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EVERY AF-ALGEBRA IS MORITA EQUIVALENT TO A GRAPH ALGEBRA

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We show how to modify any Bratteli diagram E for an AF-algebra A to obtain a Bratteli diagram KE for A whose graph algebra $C^*(KE)$ contains both A and $C^*(E)$ as full corners.

An elegant theorem of Drinen says that every AF-algebra A is isomorphic to a corner in a graph algebra [**3**, Theorem 1], and hence is Morita equivalent to the graph algebra. The graph in question is a Bratteli diagram for A, but it needs to be a carefully chosen one; two constructions of such a diagram were described in [**3**], one attributed to Kumjian. Here we show that applying Kumjian's construction to an arbitrary Bratteli diagram E for A gives a graph KE whose C^*-algebra contains both A and $C^*(E)$ as full corners, so that A is Morita equivalent to the C^*-algebra $C^*(E)$ of the original Bratteli diagram E.

A *directed graph* E consists of countable sets E^0 of vertices and E^1 of edges, along with functions $r, s : E^1 \to E^0$ which map edges to their range and source vertices. The graph is *row-finite* if each vertex emits at most finitely many edges. Given a row-finit graph E, a *Cuntz-Krieger E-family* in a C^*-algebra consists of a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*:

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{e \in s^{-1}(v)} s_e s_e^* \text{ whenever } s^{-1}(v) \neq \emptyset.$$

The graph algebra $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$ ([**4**, Theorem 1.2]). We denote by E^* the set of all finite paths in E; that is, sequences of edges $\mu_1\mu_2\dots\mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leqslant i < n$. We include the vertices as paths of length zero. Given $\mu = \mu_1\mu_2\dots\mu_n \in E^*$, define $s_\mu := s_{\mu_1}s_{\mu_2}\dots s_{\mu_n}$. It follows from [**4**, Lemma 1.1] that

$$C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}.$$

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A *Bratteli diagram* is a directed graph E such that:

1. E^0 is the disjoint union of finite sets $\{V_n\}$,
2. every edge with source in V_n has range in V_{n+1}, and
3. each $v \in E^0$ is labelled with a positive integer d_v which satisfies

$$d_v \geqslant \sum_{e \in r^{-1}(v)} d_{s(e)}.$$

We say that E is a Bratteli diagram for a sequence of C^*-algebras $A_1 \subset A_2 \subset \dots$ if each A_n is isomorphic to $\bigoplus_{v \in V_n} M_{d_v}(\mathbb{C})$ and the embedding of each $M_{d_v}(\mathbb{C}) \subset A_n$ in each $M_{d_w}(\mathbb{C}) \subset A_{n+1}$ scales the trace by $\#\big(s^{-1}(v) \cap r^{-1}(w)\big)$. We say that E is a Bratteli diagram for an AF-algebra A if there exists a sequence of C^*-subalgebras $\{A_n\}$ of A such that $A = \overline{\cup A_n}$ and E is a Bratteli diagram for $\{A_n\}$.

Theorem 1. *Let E be a Bratteli diagram for an* AF*-algebra A. Then there exists a Bratteli diagram KE for A such that $C^*(KE)$ contains A and $C^*(E)$ as complementary full corners.*

The projection p defining the corner is the sum $p = \sum_{v \in S} p_v$ where $S \subset KE^0$; this sum converges strictly to a projection in $M\big(C^*(KE)\big)$ by [**1**, Lemma 1.1]. Crucial for us is the observation that for $\mu,\ \nu \in KE^*$,

$$ps_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } s(\mu) \in S \\ 0 & \text{otherwise} \end{cases}$$

so that

$$pC^*(KE)p = \overline{\text{span}}\big\{s_\mu s_\nu^* : s(\mu),\ s(\nu) \in S,\ r(\mu) = r(\nu)\big\}.$$

Proof of the Theorem: For $n > 0$, denote by V_n the set of vertices on the n^{th} level of E, and let $V_0 = \emptyset$. For each $v \in E^0$, let d_v be the rank of the matrix algebra corresponding to v. For every vertex $v \in E^0$, calculate $\sigma_v := d_v - \sum_{e \in r^{-1}(v)} d_{s(e)}$. We define $KE^0 = \bigcup_{n=0}^{\infty} KV_n$, where

$$KV_n := \begin{cases} V_n & \text{if } \sigma_v = 0 \ \text{ for all } \ v \in V_{n+1} \\ V_n \cup \{w_n\} & \text{if } \sigma_v > 0 \ \text{ for some } \ v \in V_{n+1}, \end{cases}$$

and define KE^1 to be E^1 together with, for every w_n and $v \in V_{n+1}$, σ_v edges from w_n to v. Denote by S the set $KE^0 \backslash E^0 = \cup\{w_n\}$, and set $d_w = 1$ for all $w \in S$. Constructing KE in this fashion ensures that for all $v \in KE^0$, the number of paths beginning in S and ending at v is d_v.

Since E is a Bratteli diagram for A, there is an increasing sequence of C^*-subalgebras F_n of A such that $A = \overline{\cup F_n}$ and E is a Bratteli diagram for the sequence $\{F_n\}$. For those n where $KV_n \neq V_n$, we define a subalgebra F_n' of A by $F_0' := \mathbb{C}1$ and

$$F_n' := F_n \oplus \mathbb{C}\big(1_{F_{n+1}} - 1_{F_n}\big) \cong \bigoplus_{v \in V_n} M_{d_v}(\mathbb{C}) \oplus \mathbb{C} \quad \text{for} \quad n > 0.$$

For all other n, define $F_n' = F_n$. The graph KE is then a Bratteli diagram for the sequence $\{F_n'\}$. Since $F_n \subseteq F_n' \subseteq F_{n+1}$ for all n, we have $\overline{\cup F_n'} = \overline{\cup F_n} = A$; thus KE is a Bratteli diagram for A.

Let $\{s_e, p_v\}$ be the universal Cuntz-Krieger KE-family generating $C^*(KE)$. Define a projection $p \in M\big(C^*(KE)\big)$ by $p := \sum_{v \in S} p_v$. We aim to show that the corner $pC^*(KE)p$ is isomorphic to A. Since two algebras with the same Bratteli diagram are isomorphic ([**2**, Proposition III.2.7]), we can achieve this by identifying a sequence of subalgebras of $pC^*(KE)p$ for which E is a Bratteli diagram and whose union is dense in $pC^*(KE)p$. For each $n > 0$ define $D_n := \operatorname{span}\{D^v : v \in V_n\}$, where

$$D^v := \operatorname{span}\big\{s_\mu s_\nu^* : \mu,\, \nu \in KE^*,\, s(\mu),\, s(\nu) \in S,\, r(\mu) = r(\nu) = v\big\}$$

for each $v \in KE^0$. Note that

$$pC^*(KE)p = \overline{\operatorname{span}}\big\{s_\mu s_\nu^* : \mu,\, \nu \in KE^*,\, s(\mu),\, s(\nu) \in S,\, r(\mu) = r(\nu)\big\} = \overline{\cup D_n}.$$

Given $v \in E^0$ and paths μ, ν, α, β with source in S and range v, observe that none of μ, ν, α, β can extend any other since KE contains no loops; [**4**, Lemma 1.1] then gives

$$s_\mu s_\nu^* s_\alpha s_\beta^* = \begin{cases} s_\mu s_\beta^* & \text{if } \nu = \alpha \\ 0 & \text{otherwise}. \end{cases}$$

Also, $(s_\mu s_\nu^*)^* = s_\nu s_\mu^*$, so

$$\big\{s_\mu s_\nu^* : \mu,\, \nu \in KE^*,\, s(\mu),\, s(\nu) \in S,\, r(\mu) = r(\nu) = v\big\}$$

is a family of matrix units. Since there are d_v paths μ with $s(\mu) \in S$ and $r(\mu) = v$, D^v is isomorphic to $M_{d_v}(\mathbb{C})$. Further, note that for distinct v, $w \in V_n$, no path ending at v may extend one ending at w, so $D^v D^w = 0$ and $D_n = \bigoplus_{v \in V_n} D^v \cong \bigoplus_{v \in V_n} M_{d_v}(\mathbb{C})$. It remains only to check that the embedding of each D_n in D_{n+1} matches that described by E; specifically, for $v \in V_n$ and $w \in V_{n+1}$ we need that D^v is embedded in D^w with multiplicity $\#\big(s^{-1}(v) \cap r^{-1}(w)\big)$. This follows from the Cuntz-Krieger relations at v:

take paths μ, ν with source in S and range v, decompose the matrix unit $s_\mu s_\nu^* \in D^v$ as

$$s_\mu s_\nu^* = s_\mu p_v s_\nu^* = s_\mu \Big(\sum_{e \in s^{-1}(v)} s_e s_e^* \Big) s_\nu^* = \sum_{e \in s^{-1}(v)} s_{\mu e} s_{\nu e}^*$$

and note that $s_{\mu e} s_{\nu e}^*$ is a matrix unit in D^w precisely when $e \in r^{-1}(w)$.

Consider now the complementary corner

$$(1-p)C^*(KE)(1-p) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu,\, \nu \in KE^*,\ s(\mu),\, s(\nu) \in E^0,\ r(\mu) = r(\nu)\}.$$

Since $KE^1 \backslash E^1$ contains only edges from S to E^0, paths beginning in E^0 never leave E^0. Thus $(1-p)C^*(KE)(1-p)$ is generated by the Cuntz-Krieger E-family

$$\{s_e,\, p_v : e \in E^1,\, v \in E^0\}.$$

Further, E contains no loops, so the Cuntz-Krieger uniqueness theorem [**1**, Theorem 3.1] implies that $(1-p)C^*(KE)(1-p)$ is isomorphic to $C^*(E)$.

Finally, we must show that p and $1-p$ are full. Note that for every $v \in KE^0$ there is a path beginning in S and ending at v. Suppose that I is an ideal in $C^*(KE)$ containing $pC^*(KE)p$; then I certainly contains the projections $\{p_w : w \in S\}$. Given a vertex v in E^0, choose a path α beginning at some $w \in S$ and ending at v. Then $s_\alpha = p_w s_\alpha \in I$, so $p_v = s_\alpha^* s_\alpha \in I$, every generator $\{s_e, p_v\}$ of $C^*(KE)$ is in I, and $I = C^*(KE)$. Now suppose that J is an ideal in $C^*(KE)$ containing $(1-p)C^*(KE)(1-p)$, so for every $v \in E^0$ we have $p_v \in J$. Given a vertex $v \in S$, note that every edge e with $s(e) = v$ satisfies $r(e) \in E^0$; so for all $e \in s^{-1}(v)$, we know that $p_{r(e)} = s_e^* s_e \in J$, implying $s_e = s_e s_e^* s_e \in J$ and $s_e s_e^* \in J$. Thus $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^* \in J$, the universal KE-family $\{s_e, p_v\}$ is contained in J, and $J = C^*(KE)$. ∎

References

[1] T. Bates. D. Pask, I. Raeburn and W. Szymański, 'The C^*-algebras of row-finite graphs', *New York J. Math.* **6** (2000), 307–324.

[2] K. Davidson, *C^*-algebras by example* (American Mathematical Society, Providence R.I., 1996).

[3] D. Drinen, 'Viewing AF-algebras as graph algebras', *Proc. Amer. Math. Soc.* **128** (1999), 1991–2000.

[4] A. Kumjian, D. Pask and I. Raeburn, 'Cuntz-Krieger algebras of directed graphs', *Pacific J. Math* **184** (1998), 161–174.

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