated differently, the arithmetic method would be the full equivalent of the analogue procedure, if only a suitable radial term were added.

A secondary implication is that the full effect of case 2 may be obtained by adding Figures 1 and 2.

It is customary in discussions of lunar ranging that attention be concentrated on the effect on the range observable of the acceleration in longitude, an effect that varies as the sine of the local topographic hour angle of the observed point; this is often near zero. Examination of the above figures shows that, in fact, the direct effect in Earth-Moon radial distance is also extremely important, reaching a maximum peak-to-peak amplitude during this 400-day span of about 35 cm. On the other hand, the effect on the mean motion is essentially trivial -- about 0.06 arcsec/century, within the current noise on mass determinations. The structural features of the graphs suggest that the periodic behavior is dominated by at least two near-monthly frequencies and a semi-annual one, which may be caused by phase interference between them. These conjectures should be tested analytically.

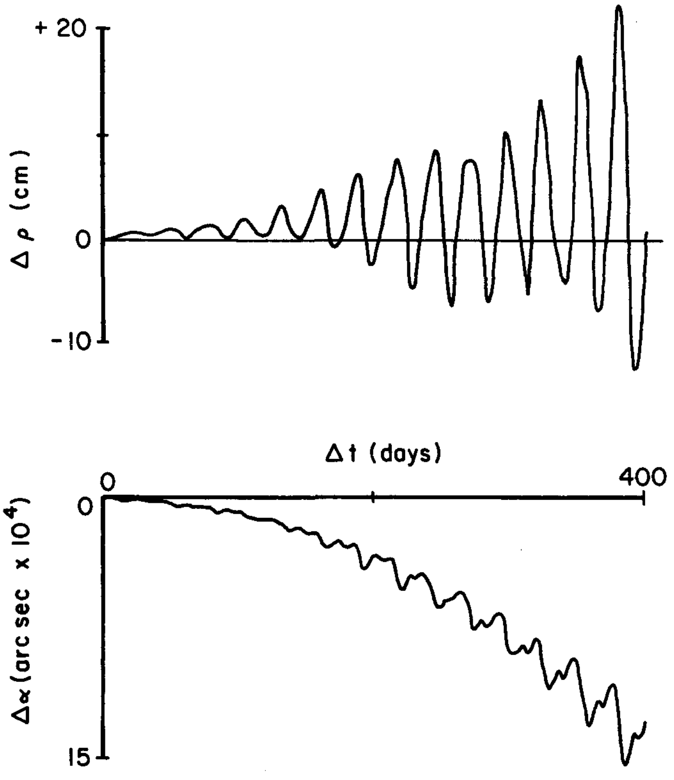

Figure 1: The differential effect of modelling tidal friction with the arithmetic method, with $W = -26$ arcseconds/century^2. This is virtually identical with the effect of the transverse component of the analogue approach.

5. ACKNOWLEDGEMENTS

Portions of this work were performed while one of us (JDM) was on temporary leave from the University of Texas, in residence at CERGA under appointments as Astronome Associé accorded by the Ministère de l'Enseignement Supérieur (France). The other author is attached to the Centre National de la Recherche Scientifique, while the computations were supported by the Groupe de Recherches de Géodésie Spatiale.

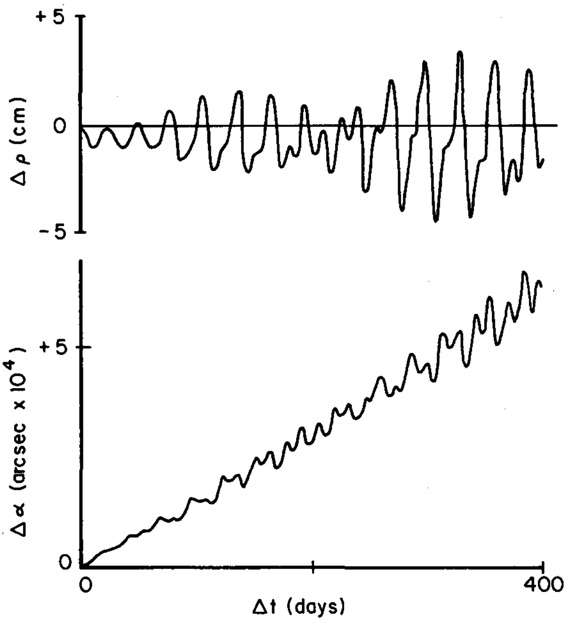

Figure 2: The effect of the elastic tide for k_2 = 0.30, which is also virtually identical with the difference between the full analogue model and the arithmetic model.

6. REFERENCES

Brouwer, D., and Clemence, G. M.: 1961, *Methods of Celestial Mechanics* (Academic Press, New York).

Darwin, G. H.: 1898, *The Tides* (Houghton, Mifflin & Co, New York; reprinted by Freeman, San Francisco, 1962).

Kaula, W. M.: 1968, *An Introduction to Planetary Physics* (Wiley, New York).

Mulholland, J. D.: 1980, Rev. Geophys. & Space Phys. 18, pp. 549-564.

Munk, W. H., and Macdonald, G. J. F.: 1960, *The Rotation of the Earth* (Cambridge Univ. Press, London).

Oesterwinter, C., and Cohen, C. J.: 1972, Celest. Mech. 5, pp. 317-395.

Van Flandern, T. C.: 1981, Astrophys. Jour. 248, in press.

Williams, J. G., Sinclair, W. S., and Yoder, C. F.: 1978, Geophys. Res. Lett. 5, pp. 943-946.

DISCUSSION

King : Williams et al. (1978) pointed out a 9-cm term in longitude with 18.6 year period. In principle, this permits eventual separation of the diurnal and semi-diurnal tidal components, which is an exciting prospect for the future.