ON OPEN EXTENSIONS OF MAPS

S. P. FRANKLIN AND J. K. KOHLI

0. Introduction. In recent years there has been some interest in trying to improve the behaviour of maps by extending their domains. For example, in 1953 Whyburn showed that every map is the restriction of a compact map [**17**]. Similarly, Krolevec proved in 1967 that each locally perfect map can be extended to a perfect map [**12**] and in an as yet unpublished paper, Dickman obtained the same result for arbitrary maps [**4**]. In this paper we show that every map can be extended to an open map so that certain properties of the domain and range are preserved in the new domain. These results are then used to obtain analogues and improvements of recent theorems of Arhangel'skiĭ, Čoban, Hodel, and Proizvolov.

1. Open extensions. Let $f: X \to Y$ be a function, not necessarily continuous, from a topological space X into a topological space Y. We shall call a point $x \in X$ and its image $f(x) \in Y$ singular points of X and Y, respectively, if there is an open set U of X containing x whose image $f(U)$ is not a neighbourhood of $f(x)$. The function f is open just in case there are no singular points of X (or equivalently of Y).

For each singular point x of X, let Y_x be a copy of Y. Let $W = X \oplus (\oplus Y_x)$, where the second disjoint topological sum is taken over all singular points of X. By identifying each singular point $x \in X$ (with X thought of as a subset of W) with its image $f(x)$ (as a point of $Y_x \subseteq W$), we arrive at a quotient space X^* of W. The inclusion map $i: X \to W$ composes with the quotient map $q: W \to X^*$ to give an embedding of X into X^*. Hence we may think of X^* as an extension of X.

Let $f_1: W \to Y$ be the function whose restriction to X is f and whose restriction to each Y_x is the identity map $1_Y: Y_x \to Y$. We leave it to the reader to verify that the unique function $f^*: X^* \to Y$ satisfying $f^* \circ q = f_1$ is an open extension of f. Recapitulating, we have the following.

1.1. *There is an overspace X^* of X and an open function $f^*: X^* \to Y$ whose restriction to X is f; f^* is continuous just in case f is.*

Since only sums and quotients were used in the construction of X^*, the following follows at once,

1.2. *Any coreflexive property† of X and Y is preserved in X^*.*

†See Herrlich [**10**] for the definition and properties of coreflective subcategories.

In particular, if X and Y are countably, sequentially or compactly generated,†† so is X^*. If X and Y are locally connected, or are P-spaces, or are chain net spaces,††† so is X^*.

It is routine to verify the following.

1.3. X^* *satisfies the separation axioms* T_0, T_1, T_2, T_3, *and* $T_{3\frac{1}{2}}$ *whenever* X *and* Y *do.*

We shall prove only the last case, that of complete regularity. Suppose that F is a closed subset of X^* and that $p \notin F$. If $q^{-1}(p) \cap X = \emptyset$ (where $q \colon W \to X^*$ is the quotient map), then $q^{-1}(p) \in Y_x$ for some x. Then there is a real-valued function Φ on Y_x which is zero at $q^{-1}(p)$ and one at $x = f(x) \in Y_x$ and on $q^{-1}(F) \cap Y_x$. Extend Φ continuously to all of W by taking it constantly one on X and on each $Y_{x'}$, $x' \neq x$. This extended Φ defines a real-valued function on X^* which separates p and F. In the other case, if $x_0 \in q^{-1}(p) \cap X$, let $\Phi_0 \colon X \to R$ be zero at x_0 and one on $q^{-1}(F) \cap X$. For each singular x, choose $\Phi_x \colon Y_x \to R$ which is one on $q^{-1}(F) \cap Y_x$ and such that $\Phi_x(f(x)) = \Phi_0(x)$. These functions combine to form one $\Phi \colon W \to R$ which in turn induces a real-valued function on X^* separating p and F.

X^* can also be realized as an adjunction space. Let F be the closed discrete subset of $\bigoplus Y_x$ whose intersection with each Y_x is its singular point $f(x)$. The map $g \colon F \to X$ which sends each $f(x)$ to x yields the adjunction space $\bigoplus Y_x \cup_g X$ which is homeomorphic to X^*. Using this representation we have the following.

1.4. X^* *is normal, hereditarily normal, perfectly normal, collectionwise normal, or fully normal (i.e. paracompact) whenever* X *and* Y *are.*

The first three of these properties are preserved under sums and adjunctions. For the other three, the assertion follows from a theorem of Tsuda [16].

If x is a singular point of X, then $f(x) \in Y_x$ and $q(x) = q(f(x))$. If X and Y are (pathwise) connected, so are their continuous images $q(X)$ and $q(Y_x)$. Since each $q(X) \cap q(Y_x) \neq \emptyset$, $\bigcup(q(X) \cup_q (Y_x))$ is (pathwise) connected. Thus we have the following.

1.5. X^* *is (pathwise) connected whenever* X *and* Y *are.*

Let X be the plane set consisting of the union of the closed intervals $[-1, 1]$ on the two axes. Let $Y = [-1, 1]$ on one axis and let f be the restriction to X of the projection onto this same axis. Each point on one axis, except the origin, is a singular point of X. This example shows the following.

---

†† A countably generated space is one "determined by countable subsets" in the sense of Moore and Mrówka [14]. Sequentially generated spaces are the sequential spaces [6] and compactly generated spaces the k-spaces.

††† For the coreflexivity of local connectedness, see Gleason [9]. A P-space is one in which each G_δ is open (see [8]). A chain net is one whose underlying directed set is a chain. Chain nets are said to suffice if any set containing the limits of all its convergent chain nets is closed (see [13]).

1.6. X^* *need not preserve metrizability, either axiom of countability, weight or local weight, separability or density, the Lindelöf property, or (countable, sequential, pseudo) compactness.*

(Another open extension of a map f can be given whose domain will preserve many of these properties. Since the family $\{1_X, f\}$ separates points and also separates points from closed sets, the evaluation map $e: X \to X \times Y$ (given by $e(x) = (x, f(x))$) is an embedding. Hence X is homeomorphic to $e(X)$, the graph of f, and the projection $\pi_Y: X \times Y \to Y$ restricted to $e(X)$ is essentially f. Thus *each map is the restriction of a projection map. If X is compact, f is the restriction of a clopen map. This is also the case if X is countably compact and Y is a subspace of a sequential space* [5]. Clearly, any finitely productive property of X and Y is preserved in the domain of the projection. Hence most of the properties mentioned in 1.6 are preserved.)

By imposing restrictions on the set S of singular points of X, X^* may be induced to preserve many other properties. For example, we have the following.

1.7. *If the singular points of X do not accumulate, then metrizability, local compactness, and local weight are preserved in X^*. Further, if the cardinality of S is not more than the larger of the weights of X and Y, then neither is the weight of X^*.*

(Here we assume that X and Y are T_1 spaces.)

We first show that the quotient map $q: W \to X^*$ is a closed mapping under the given hypothesis. Suppose that F is a closed subset of W and $\{p_\alpha\}$ is a net in $q(F)$ converging to a point p in X^*. Now

$$q(F) = q(F \cap X) \cup (\cup q(F \cap Y_x))$$

and the restriction of q to X and to each Y_x is an embedding. Since $q(X)$ and each $q(Y_x)$ is closed in X^*, we conclude that $q(F \cap X)$ and each $q(F \cap Y_x)$ is closed in X^* also. Thus if $\{p_\alpha\}$ is frequently in $q(F \cap X)$ or in some $q(F \cap Y_x)$, p must belong to $q(F)$, and our proof is complete. If $q^{-1}(p) = \{y\}$ with $y \in Y_x$, let $U = Y_x$. If $q^{-1}(p) = \{x\}$, where x is a non-singular point of X, let U be a neighbourhood of x in X which is free of singular points. If $q^{-1}(p) = \{x, f(x)\}$, let U be the union of Y_x and a neighbourhood of x in X which is free of singular points of X other than x. In any case, $q(U)$ is a neighbourhood of p which has a non-empty intersection with at most one $q(Y_x \cap F)$. But $\{p_\alpha\}$ is eventually in $q(U)$, and hence is frequently in either

$$q(U) \cap q(F \cap X) \quad \text{or} \quad q(U) \cap q(F \cap Y_x).$$

Hence q is a closed map. Since each $q^{-1}(p)$ is at most a doubleton, q is a perfect map. Since metrizability and local compactness are preserved under sums and perfect maps, the first two assertions of 1.7 are proved.

Suppose that $\mathfrak{m}$ is the larger of the local weights of X and Y. If $q^{-1}(p) = \{y\}$ where $y \in Y_x \backslash \{f(x)\}$, then the image of a base at y under q is a base at p. If

$q^{-1}(p) = \{x\}$, where x is a non-singular point of X, a base can be chosen at x whose members contain no singular points. The image of this base under q is a base at p. If $q^{-1}(p) = \{x, f(x)\}$, choose a base $\mathscr{B}$ at x whose members contain only one singular point, and choose a base $\mathscr{V}$ at $f(x) \in Y_x$, with the cardinality of $\mathscr{V}$ and $\mathscr{B}$ no larger than $\mathfrak{m}$. The images under q of sets of the form $B \cup V$ with $B \in \mathscr{B}$ and $V \in \mathscr{V}$, form a base at p of cardinality no larger than $\mathfrak{m}$.

For what remains we need only note that

$$\text{weight } W = \text{weight } X + \text{card } S \cdot \text{weight } Y \leqq \mathfrak{m} + \mathfrak{m}^2 = \mathfrak{m}$$

and that perfect maps do not increase weight.

Example 1.6 shows that the restrictions imposed on the set S of singular points in 1.7 are not superfluous. By replacing one of the intervals $[-1, 1]$ in 1.6 by a sequence converging to zero, one can easily see that these cannot be weakened even to S being a countable discrete subset.

2. Finite-to-one mappings. In 1966, Proizvolov [**15**] showed that weight and metrizability are inversely preserved in locally compact spaces under open finite-to-one maps. Later that year, Arhangel′skiĭ [**1**; **2**] showed that they were always inversely preserved under clopen finite-to-one maps. In 1967, Čoban [**3**] proved that hereditary paracompactness (metacompactness, Lindelöf) are inversely preserved under open finite-to-one maps. (Some separation axioms are required for all these results.)

If f is a finite-to-one mapping and the set S of singular points of X is finite (i.e., f is open except at finitely many points), then $f^*: X^* \to Y$ is an open, finite-to-one map and hence X^* will inherit properties from Y by the results quoted in the paragraph above. (1.3 shows that the needed separation axioms (see below) also lift properly.) Since these properties are all hereditary, X must also enjoy them. Thus we see that it is sufficient to require that f be open except at finitely many points to arrive at the desired conclusion. For the convenience of the reader we list precise statements of the improved theorems. (Assume all spaces to be Hausdorff, and that f is continuous and onto.)

2.1 (Proizvolov). *If X and Y are locally compact, and f is finite-to-one and open except at finitely many points, then* weight $X \leqq$ weight Y. *If Y is metrizable, so is X.*

In the proof, 1.7 as well as Proizvolov's original theorem must be used.

2.2 (Arhangel′skiĭ). *If X and Y are completely regular and f is a finite-to-one closed map which is open except at finitely many points, then* weight $X \leqq$ weight Y. *If Y is metrizable, so is X.*

Here, in addition to 1.3, we need only note that with S finite, f^* is closed if and only if f is closed.

2.3 (Čoban). *If f is finite-to-one and open except at finitely many points, then X is hereditarily paracompact (metacompact, Lindelöf) whenever Y is.*

(For paracompactness, X is required to be regular.)

Simple examples can be given to show that the conditions on the singular points of f cannot simply be omitted in 2.1, 2.2, and 2.3. First let $X = \beta N$, $Y = \{0\} \cup \{1/n \mid n \in N\} \subseteq R$ and $f: X \to Y$ arise from $n \to 1/n$. Y is locally compact, second countable, and metrizable. X is locally compact, has weight c, and is not metrizable, even though f is a perfect map. This covers 2.1 and 2.2.

For 2.3 let Y be as before and let X' be the ordinal compactification of N recently constructed by Franklin and Rajagopalan [7], i.e. $X' = N \cup (\omega_1 + 1)$ with N embedded as an open dense subspace, $\omega_1 + 1$ embedded as a closed subspace, $N \cap (\omega_1 + 1) = \emptyset$, and X' compact Hausdorff. Let $X = X' \backslash \{\omega_1\}$ and define $f: X \to Y$ by $f(n) = 1/n$ and $f(x) = 0$ otherwise. Y is hereditarily Lindelöf (and much more) while X fails to be metacompact.

3. Dimension. In this section all spaces are assumed to be metric, and f is continuous and onto. In 1963 Hodel [11] showed that dimension cannot be lowered by open maps f such that each $f^{-1}(y)$ is discrete. The technique of the last section can be used to improve this result also.

3.1 (Hodel). *If the singular points of X do not accumulate and if each $f^{-1}(y)$ is discrete, then $\dim X \leqq \dim Y$.*

For the proof we use 1.7, Hodel's original theorem, and the fact that X is a closed subspace of X^*.

Hodel's theorem (in both the original and the improved versions) holds true for not necessarily continuous f if Y is taken to be locally compact and separable.

To show that some hypothesis is needed on f, one need only look at Peano's map of the interval onto the square.

REFERENCES

1. A. B. Arhangel'skiï, *Existence criterion of a bicompact element in a continuous decomposition. A theorem on the invariance of weight for open-closed finitely multiple mappings*, Dokl. Akad. Nauk SSSR *166* (1966), 1263–1266 (Russian); translated as Soviet Math. Dokl. *7* (1966), 249–253.
2. ——— *A theorem on the metrizability of the inverse image of a metric space under an open-closed finite-to-one mapping: Example and unsolved problems*, Dokl. Akad. Nauk SSSR *170* (1966), 759–762 (Russian); translated as Soviet Math. Dokl. *7* (1966), 1258–1262.
3. M. M. Čoban, *Open finite-to-one mappings*, Soviet Math. Dokl. *8* (1967) 603–605.
4. R. F. Dickman, Jr., *On closed extensions of functions* (to appear).
5. I. Fleischer and S. P. Franklin, *On compactness and projections*, pp. 70–79 in *Contributions to extension theory of topological structures*, Symposium, Berlin, 1967 (Academic Press, New York, 1969).
6. S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. *57* (1965), 107–115.

7. S. P. Franklin and M. Rajagopalan, *Some examples in topology* (to appear in Trans. Amer. Math. Soc.).

8. L. Gilman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, N.J., 1960).

9. A. M. Gleason, *Universal locally connected refinements*, Illinois J. Math. 7 (1963), 521–531.

10. H. Herrlich, *Topologische Reflexionen und Coreflexionen* (Springer-Verlag, Berlin, 1968).

11. R. E. Hodel, *Open functions and dimension*, Duke Math. J. 30 (1963), 461–467.

12. N. Krolevec, *Locally perfect mappings*, Dokl. Akad. Nauk SSSR 175 (1967), 1008–1011. (Russian)

13. A. K. Misra, *Spaces in which chain nets suffice* (to appear).

14. R. C. Moore and S. G. Mrówka, *Topologies determined by countable objects*, Notices Amer. Math. Soc. 11 (1964), 554.

15. V. V. Proizvolov, *On finite-to-one open mappings*, Dokl. Akad. Nauk SSSR 166 (1966), 38–40 (Russian); translated as Soviet Math. Dokl. 7 (1966), 35–38.

16. M. Tsuda, *On adjunction spaces*, Proc. Japan Acad. 38 (1962), 23–26.

17. G. T. Whyburn, *A unified space for mappings*, Trans. Amer. Math. Soc. 74 (1953), 344–350.

Indian Institute of Technology,
Kanpur, India;
Carnegie-Mellon University,
Pittsburgh, Pennsylvania