ON TOTALLY FREE CROSSED MODULES

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0. Introduction. In **[10]** we associate to a crossed module (T, G, ∂) an invariant abelian crossed module $H_2(T, G, \partial)$. The construction uses presentations by Set-free crossed modules. Now, Set-free crossed modules are special cases of totally free crossed modules, which are algebraic models of 2-dimensional CW complexes used by several authors (see **[1]** and **[6]**). The aim of this paper is to show that $H_2(T, G, \partial)$ can also be constructed from presentations by arbitrary totally free crossed modules.

Section 1 contains some standard definitions and results on crossed modules. In Section 2 we characterize a class $\mathcal{E}$ of epimorphisms, with respect to which totally free crossed modules are projective, and we prove the existence of $\mathcal{E}$-projective presentations. We show that this class is stable under pullbacks. With all the previous results we can use the proof given in **[10]** to get in Theorem 5 the invariant $H_2(T, G, \partial)$. In particular we obtain a formula for the second integral homology of a crossed module which generalizes the Hopf formula in group homology. For instance, it would allow us to obtain, for a crossed module, a generalized Hopf formula similar to the one obtained in **[2]** for a group. Also if we take a G-module A, we get that $H_2(A, G, 0) = (H_1(G, A), H_2(G), 0)$.

Results of this type have also been obtained in **[6]**, using topological methods.

1. Some results on crossed modules. *A crossed module* (T, G, ∂) is a group homomorphism $\partial : T \to G$ together with an action of G on T satisfying:

(i) ∂ is a precrossed module, i.e., $\partial({}^{g}t) = g\partial tg^{-1}$, for all $g \in G, t \in T$.
(ii) The Peiffer subgroup is trivial, i.e.,: ${}^{\partial t}s = tst^{-1}$, for all $t, s \in T$.

Example (1) If X is a path connected topological space and Y is a path connected subspace, $Y \subset X$, then $\partial : \pi_2(X, Y) \to \pi_1(Y)$ is a crossed module. This was the motivating example for Whitehead **[15]**.

(2) $(G, \operatorname{Aut} G, c)$ is a crossed module, where c assigns to each element $g \in G$, the inner automorphism of G, $c(g) : x \to gxg^{-1}$ for all $x \in G$.

(3) (N, G, i), where N is a normal subgroup of a group G, i is the inclusion and G acts on N by conjugation. This way, every group G can be seen as as crossed module in the two obvious ways: $(1, G, i)$ or (G, G, id).

(4) $(A, G, 0)$, where A is a G-module and the boundary operator is the zero map.

A *morphism of crossed modules* $(f, \phi) : (T, G, \partial) \to (T', G', \partial')$ is a pair of group morphisms $f: T \to T'$ and $\phi : G \to G'$, such that

(i) $\partial' f = \phi\partial$,
(ii) f is a G-group morphism, via $\phi, f({}^{g}t) = {}^{\phi(g)}f(t)$, for all $g \in G, t \in T$.

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Taking objects and morphisms as defined above we obtain the category $\mathcal{CM}$ of crossed modules. A morphism (f, ϕ) in $\mathcal{CM}$ is called *injective* if both f and ϕ are injective as group morphisms. A morphism (f, ϕ) in $\mathcal{CM}$ is called *surjective* if both f and ϕ are onto maps.

We denote by Aut (T, G, ∂) the group of automorphisms of an object (T, G, ∂). A crossed module (T', G', ∂') is a *crossed submodule* of a crossed module (T, G, ∂) if:

(i) T' is a subgroup of T and G' is a subgroup of G.
(ii) $\partial' = \partial|_{T'}$
(iii) The action of G' on T' is induced by that of G on T.

A crossed submodule (T', G', ∂') of a crossed module (T, G, ∂) is a *normal crossed submodule* if:

(i) G' is a normal subgroup of G
(ii) ${}^{g}t' \in T'$, for all $g \in G, t' \in T'$
(iii) ${}^{g}t \cdot t^{-1} \in T'$, for all $g' \in G', t \in T$.

$\mathcal{CM}$ has pullbacks, zero object, kernels and cokernels **[3]**, **[9]**.

A sequence of crossed module morphisms

$$(T', G', \partial') \overset{(f,\phi)}{\longrightarrow} (T, G, \partial) \overset{(f',\phi')}{\longrightarrow} (T'', G'', \partial'')$$

is called *exact* if the crossed submodules of (T, G, ∂), Im (f, ϕ) and Ker (f', ϕ'), coincide.

If K is a subgroup of G and S is a subgroup of a crossed G-module T we denote by $[K, S]$ the smallest subgroup of T containing the elements $(k_s)s^{-1}$, with $k \in K$ and $s \in S$

The definition of commutator subgroup can be generalized in the following way.

If (S, H, ∂) and (R, K, ∂) are two normal crossed submodules of a crossed module (T, G, ∂), then we define the commutator crossed submodule of (S, H, ∂) and (R, K, ∂) as the crossed submodule $([K, S][H, R], [H, K], \partial)$. This crossed submodule is denoted by $[(S, H, \partial), (R, K, \partial)]$, **[11]**. In particular the *commutator crossed submodule* **[11]** of (T, G, ∂), denoted by $(T, G, \partial)' = [(T, G, \partial), (T, G, \partial)]$, is defined as the crossed submodule $([G, T], G', \partial)$, where $[G, T] = < \{{}^{g}tt^{-1} / t \in T, g \in G\} >$ is the displacement subgroup of T relative to G, and $G' = [G, G]$ is the commutator subgroup of G.

EXAMPLES. (1) Let N be a normal subgroup of G. The commutator of (N, G, i) is $[(N, G, i), (N, G, i)] = ([G, N], G', i)$.

(2) Regarding a group G as a crossed module in the two usual ways, $N = 1$ or $N = G$, then $[(G, G, \mathrm{Id}), (G, G, \mathrm{Id})] = (G', G', \mathrm{Id})$ or $[(1, G, i), (1, G, i)] = (1, G', i)$.

(3) If A is a G-module, then $(A, G, 0)' = (A \cdot IG, G', 0)$, where IG is the augmentation ideal of G, [7].

We define the *first homology crossed module of* a crossed module (T, G, ∂) by

$$H_1(T, G, \partial) = (T, G, \partial)/(T, G, \partial)' = \big(T/[G, T], G/[G, G], \bar{\partial}\big).$$

EXAMPLES. (1) If N is a normal subgroup of G, then $H_1(N, G, i) = (N/[G, N], H_1(G), \bar{i})$.

(2) Viewing a group G as a crossed module in the two usual ways, we have $H_1(1, G, i) = (1, H_1(G), i)$, and $H_1(G, G, \mathrm{Id}) = (H_1(G), H_1(G), \mathrm{Id})$,

which gives the first integral homology group of a group as a particular case.

(3) If A, is a G-module, then $H_1(A, G, 0) = (H_0(G, A), H_1(G), 0)$.

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For a crossed module (T, G, ∂), we denote by Der (G, T) the set of all derivations from G to T, i.e., the set of maps $d : G \to T$ satisfying

$$d(xy) = d(x)^x d(y) \quad (x, y \in G)$$

Each derivation d defines endomorphisms $\sigma(= \sigma_d)$ and $\theta(= \theta_d)$ of G and T respectively, given by $\sigma(x) = \partial d(x)x$, $\theta(t) = d\partial(t)t$, $x \in G$, $t \in T$.

There is a monoid structure on Der(G, T), given by $d_1 \cdot d_2 = d$, where

$$d(x) = d_1\sigma_{d_2}(x)d_2(x) \left(= \theta_{d_1}d_2(x)d_1(x)\right);$$

the identity is the trivial derivation which sends every element of G to the identity of T. The Whitehead group $D(G, T)$ is defined as the group of units of Der (G, T), and its elements are called *regular derivations* [**12**].

In [**12**], Norrie defines the *actor* of a crossed module (T, G, ∂), which is denoted by $A(T, G, \partial)$, as the crossed module $(D(G, T), \mathrm{Aut}(T, G, \partial), \Delta)$, where $\Delta(d) = (\theta, \sigma)$ and the action of Aut (T, G, ∂) on the group $D(G, T)$ is defined by:

$$\left({}^{(\alpha,\phi)}d\right)(x) = \alpha d\phi^{-1}(x), (\alpha, \phi) \in \mathrm{Aut}(T, G, \partial), d \in D(G, T), \ x \in G.$$

There exists a morphism of crossed modules $(\eta, \gamma) : (T, G, \partial) \to A(T, G, \partial)$, where $\eta(t)(x) = t^x t^{-1}$, $\gamma(y) = (\alpha_y, \phi_y)$, where $\alpha_y(s) = {}^y s$, $\phi_y(x) = yxy^{-1}$ for $s, t \in T$, $x, y \in G$.

In the same way as in group theory, we define the *center* of the crossed module $Z(T, G, \partial)$ as Ker (η, γ) which is the crossed module $\left(T^G, Z(G) \cap \mathrm{st}_G(T), \partial\right)$ where $T^G = \{t \in T / {}^g t = t$ for all $g \in G\}$ and $\mathrm{st}_G(T)$ is the stabilizer in G of T, i.e. $\mathrm{st}_G(T) = \{g \in G / {}^g t = t$ for all $t \in T\}$ [**11**].

One says that the crossed module (T, G, ∂) is *abelian* if $(T, G, \partial) = Z(T, G, \partial)$, [**11**]. The crossed module (T, G, ∂) is abelian if and only if G is abelian and the action of the crossed module is trivial, which implies that T is also abelian.

We say that a crossed module (T, G, ∂) *acts* on (S, H, μ) if there exists a morphism of crossed modules $(T, G, \partial) \to A(S, H, \mu)$. If (S, H, ∂) is a normal crossed submodule of (T, G, ∂), then there exists a canonical morphism $(\eta, \gamma) : (T, G, \partial) \to A(S, H, \partial)$, where $\eta : T \to D(H, S)$ is given by $\eta(t)(h) = t^h t^{-1}$, and $\gamma : G \to \mathrm{Aut}(S, H, \partial)$ is given by $\gamma(g) = (\alpha_g, \phi_g)$, with $\alpha_g(s) = {}^g s$, $\phi_g(h) = ghg^{-1}$ for $s \in S$, $t \in T$, $h \in H$, $g \in G$.

Let (M, P, μ) and (N, V, ν) be two crossed modules, and let $(\varepsilon, \rho) : (N, V, \nu) \to A(M, P, \mu)$ be an action of (N, V, ν) on (M, P, μ), i.e., the following diagram is commutative.

$$\begin{array}{ccc} N & \xrightarrow{\nu} & V \\ \varepsilon \downarrow & & \rho \downarrow \\ D(P, M) & \xrightarrow{\Delta} & \mathrm{Aut}(M, P, \mu) \end{array}$$

If $\rho_1 : V \to \mathrm{Aut}(M)$, and $\rho_2 : V \to \mathrm{Aut}(P)$ are the two components of ρ, then N acts on M via $\rho_1 \cdot \nu$ and V acts on P via ρ_2, and so we can consider the semi-direct products $M \rtimes N$ and $P \rtimes V$.

Now, there exists an action of $P \rtimes V$ on $M \rtimes N$ defined as follows:

$$^{(p,v)}(m, n) = \left(p({}^v m)(\varepsilon({}^v n)(p))^{-1}, {}^v n\right)$$

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for $(p, v) \in P \rtimes V$ and $(m, n) \in M \rtimes N$, where ${}^{v}m$ means $\rho_1(v)(m)$. Then $(M \rtimes N, P \rtimes V, \pi)$ is a crossed module, where $\pi : M \rtimes N \to P \rtimes V$ is defined by $\pi(m, n) = (\mu(m), \nu(n))$. This crossed module [13] is called the *semi-direct product* of (M, P, μ) and (N, V, ν) relative to (ε, ρ) and it is denoted by $(M, P, \mu) \rtimes (N, V, \nu)$.

If (T, G, ∂) is a semi-direct product $(S, H, \partial) \rtimes (R, K, \partial)$, then there exists a short exact sequence of crossed modules split by $(i, j) : (R, K, \partial) \to (T, G, \partial)$:

$$(S, H, \partial) \to (T, G, \partial) \to (R, K, \partial),$$

where (i, j) is the inclusion morphism. Conversely, given any such split short exact sequence of crossed modules we have $(T, G, \partial) \cong (S, H, \partial) \rtimes (R, K, \partial)$, where the action of (R, K, ∂) on (S, H, ∂) is given by the composite $(\eta, \gamma) \cdot (s_1, s_2)$ where $(s_1, s_2) : (R, K, \partial) \to (T, G, \partial)$ is the section and $(\eta, \gamma) : (T, G, \partial) \to A(S, H, \partial)$ is the morphism defined above [13].

2. Totally Free Crossed Modules. Let $h : X \to F$ be a function from a set X to a free group F. A crossed module (T, F, ∂) is called *totally free* on h if

(i) X is a subset of T with h the restriction of ∂ and,

(ii) for any crossed module (T', G', ∂'), function $\nu : X \to T'$ and morphism $\phi : F \to G'$ satisfying $\partial'\nu = \phi h$ there is an unique morphism of crossed modules,

$$(f, \phi) : (T, F, \partial) \to (T', G', \partial'),$$

extending ν. The totally free crossed module on $h : X \to F$ always exists: let $\partial : < X \times F > \to F$ be the totally free precrossed module on h[5], that is, $< X \times F >$ is the free group with basis the set $X \times F$ with action of F defined by ${}^{f}(x, f') := (x, f'f)$ and $\partial(x, f) = fh(x)f^{-1}$ for $x \in X, f, f' \in F$. ∂ is zero on the Peiffer subgroup P and then $(< X \times F > /P, F, \partial)$ is the totally free crossed module on h [4].

The totally free crossed module on $h : X \to F$ is clearly unique up to isomorphism. The Set-free crossed module on a function $h : X \to Y$ is the totally free crossed module on $h : X \to Y \subset F$, where F is the free group with basis the set Y. The set-free crossed module can be interpreted by adjoint functors [10].

Proposition 1. *Let $(p, p') : (T', G', \partial') \to (T, G, \partial)$ be a surjective morphism of crossed modules. Then the following assertions are equivalent.*

(i) *The morphism* $\mathrm{Ker}\partial' \to \mathrm{Ker}\partial$ *is surjective and the morphism* $\mathrm{Coker}\partial' \to \mathrm{Coker}\partial$ *is an isomorphism.*

(ii) *The morphism $T' \to T \times_G G'$ is surjective.*

We denote by $\mathcal{E}$ the class of epimorphisms satisfying the conditions above.

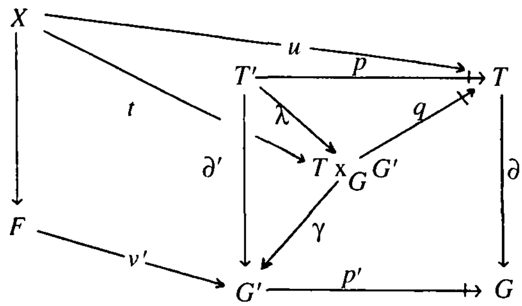
Proof. There is a commutative diagram with exact rows given by

$$\begin{array}{ccccccccc} 1 \to & \mathrm{Ker}\partial' \to & T' \to & G' \to & \mathrm{Coker}\partial' & \to 1 \\ & \downarrow & \downarrow p & \downarrow p' & \downarrow & \\ 1 \to & \mathrm{Ker}\partial \to & T \to & G \to & \mathrm{Coker}\partial & \to 1 \end{array}$$

such that p and p' are surjective. Diagram-chasing shows that $T' \to T\mathrm{x}_G G'$ is surjective if and only if $\mathrm{Ker}\partial' \to \mathrm{Ker}\partial$ is surjective and $\mathrm{Coker}\partial' \to \mathrm{Coker}\partial$ is an isomorphism.

PROPOSITION 2. *Every totally free crossed module is $\mathcal{E}$ projective.*

Proof. Let (M, F, μ) be a totally free crossed module on $h : X \to F$, $(p, p') : (T', G', \partial') \to (T, G, \partial)$ a morphism in the class $\mathcal{E}$, and $(u, u') : (M, F, \mu) \to (T, G, \partial)$ a morphism of crossed modules. If we donote by u the restriction to X, we have $\partial u = u'h$.

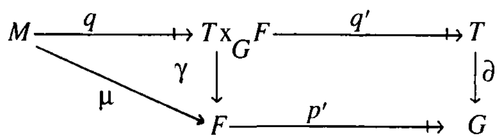


Since F is a free group, there exists $v' : F \to G'$ with $p'v' = u'$. The maps $v'h$ and u give a unique map $t : X \to T\mathrm{x}_G G'$ with $qt = u$ and $\gamma t = v'h$. Proposition 1 gives that λ is surjective, and then there exists a map $t' : X \to T'$ with $\lambda t' = t$. Then we get $\partial' t' = v'h$ and therefore a morphism $(t', v') : (M, F, \mu) \to (T', G', \partial')$ that verifies $(p, p')(t', v') = (u, u')$.

PROPOSITION 3. *Every crossed module (T, G, ∂) is the quotient of a totally free crossed module (M, F, μ) and there is an exact sequence*:

$$(V, R, \mu) \rightarrowtail (M, F, \mu) \overset{(p,p')}{\twoheadrightarrow} (T, G, \partial) \quad \text{with } (p, p') \in \mathcal{E}.$$

Proof. Let F be the free group with basis G, and $T\mathrm{x}_G F$ the pullback of ∂ and p', with $q' : T\mathrm{x}_G F \twoheadrightarrow T$. The crossed module $(T\mathrm{x}_G F, F, \gamma)$ is a quotient of the totally free crossed module (M, F, μ), $q : M \twoheadrightarrow T\mathrm{x}_G F$, on the function $\gamma : T\mathrm{x}_G F \to F$. If $p = q'q$, we have the following diagram.



We will prove that $(p, p') \in \mathcal{E}$ because p', p and q are surjective (see Proposition 1).

PROPOSITION 4. *In a pullback of crossed modules*

$$\begin{array}{ccc} (P, Q, \lambda) & \to & (T', G', \partial') \\ (q, q') \downarrow & & \downarrow (p, p') \\ (T'', G'', \partial'') & \to & (T, G, \partial) \end{array}$$

if the morphism $(p, p') \in \mathcal{E}$, then $(q, q') \in \mathcal{E}$.

Proof. One has $P = T'' \times_T T'$ and $Q = G'' \times_G G'$; also $T' \to T$, $G' \to G$ and $T' \to T \times_G G'$ are surjective. One can now check that $P \to T''$, $Q \to G''$ and $P \to T'' \times_{G''} Q$ are surjective.

3. $H_2(T,G,\partial)$. Now we will introduce the *second homology crossed module* of a crossed module using an $\mathcal{E}$-projective presentation, and we will show that this definition constitutes an invariant of the crossed module.

Given an $\mathcal{E}$-projective presentation

$$(V, R, \mu) \rightarrowtail (M, F, \mu) \twoheadrightarrow (T, G, \partial)$$

of the crossed module (T, G, ∂), we define the abelian crossed module $H_2(T, G, \partial)$ by

$$\begin{aligned} H_2(T, G, \partial) &= ((V, R, \mu) \cap [(M, F, \mu), (M, F, \mu)])/[(M, F, \mu), (V, R, \mu)] \\ &= (V \cap [F, M]/[R, M][F, V], R \cap [F, F]/[F, R], \mu_*) \end{aligned}$$

THEOREM 5. *$H_2(T, G, \partial)$ is independent up to isomorphism of the chosen $\mathcal{E}$-projective presentation and the correspondence $(T, G, \partial) \to H_2(T, G, \partial)$ defines a functor $H_2 : \mathcal{CM} \to \mathcal{ACM}$, where $\mathcal{ACM}$ denotes the category of abelian crossed modules.*

Proof. Consider the following two $\mathcal{E}$-projective presentations of the crossed module

$$(T, G, \partial) : (V, R, \mu) \rightarrowtail (M, F, \mu) \twoheadrightarrow (T, G, \partial),$$

and

$$(V', R', \mu') \rightarrowtail (M', F', \mu') \twoheadrightarrow (T, G, \partial)$$

Using the pullback construction, we get the following diagram:

$$\begin{array}{ccccc} (V'', R'', \mu'') & \rightarrowtail & (M'', F'', \mu'') & & (V', R', \mu') \\ \downarrow & & \downarrow & & \downarrow \\ & & (P, Q, \lambda) & \twoheadrightarrow & (M', F', \mu') \\ & & \downarrow & & \downarrow \\ (V, R, \mu) & \rightarrowtail & (M, F, \mu) & \twoheadrightarrow & (T, G, \partial) \end{array} ,$$

where (M'', F'', μ'') is an $\mathcal{E}$-projective presentation of (P, Q, λ), $(V'', R'', \mu'') = \mathrm{Ker}((M'', F'', \mu'') \to (T, G, \partial))$, by construction of the pullback, and $(P, Q, \lambda) \twoheadrightarrow (M, F, \mu)$ and $(V'', R'', \mu'') \twoheadrightarrow (V, R, \mu)$ both belong to $\mathcal{E}$ by Proposition 4. We obtain in this way a third $\mathcal{E}$-projective presentation

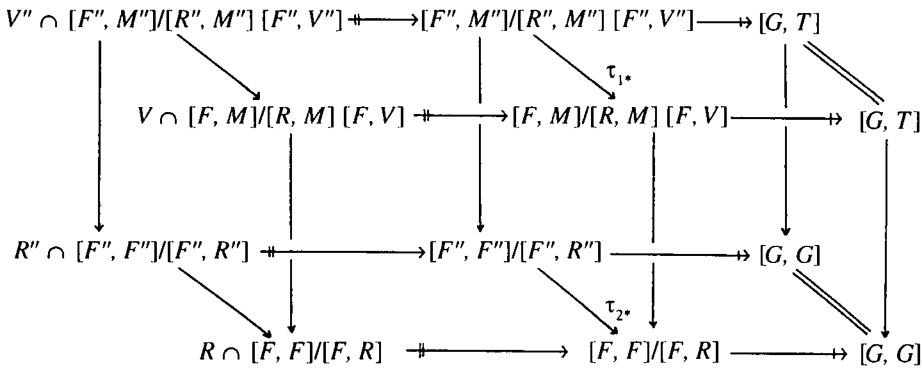
$$(V'', R'', \mu'') \rightarrowtail (M'', F'', \mu'') \twoheadrightarrow (T, G, \partial)$$

for (T, G, ∂). Since $(\tau_1, \tau_2) : (M'', F'', \mu'') \twoheadrightarrow (M, F, \mu)$ belongs to $\mathcal{E}$ and (M, F, μ) is a totally free crossed module, there exists a section $(s_1, s_2) : (M, F, \mu) \to (M'', F'', \mu'')$. By the properties of the pullback we have a section $(V, R, \mu) \to (V'', R'', \mu'')$.

Now, split short exact sequences with a chosen section (s_1, s_2) are equivalent to semi-direct products, and we have

$$(M'', F'', \mu'') \cong (N, E, \mu) \rtimes (M, F, \mu) = (N \rtimes M, E \rtimes F, \pi),$$
$$(V'', R'', \mu'') \cong (N, E, \mu'') \rtimes (V, R, \mu) = (N \rtimes V, E \rtimes R, \pi)$$

To show the independence of $H_2(T, G, \partial)$ from the $\mathcal{E}$-projective presentation it will be enough to find an isomorphism between $(V \cap [F, M])/[R, M][F, V], R \cap [F, F]/[F, R], \mu_*)$ and $\big(V'' \cap [F'', M'']/[R'', M''][F'', V''], R'' \cap [F'', F'']/[F'', R''], \mu''_*\big)$. Taking into account that the following diagram of short exact sequences commutes



it is enough to show that the induced morphism

$$(\tau_{1*}, \tau_{2*}) : [F'', M'']/[R'', M''][F'', V''], [F'', F'']/[F'', R''], \mu''_*)$$
$$\rightarrow ([F, M])/[R, M][F, V], [F, F]/[F, R], \mu_*)$$

is an isomorphism of crossed modules, as it passes to the kernels.

The classic theory of Hopf's invariant [**8**] gives us that τ_{2*} is an isomorphism of groups. τ_{1*} is also an isomorphism: given that $\tau_1 \cdot s_1 = \mathrm{id}_M$, one has $\tau_{1*} \cdot s_{1*} = \mathrm{id}[F, M])/[R, M][F, V]$, where $s_{1*}\big({}^f m m^{-1}[R, M][F, V]\big) = s_1\big({}^f m m^{-1}\big)[R'', M''][F'', V'']$ with $f \in F, m \in M$, and $\tau_{1*}\big({}^{es_2(f)}(ns_1(m))(ns_1(m))^{-1}[R'', M''][F'', V'']\big) = {}^f m m^{-1}[R, M][F, V]$, where $e \in E$, $n \in N$, $es_2(f) \in F''$, $ns_1(m) \in M''$, because (τ_1, τ_2) is a morphism of crossed modules. To see that $s_{1*} \cdot \tau_{1*} = \mathrm{id}[F'', M'']/[R'', M''][F'', V'']$, i.e., ${}^{es_2(f)}(ns_1(m))(ns_1(m))^{-1}[R'', M''][F'', V''] = s_1\big({}^f m m^{-1}\big)[R'', M''][F'', V'']$, notice that

$$\begin{aligned}
{}^{es_2(f)}(ns_1(m))(ns_1(m))^{-1} &= {}^{es_2(f)}\, n\, {}^{es_2(f)}s_1(m)\, {}^{s_1(m)^{-1}}n^{-1} s_1(m)^{-1} \\
&= {}^{es_2(f)}n\, {}^{es_2(f)}s_1(m)\, {}^{s_2(f)}s_1(m)^{-1}\, {}^{s_2(f)}s_1(m)\, {}^{s_1(m)^{-1}}n^{-1} s_1(m)^{-1} \\
&= {}^{es_2(f)}n\, {}^{es_2(f)}s_1(m)\, {}^{s_2(f)}s_1(m)^{-1}\, {}^{s_1({}^f m)}\Big({}^{s_1(m)^{-1}}n^{-1}\Big) s_1\big({}^f m\big) s_1(m)^{-1} \\
&= {}^{es_2(f)}n\, {}^{es_2(f)}s_1(m)\, {}^{s_2(f)}s_1(m)^{-1}\, {}^{\mu'' s_1({}^f m m^{-1})}n^{-1} s_1\big({}^f m m^{-1}\big) \\
&= {}^{es_2(f)}n\, {}^{es_2(f)}s_1(m)\, {}^{s_2(f)}s_1(m)^{-1}\, {}^{s_2[f, \mu(m)]}n^{-1} s_1\big({}^f m m^{-1}\big) \\
&= {}^{es_2(f)}n\, {}^{es_2(f)}s_1(m)\, {}^{s_2(f)}s_1(m)^{-1}\, {}^{s_2(f)}n^{-1}\, {}^{s_2(f)}n^{\,s_2([f, \mu(m)]f^{-1})}\big({}^{s_2(f)}n^{-1}\big) s_1\big({}^f m m^{-1}\big) \\
&\equiv s_1\big({}^f m m^{-1}\big)
\end{aligned}$$

since ${}^{es_2(f)}s_1(m)\, {}^{s_2(f)}s_1(m)^{-1} \in [R'', M'']$, ${}^{es_2(f)}n\, {}^{s_2(f)}n^{-1} \in [F'', V'']$, and ${}^{s_2(f)}n^{\,s_2([f, \mu(m)]f^{-1})}\big({}^{s_2(f)}n^{-1}\big) \in [F'', V'']$.

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To see the action on the arrows, let $(f,\phi):(T',G',\partial')\to(T,G,\partial)$ be a morphism of crossed modules, and consider an $\mathcal{E}$-projective presentation for each of the two crossed modules. Since (M',F',μ') is $\mathcal{E}$-projective, one can consider the following commutative diagram.

$$\begin{array}{ccccc} (V,R',\mu') & \rightarrowtail & (M',F',\mu') & \twoheadrightarrow & (T',G',\partial') \\ \downarrow & & \downarrow & & \downarrow \\ (V,R,\mu) & \rightarrowtail & (M,F,\mu) & \twoheadrightarrow & (T,G,\partial) \end{array}$$

In the same way as above, we get the morphism

$$\begin{aligned} H_2(T',G',\partial') &= \big(V'\cap[F',M']/[R',M'][F',V'],\ R'\cap[F',F']/[F',R'],\ \mu'_*\big) \\ &\to (V\cap[F,M]/[R,M][F,V],\ R\cap[F,F]/[F,R],\ \mu_*) = H_2(T,G,\partial) \end{aligned}$$

Checking the conditions of functoriality is now routine.

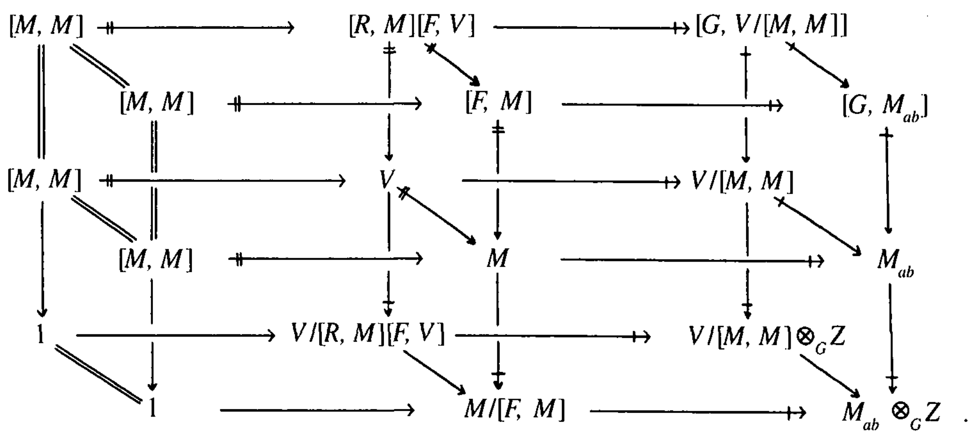
COROLLARY 6. *If (M,F,μ) is a totally free crossed module, then $H_2(M,F,\mu)=(1,1,1)$.*

Proof. $(1,1,1)\rightarrowtail(M,F,\mu)\to(M,F,\mu)$ is an $\mathcal{E}$ projective presentation of (M,F,μ).

EXAMPLES (1) If we consider a group G as a crossed module in the two usual ways, (G,G,id) or $(1,G,i)$, then from the classic formula of Hopf **[8]** we obtain $H_2(G,G,\mathrm{id})=(H_2(G),H_2(G),\mathrm{id})$, or $H_2(1,G,i)=(1,H_2(G),i)$.

(2) If A is a G-module, then $H_2(A,G,0)=(H_1(G,A),H_2(G),0)$. Indeed, let $R\rightarrowtail F\twoheadrightarrow G$ be a free presentation of G and $(V,R,\mu)\rightarrowtail(M,F,\mu)\twoheadrightarrow(A,G,0)$ a totally free presentation as in Proposition 3, where $A\times_G F=A\times R$. Then $\mu(M)=R$, M_{ab} is a free G-module **[4]** and $V/[M,M]\twoheadrightarrow M_{ab}\twoheadrightarrow A$ is a projective presentation of G-modules for A. So $H_1(G,A)=\mathrm{Ker}(V/[M,M]\otimes_G Z\to M_{ab}\otimes_G Z)$, where $V/[M,M]\otimes_G Z=(V/[M,M])/[G,V/[M,M]]$ and $M_{ab}\otimes_G Z=M_{ab}/[G,M_{ab}]$ **[4]**.

As $V\cap[F,M]/[R,M][F,V]=\mathrm{Ker}(V/[R,M][F,V]\to M/[F,M])$, the following commutative diagram, obtained by using the cross lemma**[14]**, gives the result.



(3) If $R\overset{i}{\rightarrowtail}F\twoheadrightarrow G$ is a free presentation of G and $(V,0,0)\rightarrowtail(M,F,\mu)\twoheadrightarrow(R,F,i)$ a totally free presentation as in Proposition 3, then $V\rightarrowtail M_{ab}\twoheadrightarrow R_{ab}$ is a free presentation of

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G-modules for R_{ab} [4]. So $H_1(G, R_{ab}) = \mathrm{Ker}(V \otimes_G Z \to M_{ab} \otimes_G Z) = \mathrm{Ker}(V/[F, V] \to M/[F, M]) = V \cap [F, M]/[F, V]$, using the same reasoning as in (2). By the reduction theorem [7] $H_1(G, R_{ab}) = H_3(G)$ and so $H_2(R, F, i) = (H_3(G), 0, 0)$.

THEOREM 7. *Let* $(P, N, \partial) \rightarrowtail (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$ *be a short exact sequence of crossed modules, such that the epimorphism* $(T, G, \partial) \to (U, Q, \omega)$ *belongs to* $\mathcal{E}$. *Then there exists the following five term exact (and natural) sequence in homology:*

$$H_2(T, G, \partial) \to H_2(U, Q, \omega) \to (P/[G, P][N, T], N/[G, N], \bar{\partial}) \to H_1(T, G, \partial)$$

$$\to H_1(U, Q, \omega) \to (1, 1, 1)$$

Proof. See 4.1 Theorem in [10].

EXAMPLES. (1) If we consider a group G as a crossed module in any of the two usual ways, we get the five term exact sequence in integral homology of groups [7]:

$$H_2(G) \to H_2(Q) \to N/[G, N] \to H_1(G) \to H_1(Q) \to 1$$

where $1 \to N \to G \to Q \to 1$ is a short exact sequence of groups.

(2) Considering the sequence $(A', 0, 0) \rightarrowtail (A, G, 0) \twoheadrightarrow (A'', G, 0)$ we get in the first component the last five terms of the long exact sequence of homology associated to a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of G-modules [7].

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