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EXISTENCE OF SOLUTIONS OF AN ILL-POSED PROBLEM FOR THE VIBRATING STRING

BY
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ABSTRACT. The Dirichlet problem is examined for the vibrating string equation on a rectangle with commensurable sides. As is well-known, a solution, if it exists, is not unique. A necessary and sufficient condition is obtained on the boundary values for existence of solutions. A simple formula for the solution is obtained.

1. **Introduction.** In 1939 Bourgin and Duffin [2] examined the Dirichlet problem for the vibrating string equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0, \qquad (x, t)\in D \tag{1}$$

where D is the rectangle $\{(x, t): 0\le x\le L, 0\le t\le T\}$. They showed that if T/L is irrational and u vanishes on the boundary of D, then u vanishes throughout D. On the other hand, if $T/L=k/l$ where k and l are integers, then $\sin l\pi x/L \sin l\pi t/L$ is a solution of (1) which vanishes on the boundary of D. Thus the uniqueness of solutions of the Dirichlet problem (and similarly of many other boundary value problems with data given on all four sides of the rectangle) for (1) depends on the commensurability or otherwise of the lengths of the sides.

This result is often "explained" by saying that the Dirichlet problem for (1) is ill-posed. Physically it is more appropriate to assign the initial position and velocity of the points of the string; i.e. to give Cauchy data at $t=0$, and to leave u unrestricted on the side $t=T$. However there are some inverse problems where one might observe $u(x, T)$ for $0<x<L$ and attempt to infer from this the values of $u_t(x, 0)$. Or, equivalently, one might consider a very simple control problem in which one wishes to choose $u_t(x, 0)$ in order to achieve a target displacement $u(x, T)$. The variety of applications which lead to ill-posed problems [4, 7] suggest that a more complete understanding of the Dirichlet problem for (1) might be useful.

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In fact [2] has inspired a number of extensions to other hyperbolic and ultrahyperbolic equations, to more general boundary conditions, and to different methods of proof. See, for example, [1, 5, 9] where some recent results and references to earlier work appear.

Much of the work to date has dealt with uniqueness questions. Bourgin and Duffin [2] give one condition for existence when T/L is irrational and Travis [8] deals with existence of a solution of the non-homogeneous version of (1) with zero boundary conditions. Fox and Pucci [6] deal with uniqueness, existence, and continuous dependence on the data for solutions of (1) which vanish on the sides $x=0$ and $x=L$ of the rectangle D and which assume given values on the sides $t=0$ and $t=T$.

In the present paper we extend some of Fox and Pucci's results [6]. For T/L rational, we extend their criterion for existence of a solution to include the case where u assumes arbitrary values on all four sides of the rectangle. We also obtain a more explicit representation of the solution when it exists (see (8), (12), (13)). Our existence condition (14), with $g_n^{(i)}=0$, is equivalent to (32) in [6], while our condition $L\tilde{f}=T\tilde{g}$, together with (12), reduces to (25) in [6] when $g=0$.

Since the derivation of our main result depends rather heavily on the explicit form of (1), we indicate in Section 4, two alternative approaches to the problem which offer more prospect of generalization. One of these alternatives is to use an integral representation of the solutions of (1) to convert the problem to a Fredholm integral equation of the first kind. The condition for existence of a solution of the Dirichlet problem in this approach becomes the condition for existence of a solution of the integral equation in the case of non-uniqueness. The particular integral equation which arises is closely related to one arising in a Wiener filtering problem which was studied by Brown [3].

2. **Preliminary definitions and results.** Let f be an odd periodic function with period $2L$. Let $T/L=k/l$ where k and l are relatively prime integers. Define two functions $\tilde{f}$ and $\hat{f}$ by

$$\tilde{f}(x)=\frac{1}{l}\sum_{m=0}^{l-1} f(x+(2m+1)T) \tag{2}$$

and

$$\hat{f}(x)=\frac{1}{l}\sum_{m=0}^{l-1}\left(m-\frac{l-1}{2}\right)f(x+(2m+1)T). \tag{3}$$

Fairly straightforward calculations using the oddness and periodicity of f amd the relative primeness of k and l show the following:

(i) $\tilde{f}$ is an odd function

(ii) $\tilde{f}$ has period $2L/l=2T/k$

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(iii) $\hat{f}$ is an even function
(iv) $\hat{f}$ has period $2L$
(v) $\hat{f}(x+T)-\hat{f}(x-T)=f(x)-\tilde{f}(x-T)$ for all x.

Moreover, if f is integrable and has associated Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L}, \tag{4}$$

then $\tilde{f}$ and $\hat{f}$ have the associated Fourier series

$$\tilde{f}(x) \sim \sum_{r=1}^{\infty} f_{rl} \sin \frac{rl\pi}{L}(x+T) = \sum_{r=1}^{\infty} (-1)^{rk} f_{rl} \sin \frac{rl\pi x}{L} \tag{5}$$

and

$$\hat{f}(x) \sim -\frac{1}{2} \sum_{n=1}^{\infty}{}' f_n \csc \frac{n\pi T}{L} \cos \frac{n\pi x}{L}, \tag{6}$$

where $\sum'$ denotes a sum over terms for which $\sin n\pi T/L \neq 0$. Thus the series for $\tilde{f}$ and $\hat{f}$ involve, respectively, those terms from (4) for which $n=0 \pmod{l}$ and $n \neq 0 \pmod{l}$.

If g is odd and periodic with period $2T$, we define similarly

$$\tilde{g}(x) = \frac{1}{k} \sum_{m=0}^{k-1} g(x+(2m+1)L) \tag{2$'$}$$

and

$$\hat{g}(x) = \frac{1}{k} \sum_{m=0}^{k-1} \left(m - \frac{k-1}{2}\right) g(x+(2m+1)L). \tag{3$'$}$$

These functions have analogous properties to those of $\tilde{f}$ and $\hat{f}$. In particular, we remark that $\tilde{f}$ and $\tilde{g}$ both have the same period $2L/l = 2T/k$.

3. **Existence of solutions.** Throughout this paper we shall use the term "solution" of (1) to mean any function

$$u(x,t) = \phi_1(x-t) + \phi_2(x+t)$$

where ϕ_1 and ϕ_2 are integrable functions. We consider first the boundary values

$$\begin{cases} u(x,0)=0, & u(x,T)=f(x) \quad \text{for} \quad 0<x<L \\ u(0,t)=0, & u(L,t)=g(t) \quad \text{for} \quad 0<t<T, \end{cases} \tag{7}$$

where f and g are integrable functons. A "solution" of the Dirichlet problem (1), (7) will mean a solution of (1) which satisfies (7) almost everywhere. Let the boundary functions f and g be extended to the real line so that they are odd and periodic with period $2L$ and $2T$ respectiviely. Let $\tilde{f}, \hat{f}, \tilde{g}, \hat{g}$ be defined by (2), (3), (2)$'$, (3)$'$.

THEOREM. *Let $T/L = k/l$ where k and l are relatively prime integers. The Dirichlet problem* (1), (7) *has a solution if and only if $L\tilde{f} = T\tilde{g}$ almost everywhere. When this condition is satisfied, one solution is*

$$u_0(x,t) = \hat{f}(t+x) - \hat{f}(t-x) + \hat{g}(t+x) - \hat{g}(t-x) + (2L)^{-1}[(t+x)\tilde{g}(t+x) - (t-x)\tilde{g}(t-x)]. \tag{8}$$

Proof. We show the necessity of the condition $L\tilde{f} = T\tilde{g}$ by showing that $L(-1)^{rk}f_{rl} = T(-1)^{rl}g_{rk}$ for $r = 1, 2, \ldots$ is a necessary condition for the existence of a solution, where f_n and g_m are the Fourier coefficients of f and g. Let u be any solution of (1), (7) of the form described above. Then

$$L(-1)^{rk}f_{rl} - T(-1)^{rl}g_{rk} = 2(-1)^{rk}\int_0^L [\phi_1(x-T) + \phi_2(x+T)]\sin\frac{rl\pi x}{L}\,dx - 2(-1)^{rl}\int_0^T [\phi_1(L-t) + \phi_2(L+t)]\sin\frac{rk\pi t}{T}\,dt.$$

Make the substitution $k/T = l/L$ and some obvious changes of variables in the integrals. The result, after a little simplification, is

$$\begin{aligned} L(-1)^{rk}f_{rl} - T(-1)^{rl}g_{rk} &= 2\int_{-T}^{L} \phi_1(x)\sin\frac{rl\pi x}{L}\,dx + 2\int_{T}^{L} \phi_2(x)\sin\frac{rl\pi x}{L}\,dx \\ &= 2\int_0^L [\phi_1(x) + \phi_2(x)]\sin\frac{rl\pi x}{L}\,dx \\ &\quad - 2\int_0^T [\phi_1(-t) + \phi_2(t)]\sin\frac{rl\pi t}{L}\,dt. \end{aligned} \tag{9}$$

But $\phi_1(x) + \phi_2(x) = u(x, 0) = 0$ for $0 < x < L$ and $\phi_1(-t) + \phi_2(t) = u(0, t) = 0$ for $0 < t < T$. Hence the expression on the left in (9) vanishes. Since $(-1)^{rk}f_{rl}$ and $(-1)^{rl}g_{rk}$ are the Fourier coefficients of $\tilde{f}$ and $\tilde{g}$ respectively, it follows that $L\tilde{f} = T\tilde{g}$ almost everywhere.

To prove the sufficiency, we assume that $L\tilde{f} = T\tilde{g}$ and show that (8) gives a solution of (1), (7). By our definition, u_0 is a solution of (1). Obviously $u_0(0, t) = 0$, and the property $u_0(x, 0) = 0$ follows from the evenness of $\hat{f}$ and $\hat{g}$ and the oddness of $\tilde{g}$. Next,

$$u_0(x, T) = \hat{f}(x+T) - \hat{f}(T-x) + \hat{g}(T+x) - \hat{g}(T-x) + (2L)^{-1}[(T+x)\tilde{g}(T+x) - (T-x)\tilde{g}(T-x)].$$

But $\hat{g}$ is even and has period $2T$ so that

$$\hat{g}(T+x) - \hat{g}(T-x) = \hat{g}(T+x) - \hat{g}(x-T) = 0.$$

Similarly $\tilde{g}$ is odd and has period $2T/k$, where k is an integer, so that $-\tilde{g}(T-x) = \tilde{g}(T+x) = \tilde{g}(x-T)$. Finally, by the evenness of $\hat{f}$ and by property

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(v) of Section 2,

$$\hat{f}(x+T)-\hat{f}(T-x)=f(x)-\tilde{f}(x-T). \tag{10}$$

Making these substitutions, we get

$$u_0(x, T)=f(x)-\tilde{f}(x-T)+(T/L)\tilde{g}(x-T).$$

Since $T\tilde{g}=L\tilde{f}$ almost everywhere, $u_0(x, T)=f(x)$ almost everywhere. The property $u_0(L, t)=g(t)$ follows from a similar calculation which uses the periodicity of $\tilde{g}$ and $\hat{f}$ and the identity analogous to (10) for g. This completes the proof.

Note that the solution u_0 given by (8) is not unique, since we can add to it any convergent series of the form

$$\sum_{r=1}^{\infty} b_r \sin\frac{rl\pi x}{L}\sin\frac{rk\pi t}{T}$$

and obtain another solution.

The extension of the Theorem to more general Dirichlet conditions is now straightforward. Let the boundary conditions be

$$\begin{aligned} &u(x, 0)=f_1(x), \qquad u(x, T)=f_2(x) \quad \text{for} \quad 0<x<L \\ &u(0, t)=g_1(t), \qquad u(L, t)=g_2(t) \quad \text{for} \quad 0<t<T. \end{aligned} \tag{11}$$

Extend f_1, f_2, g_1, g_2 to odd periodic functions with periods $2L$ and $2T$. If we put

$$v(x, t)=u(x, t)-\tfrac{1}{2}[f_1(x-t)+f_1(x+t)]-\tfrac{1}{2}[g_1(t-x)+g_1(t+x)]$$

then v is a solution of (1) whenever u is a solution. Moreover, v satisfies the conditions (7) with

$$f(x)=f_2(x)-\tfrac{1}{2}[f_1(x-T)+f_1(x+T)] \tag{12}$$

and

$$g(t)=g_2(t)-\tfrac{1}{2}[g_1(t-L)+g_1(t+L)]. \tag{13}$$

If we write $\{f_n^{(1)}\}$, $\{f_n^{(2)}\}$, $\{g_n^{(1)}\}$, and $\{g_n^{(2)}\}$ for the Fourier series coefficients of f_1, f_2, g_1, and g_2 respectively, the condition for existence, $L(-1)^{rk}f_{rl}=T(-1)^{rl}g_{rk}$ for $r=1, 2, \ldots,$ becomes

$$L[f_{rl}^{(1)}-(-1)^{rk}f_{rl}^{(2)}]=T[g_{rk}^{(1)}-(-1)^{rl}g_{rk}^{(2)}] \qquad (r=1, 2, 3, \ldots), \tag{14}$$

where the subscripts rk and rl are ordinary products. That is, the solution condition (14) depends only on coefficients $f_n^{(i)}$ for which $n=0$ (mod l) and on coefficients $g_n^{(i)}$ for which $n=0$ (mod k).

Although their principal interest was in the case that T/L is irrational, Bourgin and Duffin [2, p. 857] indicated that the vanishing of each side of equation (14) is a necessary condition for the existence of a solution of (1), (11). While this is a sufficient condition, it is not quite a necessary condition, as the example

$$U(x,t)=(x+t)\sin\frac{l\pi}{L}(x+t)-(x-t)\sin\frac{l\pi}{L}(x-t)$$

shows. This is a solution of (1), (7) with

$$f(x)=2T(-1)^k\sin\frac{l\pi x}{L},\qquad g(t)=2L(-1)^l\sin\frac{k\pi t}{T}.$$

Each side of (14) is equal to $-2LT$, for $r=1$.

4. **Fourier series and Green's functions.** Both to indicate possible directions for generalization and to provide the connections with a filtering problem [3], we outline alternative approaches to the Dirichlet problem. To simplify the discussion, we restrict ourselves in this section to the boundary conditions

$$\begin{cases} u(0,t)=u(L,t)=u(x,0)=0 \\ u(x,T)=f(x). \end{cases} \tag{15}$$

That is, we take $g=0$ in (7).

A more standard set of boundary conditions for (1) is obtained by replacing the condition $u(x,T)=f(x)$ by $u_t(x,0)=v(x)$, where the subscript denotes differentiation with respect to t. The well-known Fourier series solution of this problem is

$$u(x,t)=\frac{L}{\pi}\sum_{n=1}^{\infty}\frac{v_n}{n}\sin\frac{n\pi x}{L}\sin\frac{n\pi t}{L}, \tag{16}$$

where

$$v(x)\sim\sum_{n=1}^{\infty}v_n\sin\frac{n\pi x}{L}.$$

The Dirichlet problem (1), (15) is now equivalent to finding v so that (15) is satisfied. Substitution of (16) in (15) gives the equations

$$\frac{Lv_n}{n\pi}\sin\frac{n\pi T}{L}=f_n\qquad (n=1,2,\ldots). \tag{17}$$

Since $T/L=k/l$, no solution is possible unless $f_n=0$ for $n\equiv 0 \pmod{l}$. This condition is equivalent, by (5), to the condition $\tilde f=0$ almost everywhere. Moreover, if this condition is satisfied, and v_n satisfies this equation, then (16) becomes

$$u(x,t)=\sum{}' f_n\csc\frac{n\pi T}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi t}{L}, \tag{18}$$

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where, as before, $\sum'$ denotes a sum in which terms with $n=0 \pmod l$ are omitted. This series can be summed (formally) with the aid of (6) to give

$$u(x, t)=\hat{f}(t+x)-\hat{f}(t-x). \tag{19}$$

Another approach is to obtain the Riemann–Green function for the standard problem, with $u_t(x, 0)=v(x)$ replacing the condition $u(x, T)=f(x)$. It is simpler in this case to restrict ourselves to rectangles with $T<L$. This causes no real loss of generality because it is easily established that $u(x, t+2L)=u(x, t)$ and $u(x, 2L-t)=-u(x, t)$, so that the solution can be continued once it is known for $0\le t\le L$. Let p_t be defined for $0<t<L$ by

$$p_t(x)=\begin{cases}1 & \text{for} \quad |x|<t\\ 0 & \text{for} \quad t<|x|<L\\ p_t(x+2L) & \text{for all } x,\end{cases}$$

and put

$$K_t(x, y)=\tfrac{1}{2}[p_t(x-y)-p_t(x+y)].$$

Then we have the representation

$$u(x, t)=\int_0^L K_t(x, y)v(y)\,dy$$

for the solution of the standard problem. In order to choose v in such a way that $u(x, T)=f(x)$ we must solve the Fredholm integral equation of the first kind

$$\int_0^L K_T(x, y)v(y)\,dy=f(x), \qquad 0<x<L. \tag{20}$$

The kernel of this equation is symmetric and a straightforward calculation shows that

$$\int_0^L K_T(x, y)\sin\frac{n\pi y}{L}\,dy=\frac{L}{n\pi}\sin\frac{n\pi T}{L}\sin\frac{n\pi x}{L}.$$

Since the right side vanishes for $n=0 \pmod l$, the usual necessary condition for the solubility of (20) is that $f_n=0$ for $n=0 \pmod l$. A comparison of (4) and (5) shows that $\tilde{f}(x-T)$ is the orthogonal projection of $f(x)$ on the null space of K_T. Thus the function $f(x)-\tilde{f}(x-T)$ which appears in (10) is the projection of f on the orthogonal complement of this null space.

Finally, we remark that if u is assumed to have continuous second derivatives in Section 3, then the necessity of (14) can also be established by a straightforward application of Green's theorem in the plane.

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