MINIMAL OPERATORS FOR SCHRODINGER-TYPE DIFFERENTIAL EXPRESSIONS WITH DISCONTINUOUS PRINCIPAL COEFFICIENTS

M. FAIERMAN AND I. KNOWLES

1. Introduction. The objective of this paper is to extend the recent results [**7**, **8**, **9**] concerning the self-adjointness of Schrödinger-type operators with singular potentials to a more general setting. We shall be concerned here with formally symmetric elliptic differential expressions of the form

$$(1.1) \quad \mathscr{T} = - \sum_{j,k=1}^{m} (\partial_j - ib_j(x))a_{jk}(x)(\partial_k - ib_k(x)) + q(x)$$

where $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$ (and $m \geqq 1$), $i = (-1)^{1/2}$, $\partial_j = \partial/\partial x_j$, and the coefficients a_{jk}, b_j and q are real-valued and measurable on $\mathbf{R}^m$.

The basic problem that we consider is that of deciding whether or not the formal operator $\mathscr{T}$ defined by (1.1) determines a unique self-adjoint operator in the space $L^2(\mathbf{R}^m)$ of (equivalence classes of) square integrable complex-valued functions on $\mathbf{R}^m$. It is well known that when the co-efficients a_{jk}, b_j and q are sufficiently smooth (see, for example, [**8**] conditions (S1)–(S4)), this problem reduces to deciding whether or not the restriction of $\mathscr{T}$ to $C_0^\infty(\mathbf{R}^m)$ is essentially selfadjoint in $L^2(\mathbf{R}^m)$. However, when these smoothness conditions are not satisfied a priori, the problem becomes more difficult.

An indication of what may be possible in the general case is given by the known theory for the case $m = 1$ in [**11**, § 17]. Here, we may set $b_1 = 0$ without loss of generality, because the resulting operator is known to be unitarily equivalent (via a gauge transformation) to the original operator; we also assume that

$$(1.2) \quad q \in L_{\text{loc}}^1(\mathbf{R}),$$

$$(1.3) \quad 1/a_{11} \in L_{\text{loc}}^1(\mathbf{R}).$$

Let T denote the restriction of $\mathscr{T}$ to the set

$$\mathscr{D}(T) = \{u : u \in L^2(\mathbf{R}) \cap \text{AC}_{\text{loc}}(\mathbf{R}), \mathscr{T}u \in L^2(\mathbf{R})\}$$

where $\text{AC}_{\text{loc}}(\mathbf{R})$ denotes the set of locally absolutely continuous functions on $\mathbf{R}$. Let T_0 denote the restriction of T to the set of functions of compact

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support in $\mathscr{D}(T)$. Then it is known ([**11**, p. 68]) that

$$(1.4) \qquad T_0{}^* = T.$$

Thus, given (1.2) and (1.3), the problem reduces once again to deciding whether or not a certain minimal operator (T_0 in this case) is essentially self-adjoint (equivalently, the maximal operator T is symmetric).

For $m > 1$, it was shown in [**9**] that if $a_{jk} \in C^{1+\alpha}(\mathbf{R}^m)$, for some $\alpha > 0$, $b_j \in C^1(\mathbf{R}^m)$, and $q = q_1 + q_2$ where $q_i \in L_{\mathrm{loc}}{}^1(\mathbf{R}^m)$, $i = 1, 2, q_1$ is locally bounded below and q_2 is small in a certain sense, then one can define analogues of the operators T and T_0 above that satisfy the adjoint relation (1.4). It was also shown in [**9**] that, as a consequence, most of the standard self-adjointness criteria automatically hold in the wider setting. Our main objective here is to extend this theory to cover the case in which the principal coefficients of $\mathscr{T}$ may have discontinuities. In particular, we assume that the coefficients, a_{jk}, b_j, and q satisfy the following conditions:

(C1) The matrix (a_{jk}) is symmetric, and locally uniformly elliptic in the sense that for any compact set $K \subset \mathbf{R}^m$ there exists a positive number $\lambda(K)$ such that

$$\sum_{j,k=1}^{m} a_{jk}(x)\xi_j\xi_k \geqq \lambda(K)(\xi_1{}^2 + \ldots + \xi_n{}^2)$$

for all $x \in K$ and all vectors $\xi = (\xi_1, \ldots, \xi_m) \in \mathbf{R}^m$.

(C2) For all $1 \leqq j, k \leqq m, a_{jk} \in L_{\mathrm{loc}}{}^\infty(\mathbf{R}^m)$; furthermore, if B_r denotes the open ball with centre 0 and radius r, there exist sequences of positive numbers $\{r_n\}$ and $\{\epsilon_n\}$ with $r_n \to \infty$ as $n \to \infty$, such that each a_{jk} satisfies a Lipschitz condition in each of the annular regions $B_{r_n+\epsilon_n} \backslash B_{r_n}$, $n \geqq 1$.

(C3) $b_j \in L_{\mathrm{loc}}{}^\infty(\mathbf{R}^m)$ for $1 \leqq j \leqq m$.

(C4) $q \in L_{\mathrm{loc}}{}^1(\mathbf{R}^m)$.

With reference to (C2), the regions in which we require the principal coefficients to be Lipschitz have been chosen to be annuli, for convenience; we could just as easily require that the same conditions hold in more general "bands", such as those defined in [**3**, § 6], provided only that each band has non-zero width everywhere, and that every compact set K in $\mathbf{R}^m$ can be surrounded by at least one band. In any event, it is vital for the validity of the proof given below that the a_{jk} satisfy the band condition (C2). Indeed, in Lemma 3 of the sequel we shall see that it is precisely the condition (C2) that enables us to relate the operators T_0 and T (defined in Section 2) with certain truncated operators $T_0{}^{(n)}$ and $T^{(n)}$ (which are shown to possess certain properties), and it is through this relationship that we are able to arrive at our main result concerning T_0 and T. Finally, we might mention that the physical motivation for (C2) is that the coefficients a_{jk} are allowed to have at least bounded

discontinuities, a situation encountered for example in diffraction problems (c.f. [**10**, p. 205]).

The treatment follows, in essence, that given in [**9**]. A rather surprising fact that emerges is that despite the very weak conditions on the principal coefficients, the proof of the central Lemma (Lemma 2, corresponding to Lemma 1 of [**7**] and Lemma 2 of [**9**]) is essentially a direct consequence of the (local) ellipticity assumptions together with the corresponding result for the simple case when the principal part of the operator is just the negative Laplacian.

2. Analogues of the maximal and minimal operators. In the sequel we shall set $\mathscr{H} = L^2(\mathbf{R}^m)$, and denote by H^1 the Sobolev space on $\mathbf{R}^m$ of order 1 (that is, $H^{1,2}(\mathbf{R}^m)$). L_{loc}^p and H_{loc}^1 shall have their usual meanings. Observe also that, under conditions (C2) and (C3), $\mathscr{T} u$ makes sense as a distribution provided that $u \in H_{\mathrm{loc}}^1$ and $qu \in L_{\mathrm{loc}}^1$.

We now define analogues of the operators T and T_{min} of [**9**]. Let T denote the restriction of $\mathscr{T}$ to the set

$$(2.1) \qquad \mathscr{D}(T) = \{u : u \in \mathscr{H} \cap H_{\mathrm{loc}}^1, qu^2 \in L_{\mathrm{loc}}^1, \mathscr{T} u \in \mathscr{H}\}.$$

Notice that, as both q and qu^2 are in L_{loc}^1, it follows that $qu \in L_{\mathrm{loc}}^1$, and hence that $\mathscr{T} u$ is automatically defined in a distributional sense, as outlined above. Also, when $b_j, a_{jk} \in C^1(\mathbf{R}^m)$, $1 \leqq j, k \leqq m$, the operator T is a restriction of its analogue in [**9**, equation (1.5)], by [**7**, Lemma 3]. However, this difference is more apparent than real, because once we establish the analogue of (1.4) for the present situation, it follows immediately that the two operators concerned must coincide. If, in addition, $q \in L_{\mathrm{loc}}^2$ then a similar argument shows that T is just the maximal operator in the usual sense.

Let $\mathscr{D}(T_0)$ denote the set of all functions of compact support in $\mathscr{D}(T)$, and denote by T_0 the restriction of T to $\mathscr{D}(T_0)$. We define the minimal (closed) operator, T_{min}, corresponding to T by

$$(2.2) \qquad T_{\mathrm{min}} = \bar{T}_0$$

whenever T_0 is closable in $\mathscr{H}$.

3. The main result. Following [**6**], we define the norm

$$(3.1) \qquad \|u\|_{H_b{}^1}{}^2 = \|u\|^2 + \sum_{j=1}^{m} \|D_j u\|^2$$

where $\|\cdot\|$ without the subscript denotes the usual norm in $\mathscr{H}$, and $D_j = \partial_j - ib_j(x)$. The completion of $C_0^\infty(\mathbf{R}^m)$ with respect to the norm (3.1) is denoted by $H_b{}^1(\mathbf{R}^m)$ (or $H_b{}^1$).

As in [**9**] we seek an appropriate analogue of (1.4) for the present

situation. To facilitate this, we assume that the potential q satisfies either of the following additional conditions of a general nature;

(C5) q can be expressed as $q = q_1 + q_2$ where $q_1 \in L_{\mathrm{loc}}{}^1$ and is locally bounded below, and q_2 satisfies either

$$(3.2) \qquad \int_{|x| < s} |q_2(x)|^2 dx \leqq K s^{2\beta},\, 1 \leqq s < \infty,$$

and

$$(3.3) \qquad \int_{|y| < s} |q_2(x - y)| Q(y) dy \to 0 \quad \text{as } s \to 0$$

uniformly for $x \in \mathbf{R}^m$ where

$$\begin{aligned} Q(y) &= |y|^{2-m} &&\text{if } m > 2 \\ &= -\log|y| &&\text{if } m = 2 \\ &= 1 &&\text{if } m = 1 \end{aligned}$$

or, if $m \geqq 5$,

$$(3.4) \qquad q_2 \in L^{m/2}(\mathbf{R}^m).$$

(C5)′ q can be expressed as $q = q_1 + q_2$ where $q_1 \in L_{\mathrm{loc}}{}^\infty$, and q_2 satisfies

$$(3.5) \qquad \int_{\mathbf{R}^m} |q_2(x)|\, |u(x)|^2 dx \leqq \epsilon \|u\|_{H_b{}^1}{}^2 + \gamma \|u\|^2$$

for any $u \in H_b{}^1$ and $\epsilon > 0$, where γ depends only on ϵ and q_2.

Remark. Condition (C5)′ represents a variation on the standard conditions (C5) (see [6]) in that while the assumptions on q_1 are stronger, the assumptions on q_2 are somewhat weaker. Either (3.3) or (3.4) are sufficient for (3.5) (see [6] for a discussion of these conditions).

The main result of this paper is the following

THEOREM. *Under conditions* (C1)–(C4), *and* (C5) *or* (C5)′, T_0 *is a densely defined symmetric linear operator in* $\mathscr{H}$ *and*

$$T_0{}^* = T_{\min}{}^* = T.$$

We begin the proof by establishing a number of lemmas.

LEMMA 1. *Let*

$$(3.6) \qquad \mathscr{L} = \sum_{j,k=1}^{m} (\partial_j - ib_j(x)) a_{jk}(x)(\partial_k - ib_k(x))$$

where a_{jk}, $1 \leqq j, k \leqq m$, *and* b_j, $1 \leqq j \leqq m$, *are real-valued functions in* $L_{\mathrm{loc}}{}^\infty$. *Let the matrix* $(a_{jk}(x))$ *be positive definite for each* $x \in \mathbf{R}^m$. *Then, if*

$u \in H_{\text{loc}}^1$ and $\mathscr{L}u \in L_{\text{loc}}^1$

(3.7) $\mathscr{L}_0|u| \geqq \text{Re}[(\text{sgn } \bar{u})u]$

where $\mathscr{L}_0$ *denotes the operator* $\mathscr{L}$ *with* $b_k = 0$, $k = 1, \ldots, m$, *and* $\text{sgn } \bar{u}(x) = \bar{u}(x)/|u(x)|$ *if* $u(x) \neq 0$ *and is zero otherwise.*

This is the analogue in the present situation of the well-known Kato distributional inequality (see [**6**, Lemma A]). As it happens, we only require this lemma in the case $a_{jk} = \delta_{jk}$, δ_{jk} being the Kronecker delta. However, the full result has some independent interest, and its proof is no more difficult. We delay the proof until Section 4.

The next lemma is the analogue of Lemma 1 of [**9**]; as usual most of the technical difficulties of the paper are encountered here. We define formal operators $\mathscr{T}^{(n)}$ as follows: $\mathscr{T}^{(n)}$ denotes a formal operator that is identical with $\mathscr{T}$ for $|x| < r_n + \epsilon_n$, but has coefficients $a_{jk}^{(n)} = \delta_{jk}$, and $q_1^{(n)}$ constant for $|x| \geqq r_n + \epsilon_n$. As usual, b_j and q_2 remain unaltered in this construction. We will also need the quadratic forms $a^{(n)}[\cdot]$, $c_1^{(n)}[\cdot]$, and $c_2[\cdot]$ defined by

(3.8) $a^{(n)}[u] = \displaystyle\sum_{j,k=1}^m \int_{\mathbf{R}^m} a_{jk}^{(n)}(x)D_ju(x)\overline{D_ku}(x)dx$

where D_j is defined above,

(3.9) $c_1^{(n)}[u] = \displaystyle\int_{\mathbf{R}^m} q_1^{(n)}(x)|u(x)|^2dx,$

(3.10) $c_2[u] = \displaystyle\int_{\mathbf{R}^m} q_2(x)|u(x)|^2dx.$

LEMMA 2. *Let* $T^{(n)}$ *denote the operator corresponding to* $\mathscr{T}^{(n)}$ *via* (2.1). *Let* $h^{(n)} = a^{(n)} + c_1^{(n)} + c_2$, *where* $a^{(n)}$, $c_1^{(n)}$, *and* c_2 *are defined by* (3.8)–(3.10). *Then* $T^{(n)}$ *is a self-adjoint operator with*

$$\mathscr{D}(T^{(n)}) \subset \mathscr{D}(h^{(n)}) = H_b^1 \cap \mathscr{D}(\{q_1^{(n)} + q^*\}^{1/2}),$$

where $-q^*$ *is any lower bound for* $q_1^{(n)}$ *on* $\mathbf{R}^m$ *and* $\mathscr{D}(\{q_1^{(n)} + q^*\}^{1/2})$ *denotes the domain of the self-adjoint operation of multiplication by* $(q_1^{(n)} + q^*)^{1/2}$ *in* $\mathscr{H}$. *Furthermore*

(3.11) $h^{(n)}(u, v) = (T^{(n)}u, v)$

for any $u \in \mathscr{D}(T^{(n)})$ *and any* $v \in \mathscr{D}(h^{(n)})$, *and*

(3.12) $h^{(n)}[u] \geqq k\|u\|_{H_b^1}^2 - (q^* + M)\|u\|^2$

for some positive constants k *and* M, *and all* $u \in \mathscr{D}(h^{(n)})$.

Proof. We assume firstly that conditions (C1)–(C5) hold. The modifications that need be made when (C5)′ replaces (C5) will be noted at the end.

We begin by observing that (3.12) follows easily from (C1) and [**6**, Proposition 1], a consequence of (3.3) or (3.4). The constants k and M depend only on n and q_2. Clearly then, $h^{(n)}$ is densely defined, semi-bounded and symmetric. Also, using [**6**, Proposition 1], it is not hard to see that $h^{(n)}$ is closed. Hence, associated with the form $h^{(n)}$ is a unique self-adjoint operator $S^{(n)}$ with domain $\mathscr{D}(S^{(n)}) \subset \mathscr{D}(h^{(n)})$, having the same lower bound as $h^{(n)}$, and satisfying

$$(3.13) \quad h^{(n)}(u, v) = (S^{(n)}u, v)$$

for all $u \in \mathscr{D}(S^{(n)})$ and $v \in \mathscr{D}(h^{(n)})$.

Let $\mathscr{M}^{(n)}$ denote the formal operator

$$\mathscr{M}^{(n)} = - \sum_{j=1}^{m} (\partial_j - ib_j(x))^2 + q_1^{(n)} + q_2,$$

and let $l^{(n)}$ denote the form defined by

$$l^{(n)}[u, v] = \sum_{j=1}^{m} \int_{\mathbf{R}^m} |D_j u(x)|^2 dx + \int_{\mathbf{R}^m} q_1^{(n)}(x)|u|^2 dx$$
$$+ \int_{\mathbf{R}^m} q_2(x)|u|^2 dx$$

with domain $\mathscr{D}(l^{(n)}) = H_b{}^1 \cap \mathscr{D}((q_1^{(n)} + q^*)^{1/2})$. Finally, denote by K and J respectively the Hilbert spaces $\mathscr{D}(h^{(n)})$ and $\mathscr{D}(l^{(n)})$ with norms

$$\{(h^{(n)} + c^2)[u]\}^{1/2} \geqq \|u\| \quad \text{and} \quad \{(l^{(n)} + c^2)[u]\}^{1/2} \geqq \|u\|,$$

where c is some suitably large constant. Since there exist constants λ_1 and λ_2 (depending only on n) such that

$$\lambda_1|\xi|^2 \leqq \sum_{j,k=1}^{m} a_{jk}{}^{(n)} \xi_j \bar{\xi}_k \leqq \lambda_2|\xi|^2$$

for all complex m-vectors ξ, it is clear from [**6**, Proposition 1] that the two norms defined above are equivalent norms on $H_b{}^1 \cap \mathscr{D}((q_1^{(n)} + q^*)^{1/2})$. We now note that $C_0^\infty(\mathbf{R}^m)$ is a form core for $l^{(n)}$. When $(b_j) = 0$, this is essentially just Lemma 4.6b, p. 349 of [**5**]. It is a simple matter, using Lemma 1, to adapt this proof for magnetic potentials satisfying (C3). Finally, as the spaces K and J are identical, $C_0^\infty(\mathbf{R}^m)$ is also a core for $h^{(n)}$.

We are now going to show that

$$(3.14) \quad S^{(n)} \subset T^{(n)}.$$

To this end let $u \in \mathscr{D}(S^{(n)})$. Then, observing that $q_1^{(n)}u^2 \in L_{loc}{}^1$, we see from (3.13) that for every $\phi \in C_0^\infty(\mathbf{R}^m)$

$$(S^{(n)}u, \bar{\phi}) = h^{(n)}(u, \bar{\phi}) = \langle T^{(n)}u, \phi \rangle$$

where the expression in angular brackets denotes the value of the distribution $T^{(n)}u$ at ϕ; (3.14) now follows. To show that equality holds in (3.14) it suffices to show that $\mathscr{D}(T^{(n)}) \subset \mathscr{D}(S^{(n)})$. Accordingly, let $u \in \mathscr{D}(T^{(n)})$. We shall now proceed in several steps.

Step 1. Let us assume first that u has compact support. Then $u \in \mathscr{D}(h^{(n)})$, and

$$(3.15) \quad (T^{(n)}u, \phi) = h^{(n)}[u, \phi]$$

for all $\phi \in C_0^\infty(\mathbf{R}^m)$, which is a core for $h^{(n)}$. Thus by [**5**, Theorem 2.1 (iii), p. 322] $u \in \mathscr{D}(S^{(n)})$.

Step 2. Let $\zeta \in C_0^\infty(\mathbf{R}^m)$ satisfy $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ for $|x| \leq 1$, and $\zeta(x) = 0$ for $|x| \geq 2$. For $R > \max\{r_n + \epsilon_n, 1\}$ put $\zeta_R(x) = \zeta(x/R)$. Observe that there exists a positive number A, independent of x and R such that $|\partial_j \zeta_R(x)| \leq AR^{-1}$, and $|\partial_j \partial_k \zeta_R(x)| \leq AR^{-2}$ for $1 \leq j, k \leq m$. From the distributional identity

$$(3.16) \quad \mathscr{T}^{(n)}(\zeta_R u) = \zeta_R \mathscr{T}^{(n)} - 2 \sum_{j=1}^m \partial_j \zeta_R \cdot D_j u - u \Delta \zeta_R$$

where $\Delta = \sum_{j=1}^m \partial_j^2$, it is easy to see that $\zeta_R u \in \mathscr{D}(T^{(n)})$, and since $\zeta_R u$ has compact support it follows from Step 1 that $\zeta_R u \in \mathscr{D}(S^{(n)})$. Hence from (3.11) and (3.12)

$$(3.17) \quad (T^{(n)}(\zeta_R u), \zeta_R u) = h^{(n)}[\zeta_R u, \zeta_R u]$$

$$(3.18) \quad \geq k\|\zeta_R u\|_{H_b^1}{}^2 - (q^* + M)\|\zeta_R u\|^2.$$

Fixing our attention upon (3.16), we observe that

$$(3.19) \quad (a) \quad |(\zeta_R T^{(n)}u, \zeta_R u)| \leq \|T^{(n)}u\| \, \|u\|;$$

$$(b) \quad |(\partial_j \zeta_R \cdot D_j u, \zeta_R u)| = |(\partial_j \zeta_R \cdot D_j(\zeta_R u), u) - ((\partial_j \zeta_R)^2 u, u)|$$

$$\leq \frac{A}{2} \int_{\mathbf{R}^m} 2 \cdot \frac{|D_j(\zeta_R u)|}{R} \cdot |u| dx + A^2 \|u\|^2$$

$$\leq \frac{A}{2R^2} \int_{\mathbf{R}^m} |D_j(\zeta_R u)|^2 dx + \left(A^2 + \frac{A}{2}\right)\|u\|^2$$

and hence

$$(3.20) \quad \left|\left(2 \sum_{j=1}^m \partial_j \zeta_R \cdot D_j u, \zeta_R u\right)\right| \leq \frac{A}{R^2} \|\zeta_R u\|_{H_b^1}{}^2 + 2m(A^2 + \tfrac{1}{2}A)\|u\|^2;$$

$$(3.21) \quad (c) \quad |(u \, \Delta \, \zeta_R, \zeta_R u)| \leq A\|u\|^2.$$

From these results and (3.18) we conclude that

$$k\|\zeta_R u\|_{H_b^1}{}^2 \leq \|T^{(n)}u\| \, \|u\| + C_1\|u\|^2 + \frac{A}{R^2} \|\zeta_R u\|_{H_b^1}{}^2,$$

where k, A, and C_1 do not depend on R. Consequently, for R large enough we have

$$(3.22) \quad \|\zeta_R u\|_{H_b{}^1}{}^2 \leqq C_2,$$

where C_2 does not depend on R. Returning now to (3.17) we have, using (3.16) and the estimates (3.19)–(3.21) again,

$$|h^{(n)}[\zeta_R u]| \leqq \|T^{(n)} u\| \, \|u\| + C_3 \|u\|^2 + \frac{A}{R^2} \|\zeta_R u\|_{H_b{}^1}{}^2 \leqq C_4$$

by (3.22), for all R large enough, where C_3 and C_4 are also independent of R. Since $\zeta_R u \to u$ in $\mathscr{H}$, it therefore follows from [**5**, Theorem 1.16, p. 315] that $u \in \mathscr{D}(h^{(n)})$.

Step 3. Observe now that by (3.15) and the method of Step 1, $u \in \mathscr{D}(S^{(n)})$, as required.

Our final task is to discuss the consequences of replacing (C5) by (C5)′. Here we note first that $\mathscr{D}(h^{(n)}) = H_b{}^1$, provided that q_2 is a small enough perturbation of q_1; this is certainly assured by (3.5). It is then trivial that $C_0{}^\infty(\mathbf{R}^m)$ is a form core for $h^{(n)}$. Thus no other restrictions on q_2 are needed.

The next result, though simple in form, is crucial in the sequel as it enables us to relate the properties of the truncated operators $T^{(n)}$ with those of T. For $n = 1, 2, \ldots$, let $\phi_n \in C_0{}^\infty(\mathbf{R}^m)$ be chosen so that $0 \leqq \phi_n \leqq 1$ and

$$(3.23) \quad \phi_n(x) = 1 \text{ if } |x| \leqq r_n + (1/3)\epsilon_n$$
$$= 0 \text{ if } |x| \geqq r_n + (2/3)\epsilon_n.$$

By analogy with the operator T_0, let $\mathscr{D}(T_0{}^{(n)})$ denote the set of compact support functions $u \in \mathscr{D}(T^{(n)})$; for $u \in \mathscr{D}(T_0{}^{(n)})$ set $T_0{}^{(n)} u = \mathscr{T}^{(n)} u$.

LEMMA 3. *Let $u \in \mathscr{D}(T) \cup \mathscr{D}(T^{(n)})$, and let ϕ_n be defined as above. Then*

$$\phi_n u \in \mathscr{D}(T_0) \cap \mathscr{D}(T_0{}^{(n)}) \quad \text{and} \quad T_0{}^{(n)}(\phi_n u) = T_0(\phi_n u).$$

Proof. We note first the following (distributional) identity:

$$(3.24) \quad \mathscr{T}(\phi_n u) = \phi_n \mathscr{T}(u) - \sum_{j,k=1}^{m} \partial_k \phi_n \cdot D_j(a_{jk} u) - \sum_{j,k=1}^{n} a_{jk} \partial_j \phi_n \cdot D_k u$$
$$- \sum_{j,k=1}^{m} a_{jk} \cdot \partial_j \partial_k \phi_n \cdot u.$$

Let $u \in \mathscr{D}(T)$ (the proof is similar if $u \in \mathscr{D}(T^{(n)})$). Clearly $\phi_n u \in H_b{}^1 \cap H^1$. Also, it follows easily from (C2) and Theorems 3.13 and 3.15 of [**1**] that

$$\partial_k \phi_n \cdot D_j(a_{jk} u) \in \mathscr{H}.$$

The assertion now follows immediately from (3.24).

We now have

LEMMA 4. T_0 and $T_0^{(n)}$ are densely defined symmetric linear operators in $\mathcal{H}$.

Proof. To see that T_0 is densely defined, let $f \in \mathcal{H}$, and choose $\epsilon > 0$. Choose ϕ_n so that $\|\phi_n f - f\| < \epsilon/2$. Since $T^{(n)}$ is densely defined, there is a $g \in \mathcal{D}(T^{(n)})$ with $\|g - f\| < \epsilon/2$. Then

$$\|\phi_n g - f\| \leq \|\phi_n g - \phi_n f\| + \|\phi_n f - f\| < \epsilon,$$

and $\phi_n g \in \mathcal{D}(T_0)$, by Lemma 3. Similarly we can show that $T_0^{(n)}$ is densely defined; and it is clear that $T_0^{(n)}$ is symmetric. To prove that T_0 is symmetric, let $u, v \in \mathcal{D}(T_0)$ be chosen arbitrarily, and choose n so that

$$\operatorname{supp} v \cup \operatorname{supp} u \subset B_{r_n}.$$

Then, noting that $T_0 u = T_0(\phi_n u)$ and $T_0 v = T_0(\phi_n v)$ on B_{r_n},

$$\begin{aligned}
(T_0 u, v) &= (T_0(\phi_n u), \phi_n v) \\
&= (T_0^{(n)}(\phi_n u), \phi_n v) \quad \text{by Lemma 3,} \\
&= (\phi_n u, T_0^{(n)}(\phi_n v)) \quad \text{since } T_0^{(n)} \text{ is symmetric,} \\
&= (u, T_0 v).
\end{aligned}$$

Thus T_0 is closable, and $T_{\min}^*$ is a well-defined operator in $\mathcal{H}$.

LEMMA 5. For all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(T_0)$ we have

$$(3.25) \quad (Tu, v) = (u, T_0 v).$$

If we choose n so that supp $v \subset B_{r_n}$ and make appropriate use of the function ϕ_n, then the proof of this lemma is identical with that of Lemma 4 of [9].

LEMMA 6. The operator $T_0^{(n)}$ is essentially self-adjoint in $\mathcal{H}$. i.e.,

$$\overline{T_0^{(n)}} = T^{(n)} = T_0^{(n)*}.$$

Proof. Since $T^{(n)}$ is self-adjoint in $\mathcal{H}$, it is sufficient to show that $T^{(n)} \subset \overline{T_0^{(n)}}$. Let ζ denote the function in $C_0^\infty(\mathbf{R}^m)$ defined earlier. For $s > r_n + \epsilon_n$ define $\zeta_s(x) = \zeta(x/s)$. Then, as $s \to \infty$, $\zeta_s \to 1$, $\partial_k \zeta_s \to 0$, and $\partial_j \partial_k \zeta_s \to 0$ boundedly on $\mathbf{R}^m$. Note also that

$$\mathcal{T}^{(n)}(\zeta_s u) = \zeta_s \mathcal{T}^{(n)}(u) - \sum_{j,k=1}^m a_{jk}^{(n)} \cdot \partial_j \partial_k \zeta_s \cdot u$$

$$- \sum_{j,k=1}^m a_{jk}^{(n)} \cdot \partial_j \zeta_s \cdot D_k u - \sum_{j,k=1}^m \partial_k \zeta_s \cdot D_j(a_{jk}^{(n)} u).$$

The proof now follows that of [9, Lemma 5].

LEMMA 7. *If $u \in \mathscr{D}(T_0{}^*)$, then for each $n \geqq 1$,*

$$\phi_n u \in \mathscr{D}(T_0{}^*) \cap \mathscr{D}(T_0{}^{(n)*}) \quad and \quad T_0{}^*(\phi_n u) = T_0{}^{(n)*}(\phi_n u).$$

Proof. Let $u \in \mathscr{D}(T_0{}^*)$ and set $T_0{}^* u = f$. Then

$$(3.26) \quad (u, T_0 v) = (f, v)$$

for all $v \in \mathscr{D}(T_0)$. Since $\phi_n v \in \mathscr{D}(T_0)$ for any $v \in \mathscr{D}(T_0)$ by Lemma 3, it follows that

$$(3.27) \quad (u, T_0(\phi_n v)) = (f, \phi_n v)$$

for all $v \in \mathscr{D}(T_0)$. On the other hand, if $v \in \mathscr{D}(T_0{}^{(n)})$, it also follows from Lemma 3 that

$$\phi_n v \in \mathscr{D}(T_0) \cap \mathscr{D}(T_0{}^{(n)})$$

and that

$$T_0(\phi_n v) = T_0{}^{(n)}(\phi_n v).$$

Thus

$$(3.28) \quad (u, T_0{}^{(n)}(\phi_n v)) = (u, T_0(\phi_n v)) = (f, \phi_n v)$$

by (3.26), for all $v \in \mathscr{D}(T_0{}^{(n)})$. Now, let K denote the Hilbert space associated with the form $h^{(n)}$ as defined earlier, and with norm

$$(h^{(n)}[u] + c^2 \|u\|^2)^{1/2} \geqq \|u\|,$$

where $c^2 \geqq q^* + 1 + M$. Then denoting the anti-dual of K by K^* we have,

$$\mathscr{D}(T_0{}^{(n)}) \subset \mathscr{D}(T^{(n)}) \subset K \subset H_b{}^1 \subset H_b{}^{-1} \subset K^*$$

where the inclusions are continuous, and we may regard $\mathscr{D}(T_0{}^{(n)})$ as a dense subset of K^*. It is well-known, and easily confirmed (see [**7**, Lemma 2]) that $T^{(n)}$ can be extended to a continuous map $H^{(n)}$ on K to K^*, where $H^{(n)}$ is actually a restriction of $\mathscr{T}^{(n)}$. It is also known that $c^2 + H^{(n)}$ maps K onto K^* bicontinuously.

From (3.28) we have

$$(3.29) \quad (\phi_n u, T_0{}^{(n)} v) = (f^*, v)_{K^*}$$

for all $v \in \mathscr{D}(T_0{}^{(n)})$, where $(\cdot, \cdot)_{K^*}$ denotes the inner product in K^*, and f^* is given by

$$(3.30) \quad f^* = \phi_n f - \sum_{j,k=1}^{m} D_k(a_{jk} u) \cdot \partial_j \phi_n - \sum_{j,k=1}^{m} a_{jk} D_j(\partial_k \phi_n \cdot u).$$

Clearly $f^* \in H_b{}^{-1} \subset K^*$ by an argument similar to [**6**, p. 143]. Note that, by the same reasoning we also have from (3.27) that

$$(3.31) \quad (\phi_n u, T_0 v) = (f^*, v)_{K^*}$$

for all $v \in \mathscr{D}(T_0)$. Returning now to (3.29) we see that

$$\phi_n u \in \mathscr{D}(T_0^{(n)'}) \quad \text{and} \quad T_0^{(n)'}(\phi_n u) = f^*,$$

where $T_0^{(n)'}$ denotes the adjoint of $T_0^{(n)}$ regarded as an operator in K^*. Set

$$(T_0^{(n)} + c^2)'(\phi_n u) = g \in K^*.$$

By the preceding remarks there exists a function $z \in K$ with

$$(H^{(n)} + c^2)(z) = g.$$

Consequently,

$$
\begin{aligned}
(\phi_n u, (T_0^{(n)} + c^2)v) &= ((T_0^{(n)} + c^2)'(\phi_n u), v)_{K^*} \\
&= ((H^{(n)} + c^2)(z), v)_{K^*} \\
&= (h^{(n)} + c^2)(z, v)
\end{aligned}
$$

(by continuity of $H^{(n)}$ on K)

$$= (z, (T_0^{(n)} + c^2)v)$$

for all $v \in \mathscr{D}(T_0^{(n)})$. Hence

$$(\phi_n u - z, (T_0^{(n)} + c^2)v) = 0 \quad \text{for all} \quad v \in \mathscr{D}(T_0^{(n)}).$$

Now, since $T_0^{(n)}$ is essentially self-adjoint, it follows ([**12**, Theorem X.26]) that the range of $T_0^{(n)} + c^2$ is dense in $\mathscr{H}$; hence we have that $\phi_n u = z \in H_b{}^1$, and therefore that $u \in H_{\text{loc}}{}^1$. Thus, from (3.29) and (3.30),

$$\phi_n u \in \mathscr{D}(T_0^{(n)*}) \quad \text{and} \quad T_0^{(n)*}(\phi_n u) = f^* \in \mathscr{H}.$$

Finally, from (3.31)

$$\phi_n u \in \mathscr{D}(T_0^*) \quad \text{and} \quad T_0^*(\phi_n u) = f^* = T_0^{(n)*}(\phi_n u).$$

We now complete the proof of the theorem stated earlier.

Proof of theorem. By Lemmas 4 and 5 it is sufficient to show that

(3.32) $T_0^* \subset T.$

Let $u \in \mathscr{D}(T_0^*)$ and set $T_0^* u = f \in \mathscr{H}$. Choose n arbitrarily. By Lemma 7,

$$\phi_n u \in \mathscr{D}(T_0^*) \cap \mathscr{D}(T_0^{(n)*}) \quad \text{and}$$

(3.33) $T_0^*(\phi_n u) = T_0^{(n)*}(\phi_n u).$

Since $T_0^{(n)*} = T^{(n)}$ by Lemma 6, it follows that $u \in H^1(B_{r_n})$ and $qu^2 \in L^1(B_{r_n})$ for each n. We now show that for almost all $x \in B_{r_n}$,

(3.34) $(T_0^* u)(x) = T_0^*(\phi_n u)(x).$

Clearly, as $\phi_n = 1$ on $B = B_{r_n + (1/4)\epsilon_n}$

$$(T_0^* u - T_0^*(\phi_n u), v)_{L^2(B)} = 0$$

for all $v \in \mathscr{D}(T_0)$ with supp $v \subset B$. Thus (3.34) will be established if we can show that the set of all such functions v is dense in $L^2(B)$. Let $w \in L^2(B)$ and define w^* on $\mathbf{R}^m$ to be equal to w in B, and zero eslewhere. Then given $\epsilon > 0$ there is a $g \in \mathscr{D}(T_0)$ with $\|g - w^*\| < \frac{1}{2}\epsilon$. Choose $\psi \in C_0^\infty(\mathbf{R}^m)$ so that

$$0 \leqq \psi \leqq 1, \text{ supp } \psi \subset B \quad \text{and} \quad \psi = 1 \text{ on } B_{r_n + (1/4)\epsilon_n - \delta}$$

where δ is chosen small enough to ensure that

$$\|\psi w - w\|_{L^2(B)} < \frac{1}{2}\epsilon.$$

Clearly $\psi g \in \mathscr{D}(T_0)$ (by the method of Lemma 3), supp $(\psi g) \subset B$, and

$$\begin{aligned}
\|\psi g - w\|_{L^2(B)} &= \|\psi g - w^*\| \\
&\leqq \|\psi g - \psi w^*\| + \|\psi w - w\|_{L^2(B)} \\
&< \epsilon.
\end{aligned}$$

Finally, for almost all $x \in B_{r_n}$ we have by (3.33) and (3.34)

$$\begin{aligned}
f(x) = T_0^* u(x) = T_0^*(\phi_n u)(x) = T_0^{(n)*}(\phi_n u)(x) &= T^{(n)}(\phi_n u)(x) \\
&= \mathscr{T}^{(n)}(\phi_n u)(x) = \mathscr{T}(u)(x).
\end{aligned}$$

Since $f \in \mathscr{H}$ and n was chosen arbitrarily it follows that $\mathscr{T} u \in \mathscr{H}$, $u \in H_{\text{loc}}^1$ and $q u^2 \in L_{\text{loc}}^1$. Thus $u \in \mathscr{D}(T)$.

As in [**9**] we have

COROLLARY. *The operator T is self-adjoint if and only if it is symmetric. In this case T_0 is essentially self-adjoint and defines a unique self-adjoint operator $(\bar{T}_0)$ in $\mathscr{H}$.*

Thus, by the reasoning of [**9**], it follows that virtually all of the known criteria for essential self-adjointness (and in particular those given in [**2**, Theorem 2; **4**; **8**]) are automatically valid under the considerably weaker a priori smoothness assumptions (C1)–(C4), and (C5) or (C5)′.

4. Proof of lemma 1. We let $\mathscr{D}$ denote the space of test functions on $\mathbf{R}^m$ with the Schwartz topology and $\mathscr{D}'$ the corresponding space of distributions. Let

$$j(x) = \alpha \exp\{-(1 - |x|^2)^{-1}\}$$

for $|x| < 1$ and $j(x) = 0$ for $|x| \geqq 1$, where α is chosen so that

$$\int_{\mathbf{R}^m} j(x) dx = 1.$$

For $\delta > 0$, let

$$j_\delta(x) = \delta^{-m}j(x/\delta).$$

Note that $j_\delta(x) \in C^\infty(\mathbf{R}^m)$, $j_\delta(x) = 0$ for $|x| \geqq \delta$, and

$$\int_{\mathbf{R}^m} j_\delta(x)dx = 1.$$

Let

(4.1) $\quad u^\delta(x) = J_\delta u(x) = \int_{\mathbf{R}^m} j_\delta(x - y)u(y)dy,$

and for $\epsilon > 0$ let

$$u_\epsilon = (|u|^2 + \epsilon^2)^{1/2} \quad \text{and} \quad u_\epsilon{}^\delta = (|u^\delta|^2 + \epsilon^2)^{1/2}.$$

Then an argument similar to [**6**, Section 5, Lemma 3] shows that

(4.2) $\quad \mathscr{L}_0(u_\epsilon{}^\delta) \geqq \frac{1}{2}\left[\frac{\overline{u^\delta}}{u_\epsilon{}^\delta}\mathscr{L}(u^\delta) + \frac{u^\delta}{u_\epsilon{}^\delta}\overline{\mathscr{L}(u^\delta)}\right]$

where, for $v \in H_{\mathrm{loc}}{}^1$, $\overline{\mathscr{L}(v)}$ denotes the distribution defined by

$$\langle \overline{\mathscr{L}(v)}, \phi\rangle = \overline{\langle \mathscr{L}(v), \bar\phi\rangle}$$

(here $\phi \in \mathscr{D}$ and (S, ϕ) denotes the value of the distribution S at ϕ). We now complete the proof in two steps.

Step 1. Here we hold ϵ fixed and examine the behaviour of (4.2) as $\delta \to 0$. Firstly, it is clear that there exists a null sequence of values of δ, $\{\delta_p : p \geqq 1\}$ such that $u^p(x) = u^{\delta_p}(x) \to u(x)$ and $u_\epsilon{}^p(x) \to u_\epsilon(x)$ a.e. pointwise in $\mathbf{R}^m$ as $p \to \infty$. Moreover, from the relations

$$\partial_k u_\epsilon{}^p = \frac{1}{2}\left[\frac{\overline{u^p}}{u_\epsilon{}^p}\partial_k u^q + \frac{u^p}{u_\epsilon{}^p}\partial_k u^p\right]$$

and $\partial_k u^p = (\partial_k u)^p$ (where $(\ )^p \equiv (\ )^{\delta_p}$), it follows that

$$\partial_k u_\epsilon{}^p \to \mathrm{Re}\left[\frac{\bar{u}}{u_\epsilon}\partial_k u\right]$$

in $L_{\mathrm{loc}}{}^2$ as $p \to \infty$. Hence $u_\epsilon \in H_{\mathrm{loc}}{}^1$, $u_\epsilon{}^p \to u_\epsilon$ in $H_{\mathrm{loc}}{}^1$ as $p \to \infty$, and

(4.3) $\quad \partial_k u_\epsilon = \mathrm{Re}\left[\frac{\bar{u}}{u_\epsilon}\partial_k u\right].$

Turning now to the left-hand side of (4.2) we see that for $\phi \in \mathscr{D}$,

$$\langle \mathscr{L}_0(u_\epsilon{}^p), \phi\rangle = -\sum_{j,k=1}^m \int_{\mathbf{R}^m} a_{jk} \cdot \partial_k(u_\epsilon{}^p) \cdot \partial_j\phi.$$

Clearly then, $\mathscr{L}_0(u_\epsilon{}^p) \to \mathscr{L}_0(u_\epsilon)$ in $\mathscr{D}'$ as $p \to \infty$. Consider now the first term inside the bracket on the right-hand side of (4.2); i.e., we consider

the distribution

$$(4.4) \quad \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} [\partial_j(a_{jk}\partial_k u^p) - i\partial_j(a_{jk}b_k u^p) - ia_{jk}b_j\partial_k(u^p) - a_{jk}b_j b_k u^p].$$

It is clear that

$$(4.5) \quad \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} a_{jk}b_j b_k u^p \rightarrow \frac{\bar{u}}{u_\epsilon} a_{jk}b_j b_k u \quad \text{in } \mathscr{D}'$$

and

$$(4.6) \quad \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} a_{jk}b_j\partial_k(u^p) \rightarrow \frac{\bar{u}}{u_\epsilon} a_{jk}b_j\partial_k u \quad \text{in } \mathscr{D}'$$

as $p \rightarrow \infty$. Also, for $\phi \in \mathscr{D}$ we have

$$\left\langle \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} \partial_j(a_{jk}b_k u^p), \phi \right\rangle = -\left\langle a_{jk}b_k u^p, \partial_j\left(\phi \cdot \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}}\right) \right\rangle$$

$$= -\int_{\mathbf{R}^m} a_{jk} \cdot b_k \cdot u^p \cdot \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} \cdot \partial_j\phi$$

$$- \int_{\mathbf{R}^m} a_{jk} \cdot b_k \cdot u^p \cdot \left\{ \frac{\partial_j(u^p)}{u_\epsilon^{\overline{p}}} - \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} \cdot \frac{\partial_j(u_\epsilon^{\overline{p}})}{u_\epsilon^{\overline{p}}} \right\} \cdot \phi.$$

Since $u^p \rightarrow u$ and $u_\epsilon^{\overline{p}} \rightarrow u_\epsilon$ in H_{loc}^1, it follows that

$$(4.7) \quad \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} \partial_j(a_{jk}b_k u^p) \rightarrow \frac{\bar{u}}{u_\epsilon} \partial_j(a_{jk}b_k u)$$

in $\mathscr{D}'$. In a similar fashion, one can show that as $p \rightarrow \infty$,

$$(4.8) \quad \frac{\overline{u^p}}{u_\epsilon^{\overline{p}}} \partial_j(a_{jk}\partial_k u^p) \rightarrow \frac{\bar{u}}{u_\epsilon} \partial_j(a_{jk}\partial_k u)$$

in $\mathscr{D}'$. Thus from (4.4)–(4.8) we see that

$$\frac{u^p}{u_\epsilon^{\overline{p}}} \mathscr{L}(u^p) \rightarrow \frac{\bar{u}}{u_\epsilon} \mathscr{L}(u)$$

in $\mathscr{D}'$ as $p \rightarrow \infty$. One also has a similar result for the second term in (4.2). Combining these then gives

$$(4.9) \quad \mathscr{L}_0(u_\epsilon) \geqq \tfrac{1}{2}\left[\frac{\bar{u}}{u_\epsilon}\mathscr{L}(u) + \frac{u}{u_\epsilon}\overline{\mathscr{L}(u)} \right] = \text{Re}\left[\frac{\bar{u}}{u_\epsilon}\mathscr{L}(u) \right].$$

   Step 2. Our final task is to investigate (4.9) as $\epsilon \rightarrow 0$. Clearly, for $\phi \in \mathscr{D}$, we have $\langle \partial_k u_\epsilon, \phi \rangle \rightarrow \langle \partial_k|u|, \phi \rangle$ as $\epsilon \rightarrow 0$. On the other hand it follows from (4.3) that $\partial_k(u_\epsilon) \rightarrow \text{Re}[\text{sgn } \bar{u}\cdot\partial_k u]$ in L_{loc}^2 as $\epsilon \rightarrow 0$. Hence we have

$$\partial_k|u| = \text{Re}[\text{sgn } \bar{u}\cdot\partial_k u] \text{ a.e. in } \mathbf{R}^m, \text{ and}$$

$$\partial_k u_\epsilon \rightarrow \partial_k|u| \text{ in } L_{\text{loc}}^2 \text{ as } \epsilon \rightarrow 0.$$

A simple calculation now shows that $\mathscr{L}_0(u_\epsilon) \to \mathscr{L}_0(|u|)$ in $\mathscr{D}'$ as $\epsilon \to 0$. It is also clear that

$$\mathrm{Re}\left[\frac{\bar{u}}{u_\epsilon} \cdot \mathscr{L}(u)\right] \to \mathrm{Re}\left[\mathrm{sgn}\,\bar{u} \cdot \mathscr{L}(u)\right]$$

in $\mathscr{D}'$ as $\epsilon \to 0$. The assertion of the lemma now follows from (4.9).

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University of the Witwatersrand,
Johannesburg, South Africa