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GROUPS AND MONOIDS OF REGULAR GRAPHS (AND OF GRAPHS WITH BOUNDED DEGREES)

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Introduction. A graph X is a set $V(X)$ (the vertices of X) with a system $E(X)$ of 2-element subsets of $V(X)$ (the edges of X). Let X, Y be graphs and $f : V(X) \to V(Y)$ a mapping; then f is called a homomorphism of X into Y if $[f(x), f(y)] \in E(Y)$ whenever $[x, y] \in E(X)$. Endomorphisms, isomorphisms and automorphisms are defined in the usual manner.

Much work has been done on the subject of representing groups as groups of automorphisms of graphs (i.e., given a group G, to find a graph X such that the group of automorphisms of X is isomorphic to G). Recently, this was related to category theory, the main question being as to whether every monoid (i.e., semigroup with 1) can be represented as the monoid of endomorphisms of some graph in a given category of graphs. The answer is known and affirmative for the category of all graphs [**4**] and also for several of its subcategories [**3**; **6**; **7**; **9**]. The subcategories involved do not include the category of graphs with bounded degrees, and we shall show that in this case the answer is negative, namely, there are monoids (respectively groups) that cannot be represented as the monoids of endomorphisms (respectively groups of automorphisms) of a graph with bounded degrees. This answers a question raised by Z. Hedrlín, as to whether the category of graphs with bounded degrees is binding (the authors have been informed by private communication that L. Kučera has also given an answer to this problem, although he has not published his solution).

Furthermore, we shall investigate sets on which there exist rigid locally finite, and rigid regular graphs. (We talk in particular about 3-regular graphs but the results could be extended to k-regular graphs, $k \geqq 3$, using similar methods.)

This investigation allows us to extend the list of groups which can be represented as groups of automorphisms of 3-regular graphs – a list started by R. Frucht in 1949 [**1**], when he proved that it contained all finite groups. We will show that it also contains all countable groups, and that this is in a sense the best possible result.

Finally, we determine the greatest possible cardinality of a rigid or asymmetric graph with bounded degrees, and the smallest possible order of a group which cannot be represented as the monoid of endomorphisms (respectively group of automorphisms) of any graph with degrees bounded by a given cardinal.

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1. The category of graphs with α-bounded degrees. Let X be a graph, $x \in V(X)$. Put $d(x, X) = |\{y : [x, y] \in E(X)\}|$. Let α be a cardinal; we say that the graph X has *α-bounded degrees* if $d(x, X) < \alpha$ for each $x \in V(X)$. Graphs with $\aleph_0$-bounded degrees are called *locally finite*. A graph X is *k-regular*, if $d(x, X) = k$ for all $x \in V(X)$.

Let X be a graph; a sequence $x_1, x_2, \ldots x_n$ of pairwise distinct vertices, such that $[x_j, x_{j+1}] \in E(X)$ for all $j = 1, 2, \ldots, n - 1$, is called a *path of length* $n - 1$ from x_1 to x_n; if $[x_1, x_n] \in E(X)$, it is called a *cycle of length n*. A graph X is *connected* if there is a path from x to y for each pair $x, y \in V(X)$.

Let $\mathscr{R}^\alpha$ be the category of all graphs with α-bounded degrees and all their homomorphisms. In questions related to full embeddings of categories (all necessary concepts concerning categories can be found in [**3**]) an important role is played by rigid objects: a graph is called *rigid* (respectively asymmetric) if it does not possess a non-identical endomorphism (respectively automorphism). For graphs X, Y, let $M(X, Y)$ denote the set of all homomorphisms of X into Y, and let $M(X)$ stand for $M(X, X)$; $M(X)$ with composition is the monoid of endomorphisms of X.

A category is called *binding* if it contains, as a full subcategory, an isomorphic copy of the category of all graphs. If a category is binding, then for every monoid M, there is an object O such that the monoid of endomorphisms of O is isomorphic to M (i.e., in a binding category, every monoid can be represented as the monoid of endomorphisms of a suitable object, see [**3**]).

THEOREM 1. *$\mathscr{R}^\alpha$ is not binding for any cardinal α.*

By the previous remarks, it suffices to prove the following:

THEOREM 2. *Let α be any cardinal, $\beta = 2^{\max(\alpha, \aleph_0)}$; let G be a group of order greater than 2^β and without elements of order 2 (i.e., $g \neq 1 \Rightarrow g^2 \neq 1$). Then for each $X \in \mathscr{R}^\alpha$, $M(X) \ncong G$.*

Note that groups with the prescribed properties do exist (e.g., free groups over sets with cardinality greater than 2^β).

Proof of Theorem 2. Each connected graph $X \in \mathscr{R}^\alpha$ has $|V(X)| \leqq m = \max(\alpha, \aleph_0)$; therefore there are only 2^m non-isomorphic connected graphs with α-bounded degrees. Let $X \in \mathscr{R}^\alpha$. If $|V(X)| > \beta = 2^m$, then X must have more than 2^m components (each having cardinality at most m). Thus two of them are isomorphic and the automorphism of X interchanging them, while leaving everything else fixed, is involutory. If, on the other hand, $|V(X)| \leqq \beta$, then the order of its automorphism group is at most 2^β, while $|G| > 2^\beta$.

Remark. From the preceding proof we conclude that an α-bounded graph X cannot be rigid (or even asymmetric) if $|V(X)| > 2^{\max(\alpha, \aleph_0)}$. If α is an uncountable regular cardinal, then a connected $X \in \mathscr{R}^\alpha$ gives $|V(X)| < m = \alpha$. Thus $\mathscr{R}^\alpha$ has less than 2^α non-isomorphic connected graphs, and $|V(X)| \geqq 2^\alpha$ implies that X is neither rigid nor asymmetric.

2. Auxiliary constructions.

(A) *A finite rigid graph.* We define the graph Z by:

$$V(Z) = \{1, 2, \ldots, 12\}$$

$$E(Z) = \{[i, i + 1]: i = 1, 2, \ldots, 11\} \cup \{[1, 4], [1, 8], [2, 6], [3, 10], [5, 12], [7, 11]\}.$$

There are exactly 5 pentagons (i.e., cycles of length 5) in Z, namely $(1, 2, 6, 7, 8)$, $(2, 3, 4, 5, 6)$, $(1, 2, 6, 5, 4)$, $(5, 6, 7, 11, 12)$ and $(7, 8, 9, 10, 11)$; every vertex of Z belongs to at least one of them.

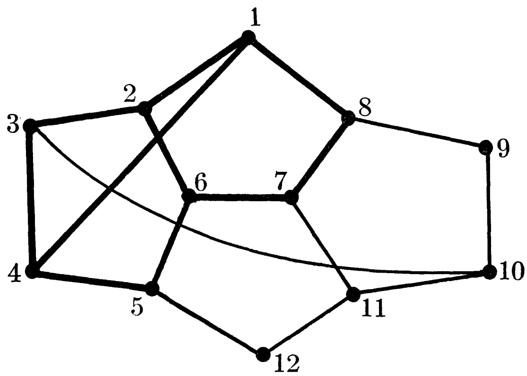

FIGURE 1

By examining the mutual position of the pentagons, we conclude that the pentagons $(1, 2, 6, 7, 8)$ and $(2, 3, 4, 5, 6)$ have a unique position (i.e., the subgraph of Z induced by these two pentagons is not isomorphic to any other subgraph induced by two pentagons).

Z has no triangles; therefore for every homomorphism $f : Z \to Z$, we have $f(a) \neq f(b)$ if there is a path of length 3 from a to b in Z (and of course also if there is a path of length 1, i.e., if $[a, b] \in E(Z)$). This condition implies, that every homomorphism $f : Z \to Z$ is one-to-one on the set $A = \{1, 2, \ldots, 8\}$ (one merely has to verify that $f(1) = f(3)$ is impossible), and because of the unique position of the two pentagons $(1, 2, 6, 7, 8)$ and $(2, 3, 4, 5, 6)$, we have $f(A) = A$.

The graph induced by A is asymmetric and thus $f|A = 1_A$. Since $(5, 6, 7, 11, 12)$ is the only pentagon containing the vertices 5, 6, 7 we have $f(11) = 11$, $f(12) = 12$; similarly $f(9) = 9$, $f(10) = 10$, i.e., $f = 1_{V(Z)}$ and Z is a rigid graph.

(B) *Rigid relations.* A binary relation R on a set X will be denoted by (X, R) and the word "relation" will apply to the couple (X, R) (some authors use the term "digraph"). Our relations will all be anti-reflexive, $(x, x) \notin R$, and anti-symmetric, $(x, y) \notin R \Rightarrow (y, x) \in R$; the transitive part of them will play a special role.

A relation (X, R) is *k-regular* if for each $x \in X$, the degree of x is

$$d(x, R) = |\{y : (x, y) \in R\}| + |\{y : (y, x) \in R\}| = k.$$

A mapping $f : X \to Y$ compatible with the relations R, S (i.e., $(x, y) \in R \Rightarrow (f(x), f(y)) \in S$) will be called a *homomorphism* of the relation (X, R) to the relation (Y, S); a *rigid* relation has no non-identical endomorphisms.

(a) *3-regular rigid relations*. Let the set N of natural numbers be partitioned, $N = A \cup B$, let $1 \in A$, and both A and B be infinite. Then we define the following relation (X_N, R_{AB}):

$$X_N = \{a, b, c, d\} \cup \{e_i : i \in N\} \cup \{f_i : i \in N\} \quad (a, b, c, d, e_i, f_i \text{ pairwise distinct})$$

$$R_{AB} = \{(e_1, a), (a, b), (a, c), (b, c), (b, d), (c, d), (d, f_1)\} \cup \{(e_i, e_{i+1}) : i \in N\} \cup \{(f_i, f_{i+1}) : i \in N\} \cup \{(e_i, f_i) : i \in A\} \cup \{(f_i, e_i) : i \in B\}.$$

For a natural number k, let $\{1, 2, \ldots, k\} = A \cup B$ be a partition, $k - 1 \in A$, $k \in B$:

$$X_k = \{a, b, c, d\} \cup \{e_i : i = 1, 2, \ldots, k\} \cup \{f_i : i = 1, 2, \ldots, k\} \cup \{x, y, z, w\} \quad (X_k \text{ a set with } 2k + 8 \text{ elements}),$$

$$\begin{aligned} R_{AB} = \{&(e_1, a), (a, b), (a, c), (b, c), (b, d), (c, d), (d, f_1)\} \\ &\cup \{(e_i, e_{i+1}) : i = 1, \ldots, k - 1\} \cup \{(f_i, f_{i+1}) : i = 1, \ldots, k - 1\} \\ &\cup \{(e_i, f_i) : i \in A\} \cup \{(f_i, e_i) : i \in B\} \\ &\cup \{(x, f_k), (x, y), (x, z), (y, z), (y, w), (z, w), (w, e_k)\}. \end{aligned}$$

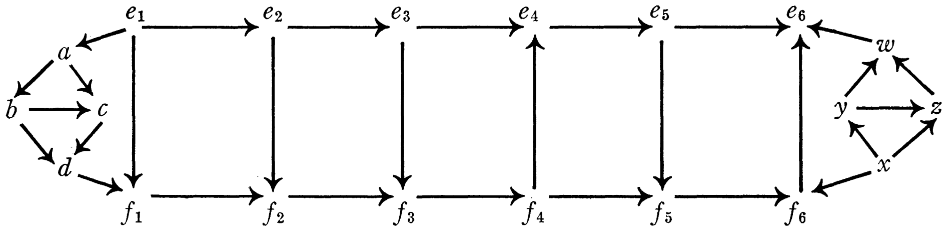

FIGURE 2

All the relations (X, R_{AB}), where X is X_N or X_k, are 3-regular.

Let (X, R) be any anti-reflexive, anti-symmetric relation and let $x, y, z \in X$. The elements x, y, z are said to form a transitive triple xyz if $(x, y) \in R$, $(y, z) \in R$ and $(x, z) \in R$. If xyz is a transitive triple and f a homomorphism of relations, then $f(x)f(y)f(z)$ is again a transitive triple; moreover if xyz and yzw are transitive triples, then x, y, z, w are pairwise distinct.

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Therefore for every homomorphism $g : (X, R_{AB}) \to (X, R_{AB})$,

$$g|\{a, b, c, d, e_1, f_1\}$$

is the identity (abc, bcd are the only transitive triples if $X = X_N$; if $X = X_k$, $g(a) = x$ is impossible, since there is no $v \in X_k$ with $(v, x) \in R_{AB}$ while $(e_1, a) \in R_{AB}$). To see that g is the identity everywhere, we suppose that $g(e_i) \neq e_i$ for an $i \in N$ ($i \leqq k$ respectively); then $g(e_i) = f_{i-1}$ and $(e_i, f_i) \in R_{AB}$. This is possible only if $g(e_{i+1}) = f_i$ and $(e_{i+1}, f_{i+1}) \in R_{AB}$, and, by induction, $g(e_j) = f_{j-1}$ and $(e_j, f_j) \in R_{AB}$ for all $j \geqq i$. This gives a contradiction. Similarly, $g(f_i) = f_i$, and so g is the identity. Thus all the relations (X, R_{AB}) are rigid.

We note that if $A \cup B$ and $A' \cup B'$ are two different partitions of the same set X, then there is no homomorphism $f : (X, R_{AB}) \to (X, R_{A'B'})$ (whether $X = X_N$ or $X = X_k$).

(b) *A modification.* We shall modify the relations (X, R_{AB}) to obtain rigid relations with some elements of degree 2. Let $A \cup B$ be a partition of N (respectively $\{1, 2, \ldots, k\}$) such that A and B are infinite (respectively such that $k - 1 \in A$, $k \in B$); let $S \subset N$ such that $A - S$ and $B - S$ are infinite (respectively $S \subset \{1, 2, \ldots, k - 2\}$). Put

$$R_{AB}{}^S = R_{AB} - \{(e_i, f_i) : i \in S\} - \{(f_i, e_i) : i \in S\}.$$

The relations $(X, R_{AB}{}^S)$ are also rigid and $\{x \in X : d(x, R_{AB}{}^S) = 2\} = S$. Moreover, if $(A - S) \cup (B - S)$ and $(A' - S) \cup (B' - S)$ are two different partitions of $N - S$ (respectively $\{1, \ldots, k\} - S$), then there is no homomorphism $(X, R_{AB}{}^S) \to (X, R_{A'B'}{}^S)$.

Note that for $X = X_N$, S may be chosen to be countable.

3. Locally finite rigid graphs and regular rigid graphs. In [**2**; **5**; **11**], it is shown that there exists a rigid graph X with $|V(X)| = \alpha$, unless α is a finite cardinal in the range $1 < n < 8$. In attempting to find a corresponding result for locally finite graphs we must realize, firstly, that "big" sets will not admit a locally finite rigid graph, according to the remark following Theorem 2, and secondly, that under the more restrictive hypothesis of 3-regularity, we cannot hope even for all finite numbers, since there are no 3-regular graphs with odd number of vertices.

However we were able to characterize all cardinals α such that there is a locally finite (respectively 3-regular) rigid graph X with $|V(X)| \geqq \alpha$.

THEOREM 3. *Let α be a cardinal. A locally finite rigid graph X with $|V(X)| \geqq \alpha$ exists if and only if there exists a 3-regular rigid graph X with $|V(X)| \geqq \alpha$.*

A 3-regular rigid graph X with $|V(X)| \geqq \alpha$ exists if and only if $\alpha \leqq 2^{\aleph_0}$.

If α is finite, X can be chosen finite; if α is countable, X can also be chosen countable.

Proof. If there is a locally finite rigid graph X with $|V(X)| \geqq \alpha$, then according to the Remark in § 1, $\alpha \leqq |V(X)| \leqq 2^{\aleph_0}$. Since a 3-regular graph is locally finite, it remains to construct 3-regular rigid graphs with the corresponding cardinalities. In § 2, (B), (a) we have some 3-regular rigid relations; to transform them into graphs we shall use the following product (which is a modification of the product introduced in [**4**]):

Let (X, R) be a relation, Z the graph from § 2, (A); define the graph $(X, R)*Z = Y$ by

$$V(Y) = X \cup (R \times V(Z))$$
$$E(Y) = \{[((x, y), z), ((x, y), z')] : (x, y) \in R, [z, z'] \in E(Z)\} \cup \{[x, ((x, y), 9)] : (x, y) \in R\} \cup \{[x, ((y, x), 12)] : (y, x) \in R\}.$$

Remark. Note that $d(x, Y) = d(x, R)$ for each $x \in X$, and $d(x, Y) = 3$ for each $x \in V(Y) - X$.

LEMMA. *Let (X, R), (Y, S) be anti-reflexive, anti-symmetric relations, let $d(x, R) \geqq 2$ for each $x \in X$.*

If (X, R) is a rigid relation, then $(X, R)*Z$ is a rigid graph; if there is no homomorphism $(X, R) \to (Y, S)$, then there is no homomorphism

$$(X, R)*Z \to (Y, S)*Z.$$

Proof. The proof is analogous to that of [**4**]. One is essentially using

(a) each vertex of Z belongs to a pentagon in Z;

(b) no vertex $x \in X$ ($y \in Y$ respectively) belongs to a pentagon in $(X, R)*Z$ (respectively $(Y, S)*Z$);

(c) Z is a rigid graph.

To complete the proof of Theorem 3 we put

$X = (X_\alpha, R_{AB})*Z$ for α finite ($A \cup B$ any partition of $1, \ldots, \alpha$, $\alpha - 1 \in A, \alpha \in B$),

$X = (X_N, R_{AB})*Z$ for $\alpha = \aleph_0$ ($A \cup B$ any partition of N into infinite sets),

$X = \cup \{(X_N, R_{AB})*Z : A \cup B$ is a partition of N into infinite sets$\}$ for $\alpha = 2^{\aleph_0}$ (the first $\cup$ denotes here the disjoint union of graphs).

Remark. A family of graphs $\{X_i : i \in I\}$ such that each X_i is rigid and such that there is no homomorphism $X_i \to X_j$ for $i \neq j$, is called *mutually rigid.* When all the graphs X_i ($i \in I$) are connected, then $\{X_i : i \in I\}$ is a mutually rigid family if and only if $\cup \{X_i : i \in I\}$ is a rigid graph (compare with the definition of X for $\alpha = 2^{\aleph_0}$ in the preceding proof). Thus by the Lemma and § 2, (B), (a) we have $2^{\aleph_0}$ mutually rigid 3-regular graphs.

Using the Lemma, the previous two Remarks, and § 2, (B), (b), we can construct a mutually rigid family $\{X_i : i \in I\}$ of $\alpha \leqq 2^{\aleph_0}$ connected graphs, such that (for $i \in I$) X_i has α_i vertices of degree 2 (and all other vertices of degree 3), where each α_i is at most countable. If α and α_i, $i \in I$, are finite, all X_i may be finite.

4. Representation of countable groups by 3-regular graphs. The reason why $\mathscr{R}^\alpha$, $\alpha \geqq 4$, is not binding is that there are groups of large order. Indeed, in [**1**] and [**10**] it was proved that all finite groups can be represented as groups of automorphisms of 3-regular graphs. Now we are in a position to generalize this result:

THEOREM 4. *Let G be any group of at most countable order. Then there exists a 3-regular graph X such that $M(X) \cong G$; if G is finite, then X may also be chosen finite.*

Note that if $M(X)$ is a group, then it is the group of automorphisms of X.

Proof. Let $G' = G - \{1\}$. According to the previous paragraph we can construct graphs Y, Y_g, $g \in G'$, which satisfy:

(i) $V(Y)$, $V(Y_g)$, $g \in G'$ are pairwise disjoint sets;

(ii) $\{Y\} \cup \{Y_g : g \in G'\}$ is a mutually rigid family of connected graphs;

(iii) there is a one-to-one correspondence denoted $g \to x_g$ between the set G' and the vertices of Y which are of degree 2; all the other vertices of Y are of degree 3;

(iv) each Y_g has two vertices, say y_g^1, y_g^2, of degree 2, and all others are of degree 3;

(v) for each $[x, y] \in E(Y)$ (respectively $[x, y] \in E(Y_g)$, $g \in G'$) either x or y belongs to a pentagon of Y (respectively Y_g);

(vi) the vertices x_g, y_g^j (for $g \in G'$, $j = 1, 2$) do not belong to any pentagons of Y or Y_g.

((v) and (vi) follow from the definition of $(X, R)*Z$).

Now we define the graph X by

$$\begin{aligned} V(X) = {} & V(Y) \times G \cup \{(g, hg, j) : g \in G, h \in G', j = 0, 1, 2\} \\ & \cup \{(y, g, hg) : y \in V(Y_h), g \in G, h \in G'\}, \\ E(X) = {} & \{[(x, g), (y, g)] : [x, y] \in E(Y), g \in G\} \\ & \cup \{[(x_h, g), (g, hg, 0)]; g \in G, h \in G'\} \\ & \cup \{[(g, hg, j), (g, hg, j + 1)] : g \in G, h \in G', j = 0, 1\} \\ & \cup \{[(g, hg, 2), (hg, g, 0)] : g \in G, h \in G'\} \\ & \cup \{[(y, g, hg), (z, g, hg)] : [y, z] \in E(Y_h), g \in G, h \in G'\} \\ & \cup \{[(y_h^j, g, hg), (g, hg, j)] : j = 1, 2, g \in G, h \in G'\} \end{aligned}$$

(X is a variation of the Cayley colour graph in which for every vertex is "substituted" a copy of the graph Y).

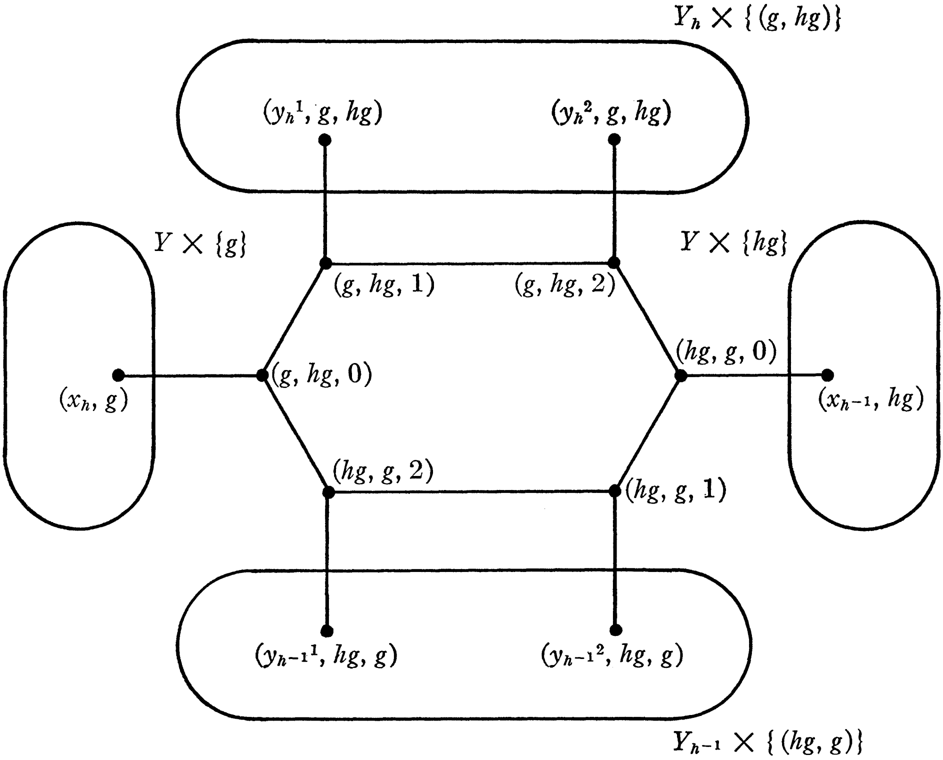

FIGURE 3

Clearly X is 3-regular.

Let $a \in G$; define $F_a : V(X) \to V(X)$ by

$$F_a((x, g)) = (x, ga) \text{ for } x \in V(Y), g \in G,$$
$$F_a((g, hg, j)) = (ga, hga, j) \text{ for } g \in G, h \in G', j = 0, 1, 2,$$
$$F_a((y, g, hg)) = (y, ga, hga) \text{ for } y \in V(Y_h), g \in G, h \in G'.$$

For each $a \in G$, F_a is an automorphism of X; if $a \neq b$, then $F_a \neq F_b$. To see that $a \to F_a$ is an isomorphism of G onto $M(X)$, it remains to show that every endomorphism F of X, is of the form $F = F_a$ for a suitable $a \in G$. According to (vi) and the definition of $E(X)$, the edges $[(x_h, g), (g, hg, 0)]$, $[(g, hg, j-1), (g, hg, j)]$, $[(g, hg, j), (y_h^j, g, hg)]$ and $[(g, hg, 2), (hg, g, 0])$ (for $g \in G$, $h \in G'$, $j = 1, 2$) do not have a vertex in common with any pentagon of X. Let $F \in M(X)$; by (v) and (ii) a copy of Y ($Y_h, h \in G'$, respectively) is mapped under F onto a copy of Y (Y_h respectively). That is, for each $g \in G$ there exists a $g' \in G$ such that $F((x, g)) = (x, g')$ (for all $x \in V(Y)$), and a $g'' \in G$ such that $F((y, g, hg)) = (y, g'', hg'')$ (for all $y \in V(Y_h)$). Let $a \in G$ be such, that

$$F((x, 1)) = (x, a) \text{ for all } x \in V(Y).$$

We see easily, that then

$$F((y, 1, g)) = (y, a, ga) \text{ for } y \in V(Y_g),\ g \in G'$$

and therefore $F((x, g)) = (x, ga)$ for all $x \in V(Y)$, $g \in G$; and $F = F_a$. Note that if G is finite, we may choose Y and Y_g finite (see § 3) and then X will also be finite.

In any case the graph X we constructed is connected, and hence at most countable.

Remark. In the proof of Theorem 2 we considered some groups of very large order, which cannot be represented by 3-regular graphs. Their order however, is unnecessarily large. In fact, Theorem 4 gives the best result concerning representation of groups by 3-regular graphs in the following sense:

For every uncountable cardinal α, there is a group G of order α, such that $M(X) \not\cong G$ for each locally finite (in particular, 3-regular) graph X.

Indeed, let $\mathrm{Alt}(\alpha)$ be the alternating group on α (i.e., the group of all those permutations on α elements, which have finite support and are even); the order of $\mathrm{Alt}(\alpha)$ is α, and $\mathrm{Alt}(\alpha)$ is a simple group (see [**8**, §§ 4 and 9]). Suppose, that $M(X) \cong \mathrm{Alt}(\alpha)$ for a locally finite graph X. Since $\mathrm{Alt}(\alpha)$ is simple, we may assume without loss of generality that X is *connected*, hence $|V(X)| \leqq \aleph_0 < \alpha$. This is impossible in view of the following statement:

If $M(X) \cong \mathrm{Alt}(\alpha)$, then $|V(X)| \geqq \alpha$.

Proof. Let $M(X) \cong \mathrm{Alt}(\alpha)$ and $|V(X)| < \alpha$. In $\mathrm{Alt}(\alpha)$ one can find two sets of permutations, say $\{f_i\}_{i\in\alpha}$ and $\{g_i\}_{i\in\alpha}$, such that

$$f_i \circ f_j = f_j \circ f_i \text{ and } g_i \circ g_j = g_j \circ g_i \text{ for all } i, j \in \alpha,$$
$$f_i \circ f_i = 1 \quad\text{ and } g_i \circ g_i = 1 \text{ for all } i \in \alpha,$$

and
$$f_i \circ g_j = g_j \circ f_i \text{ if and only if } i \neq j.$$

To see this, it is enough to partition the set α into α disjoint 5-tuples, and define, for each 5-tuple (a, b, c, d, e), the permutations f_i and g_i by

$$f_i = (a, b)(c, d), \qquad g_i = (a, b)(c, e).$$

Hence, X has 2 sets of automorphisms, to be denoted also by $\{f_i\}_{i\in\alpha}$ and $\{g_i\}_{i\in\alpha}$, satisfying the relations. For each $i \in \alpha$, let S_i be the set of all unordered pairs $[x, y]$ of distinct vertices of X, such that

$$f_i(g_i(x)) \neq g_i(f_i(x)), f_i(g_i(y)) \neq g_i(f_i(y)), \text{ and}$$
$$f_i(x) = y \text{ or } g_i(x) = y.$$

Each set S_i is non-empty and of cardinality at most equal to $|V(X)|$. Therefore, there exists a pair $[x, y]$ that belongs to more than $|V(X)|$ of the sets S_i. For each such set S_i, either $[y, z] \in S_i$ for a suitable $z \in V(X)$, or $[x, w] \in S_i$ for a suitable $w \in V(X)$ (otherwise $f_i(g_i(x)) = g_i(f_i(x))$). Let $z \in V(X)$ be such that both $[x, y]$ and $[y, z]$ belong to more than $|V(X)|$ of the sets S_i. Let

S_1, S_2, S_3 be three such sets, and let

$$f_1(y) = f_2(y) = f_3(y) = x, \; g_1(y) = g_2(y) = g_3(y) = z.$$

Then $g_1(f_2(y)) = f_2(g_1(y))$, thus $g_1(x) = f_2(z)$, and $g_2(f_1(y)) = f_1(g_2(y))$, so that $g_2(x) = f_1(z)$. Moreover, $g_3(f_1(y)) = f_1(g_3(y))$, thus $g_3(x) = f_1(z)$, and $g_3(f_2(y)) = f_2(g_3(y))$, therefore $g_3(x) = f_2(z)$. Hence $g_1(x) = g_2(x)$ and $g_1(f_1(y)) = g_1(x) = g_2(x) = f_1(z) = f_1(g_1(y))$, which is a contradiction.

COROLLARY. *The following two statements are equivalent:*

(1) *For every group G of order α there exists a 3-regular graph X such that the group of automorphisms of X is isomorphic to G.*

(2) *The cardinal α is at most countable.*

5. Large rigid and asymmetric graphs in, $\mathscr{R}^\alpha$. This paragraph should be considered as an extension of § 3, where, in effect, we proved that there exists a connected rigid graph $X \in \mathscr{R}^{\aleph_0}$ with $|V(X)| = \beta$ if and only if $\beta \leqq \aleph_0$; and that there exists a rigid graph $X \in \mathscr{R}^{\aleph_0}$ with $|V(X)| = \beta$ if and only if $\beta \leqq 2^{\aleph_0}$. (In this section the Greek letters α, β, γ, etc. denote always infinite cardinals.)

THEOREM 5. *Let α be any uncountable cardinal.*

(a) *If α is a regular cardinal, then there exists a connected rigid graph $X \in \mathscr{R}^\alpha$ such that $|V(X)| = \beta$ if and only if $\beta < \alpha$.*

(b) *If α is a singular cardinal, then there exists a connected rigid graph $X \in \mathscr{R}^\alpha$ such that $|V(X)| = \beta$ if and only if $\beta \leqq \alpha$.*

Proof. (a) As mentioned in § 1, if α is regular then each $X \in \mathscr{R}^\alpha$ which is connected has $|V(X)| < \alpha$; if $\beta < \alpha$, then by [**2**] or [**11**] there exists a connected rigid graph X with $|V(X)| = \beta$. Clearly, this X belongs to $\mathscr{R}^\alpha$.

(b) According to the proof of Theorem 2, each $X \in \mathscr{R}^\alpha$ which is connected satisfies $|V(X)| \leqq \alpha$. In view of part (a) of this proof, it remains only to construct a connected rigid graph $X \in \mathscr{R}^\alpha$ with $|V(X)| = \alpha$. Since α is singular, there exist pairwise disjoint sets B, B_b $(b \in B)$ such that $|B| < \alpha$, $|B_b| < \alpha$ $(b \in B)$ and $|B \cup \bigcup_{b \in B} B_b| = \alpha$. Let X_B and X_b be graphs such that

(1) $V(X_B) = B$, $V(X_b) = B_b$, for all $b \in B$,

(2) every edge of X_B, respectively X_b, belongs to a triangle of X_B, respectively X_b,

(3) for each $b \in B$, the pair X_B, X_b is mutually rigid.

The existence of such graphs can be deduced from [**2**] (or [**11**]) and [**6**]. Let $x_b \in V(X_b)$ for each $b \in B$. Then the graph X defined by

$$V(X) = V(X_B) \cup \bigcup_{b \in B} V(X_b) \text{ and}$$

$$E(X) = E(X_B) \cup \bigcup_{b \in B} E(X_b) \cup \{[b, x_b] : b \in B\}$$

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has α-bounded degrees and is connected. It is also rigid, for the edges $[b, x_b]$ ($b \in B$) do not belong to any triangles of X and hence $f|X_B = 1_{X_B}$ for every homomorphism $f: X \to X$; thus $f(b) = b$ (for all $b \in B$) and hence $f(x_b) = x_b$, f is the identity.

Once we know how large the connected rigid graphs can be, we will be able to determine how large the rigid ones can be also.

Let X be a graph; to change X into a relation, we define an *orientation* o of X to be an anti-reflexive, anti-symmetric relation $o(X) = (V(X), R)$ such that $[x, y] \in E(X)$ if and only if exactly one of $(x, y) \in R$, $(y, x) \in R$ holds.

Now let X be a rigid and connected graph, $|V(X)| = \beta$. Firstly, $o(X)$ is a rigid relation for every orientation o; secondly if o and o' are two different orientations, i.e., $o(X)$ and $o'(X)$ two different relations, then there is no homomorphism $o(X) \to o'(X)$. There are 2^β different orientations of X, i.e., $\{o(X)*Z : o$ an orientation of $X\}$ is a mutually rigid family of connected graphs, and it has the cardinality 2^β. Thus by choosing a sufficiently large subset O of the set of all orientations of X we can construct a rigid graph

$$Y = \bigcup_{o \in O} [o(X)*Z] \quad \text{with} \quad |V(Y)| = \gamma$$

for any γ satisfying $\beta \leqq \gamma \leqq 2^\beta$ ($|V(o(X)*Z)| = |V(X)|$ since Z is finite). Moreover if $X \in \mathscr{R}^\alpha$, then $Y \in \mathscr{R}^\alpha$ as well.

Hence, if α is singular, then by the previous Theorem we have a rigid graph $X \in \mathscr{R}^\alpha$ with $|V(X)| = \gamma$ for every $\gamma \leqq 2^\alpha$. It seems that if α is regular, there is a rigid graph $X \in \mathscr{R}^\alpha$ with $|V(X)| = \gamma$ for every $\gamma < 2^\alpha$; but the situation is not so simple and depends on the sophistication of the set theory. If we assume the generalized continuum hypothesis then $\gamma < 2^\alpha$ if and only if $\gamma \leqq \alpha$, and therefore it will suffice to find a rigid graph $Y \in \mathscr{R}^\alpha$ with $|V(Y)| = \alpha$; if there exists a $\beta < \alpha$ such that $2^\beta \geqq \alpha$, then the desired Y can be constructed as indicated, from a connected rigid $X \in \mathscr{R}^\alpha$ with $|V(X)| = \beta$ (such exists in view of Theorem 5 (a)). A regular cardinal α for which

$$\beta < \alpha \Rightarrow 2^\beta < \alpha$$

(if such a cardinal exists) is called *strongly inaccessible*. In a set theory in which the generalized continuum hypothesis is valid a cardinal is strongly inaccessible if and only if it is inaccessible. Comparison with the Remark in § 1 gives:

COROLLARY. (a) *If α is an uncountable regular cardinal, α not inaccessible, and if we assume the generalized continuum hypothesis, then a rigid graph $X \in \mathscr{R}^\alpha$ with $|V(X)| = \gamma$ exists if and only if $\gamma < 2^\alpha$.*

(b) *If α is singular or $\alpha = \aleph_0$, then a rigid graph $X \in \mathscr{R}^\alpha$ with $|V(X)| = \gamma$ exists if and only if $\gamma \leqq 2^\alpha$.*

Finally we observe that both Theorem 5 and its Corollary remain valid if we replace each word "rigid" by "asymmetric".

6. The smallest order of groups non-representable in $\mathscr{R}^\alpha$. P. Erdös suggested (in a private conversation) the following question: for a given cardinal α, what is the smallest cardinal $\beta(\alpha)$ such that there exists a group G of order $\beta(\alpha)$ which cannot be represented as the monoid of endomorphisms of any graph with α-bounded degrees (i.e., $X \in \mathscr{R}^\alpha \Rightarrow M(X) \not\cong G$). Such $\beta(\alpha)$ exists in view of Theorem 2 (and $\beta(\alpha)$ is at most the cardinality of the power-set of the power-set of $2^{\max(\alpha,\aleph_0)}$).

The Corollary in § 4 states that $\beta(4) = \aleph_1$; according to the Remark preceding it $\beta(5) = \beta(6) = \ldots = \beta(\aleph_0) = \aleph_1$. Moreover $\beta(1) = 2$, $\beta(2) = \beta(3) = 3$, as can easily be seen. Therefore it remains only to determine $\beta(\alpha)$ for α uncountable.

For α countable, we have already indicated (§§ 3 and 4) the correspondence between the possible sizes of rigid graphs in $\mathscr{R}^\alpha$ and the possible orders of groups representable in $\mathscr{R}^\alpha$. A similar correspondence exists for α uncountable.

THEOREM 6. *Let α be any uncountable cardinal.*

(a) *For every group G of order $\beta < \alpha$ there is a graph $X \in \mathscr{R}^\alpha$ such that $M(X) \cong G$.*

(b) *If $\beta > \alpha$, then there exists a group G of order β such that, for each $X \in \mathscr{R}^\alpha$, $M(X) \not\cong G$.*

(c) *If α is a regular cardinal, then there exists a group G of order α such that, for each $X \in \mathscr{R}^\alpha$, $M(X) \not\cong G$.*

(d) *If α is a singular cardinal, then for every group G of order α there is a graph $X \in \mathscr{R}^\alpha$ such that $M(X) \cong G$.*

Proof. To prove (a) and (d), we take a connected rigid $Y' \in \mathscr{R}^\alpha$ such that $|V(Y')| = \beta \leqq \alpha$ (equality holding only if α is singular) which exists by Theorem 5, and choose a set O of β different orientations of Y' and perform the product $o(Y')*Z$ for each $o \in O$. Then for every group G of order β we have graphs Y, Y_g, $g \in G' = G - \{1\}$, which satisfy (i), (ii), (v) and (vi) where x_g, $g \in G'$, are pairwise distinct vertices of Y, and $y_g^1 \neq y_g^2 \in V(Y_g)$ $(g \in G)$, and moreover Y, Y_g $(g \in G)$ belong to $\mathscr{R}^\alpha$. Therefore the graph X defined in § 4 is also in $\mathscr{R}^\alpha$ and $M(X) \cong G$.

If $\beta \geqq \alpha$ (equality holding only if α is regular), let G be the alternating group on β, and suppose, as we may, that $M(X) \cong G = \mathrm{Alt}(\beta)$ for a *connected* graph $X \in \mathscr{R}^\alpha$. Then $|V(X)| < \beta$, since $|V(X)| \leqq \alpha \leqq \beta$ and if $\alpha = \beta$, then α is regular and $|V(X)| < \alpha$. However, this is impossible according to the Remark in § 4.

COROLLARY. (a) *$\beta(\aleph_z) = \aleph_z$ if $\aleph_z$ is an uncountable regular cardinal.*

(b) *$\beta(\aleph_z) = \aleph_{z+1}$ if $\aleph_z$ is singular or $\aleph_z = \aleph_0$.*

If $\tilde{\beta}(\alpha)$ denotes the smallest cardinal such that there exists a group G of order $\tilde{\beta}(\alpha)$ which cannot be represented as the group of *automorphisms* of any graph in $\mathscr{R}^\alpha$, then $\tilde{\beta}(\alpha) = \beta(\alpha)$ for any cardinal $\alpha > 1$, $\tilde{\beta}(1) = 3$.

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