

EXISTENCE OF PERFECT PICARD SETS

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Dedicated to the memory of Professor TADASI NAKAYAMA

1. Let E be a totally disconnected compact set in the z -plane and let Ω be its complement with respect to the extended z -plane. Then Ω is a domain and we can consider a single-valued meromorphic function $f(z)$ in Ω which has a transcendental singularity at each point $\zeta \in E$. Suppose that E is a null-set of the class W in the sense of Kametani [4] (= the class $N_{\mathfrak{B}}$ in the sense of Ahlfors and Beurling [1]). Then the cluster set of $f(z)$ at each transcendental singularity is the whole w -plane and hence $f(z)$ has an essential singularity at each point of E . We shall say that a value w is exceptional for $f(z)$ at an essential singularity $\zeta \in E$ if there exists a neighborhood of ζ where the function $f(z)$ does not take this value w . If each $f(z)$ has at most n exceptional values at each singularity $\zeta \in E$, we shall call E an n -Picard set using the terminology of Lehto [5] and call a 2-Picard set a Picard set simply. For any E , by Besse's theorem, there exists a single-valued regular function $g(z)$ in Ω possessing E as the set of singularities. Therefore, considering the function $\exp g(z)$ in Ω , we see that there exists no 1-Picard set. Thus we need consider n -Picard sets only for $n \geq 2$.

For any countable E , every $f(z)$ has at most two exceptional values at each singularity $\zeta \in E$, because any neighborhood of ζ contains isolated points of E , and hence E is a Picard set. But for a non-countable E , there needs some condition in order to be an n -Picard set for some n , even if E is of logarithmic capacity zero (see Matsumoto [6]). Carleson [3] and the author [7], [8] have given sufficient conditions for sets E to be n -Picard sets for n not smaller than 3 and examples of perfect E by means of Cantor sets. There has remained a very interesting problem unsolved. *Is there a perfect Picard set?*

The purpose of this paper is to give Cantor sets which are Picard sets. The Schottky theorem will also play important roles as in papers [3], [7], [8].

Received May 26, 1965.

2. We shall consider the Riemann sphere Σ with radius $1/2$ touching the w -plane at the origin. For any two points w and w' in the w -plane we denote by $[w, w']$ the chordal distance between them, that is,

$$[w, w'] = \begin{cases} \frac{|w - w'|}{\sqrt{(1 + |w|^2)(1 + |w'|^2)}} & \text{if } w \neq \infty \text{ and } w' \neq \infty \\ \frac{1}{\sqrt{1 + |w|^2}} & \text{if } w' = \infty. \end{cases}$$

Further we denote by $C(w; \delta)$ ($\delta > 0$) the spherical open disc with center w and with chordal radius δ .

First we shall prove the following lemma which is a revised form of Carleson's [3].

LEMMA 1. *Let $w = f(z)$ be a single-valued meromorphic function on an annulus $1 \leq |z| \leq e^\mu$ ($\mu > 0$). If $f(z)$ takes there no value in a spherical disc $C(w_0; \delta)$, then there exists a positive constant A_δ depending only on δ such that the diameter of the image of $|z| = e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $A_\delta e^{-\mu/2}$ for sufficiently large μ .*

In particular, if δ is sufficiently close to 1, that is, the complementary spherical disc $C(-1/\bar{w}_0; d)$ of $C(w_0; \delta)$, $d = \sqrt{1 - \delta^2}$, has a radius sufficiently small, we have

$$A_\delta < Bd,$$

where B is a positive constant.

Proof. We may assume without any loss of generality that the center w_0 of $C(w_0; \delta)$ is the point at infinity, for otherwise we can transform w_0 to the point at infinity by the linear transformation $(1 + \bar{w}_0 w)/(w - w_0)$, under which the chordal distance remains invariant. Let $|w| > M$ be the domain in the w -plane corresponding to $C(w_0; \delta)$. Then

$$|f(z)| \leq M \quad \text{on} \quad 1 \leq |z| \leq e^\mu.$$

By Cauchy's integral theorem, we have

$$f'(z) = \frac{1}{2\pi i} \left\{ \int_{|\zeta|=e^\mu} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta - \int_{|\zeta|=1} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right\}$$

for every z on $|z| = e^{\mu/2}$ and hence, if $\mu \geq 2$,

$$|f'(z)| \leq \frac{M}{2\pi} \left\{ \frac{2\pi e^\mu}{(e^\mu - e^{\mu/2})^2} + \frac{2\pi}{(e^{\mu/2} - 1)^2} \right\} \leq \frac{2e^2}{(e-1)^2} M e^{-\mu}.$$

Therefore we have

$$\int_{|z|=e^{\mu/2}} |f'(z)| |dz| \leq \frac{2e^2}{(e-1)^2} M e^{-\mu} \cdot 2\pi e^{\mu/2} = \frac{4\pi e^2}{(e-1)^2} M \cdot e^{-\mu/2}.$$

The left side is the length of the image curve $f(|z|=e^{\mu/2})$, and hence the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the metric $|dw|$, consequently with respect to the chordal distance, is dominated by $(2\pi e^2/(e-1)^2) M e^{-\mu/2}$. We can take $(2\pi e^2/(e-1)^2) M$ as A_δ , for M depends only on δ .

If $d < 1/2$, then $M < 2d$. Hence

$$B = 4\pi e^2/(e-1)^2$$

is one of the wanted. Our lemma is established.

Now let $w = f(z)$ be a single-valued regular function in an annulus $1 < |z| < e^\mu$ ($\mu > 0$) omitting two values 0 and 1. We use Bohr-Landau's theorem [2]; if $g(z)$ is regular in $|z| < 1$ and $g(z) \neq 0, 1$ there, then

$$\max_{|z|=r} |g(z)| \leq \exp\left(\frac{K \log(|g(0)| + 2)}{1-r}\right) \quad \text{for any } r, 0 \leq r < 1,$$

where K is a positive constant (a precise form of Schottky's theorem). From this we can prove the following corollary of Lemma 1.

COROLLARY. *There exists a positive constant A not depending on μ and $f(z)$ such that the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $Ae^{-\mu/2}$ for sufficiently large μ .*

Proof. From Bohr-Landau's theorem, we can see easily that if $w = g(z)$ is a regular function in $1 < |z| < e^\sigma$ ($\sigma > 0$) such that

$$g(z) \neq 0, 1 \quad \text{and} \quad \min_{|z|=e^{\sigma/2}} |g(z)| < a \quad \text{for a positive } a,$$

then there is a positive constant b depending only on a and σ such that

$$\max_{|z|=e^{\sigma/2}} |g(z)| \leq b.$$

For a fixed $\sigma > 0$, we shall show, supposing $\mu > \sigma$, that there exists a positive number δ not depending on μ and $f(z)$ such that the image curves of $|z|=e^{\sigma/2}$ and $|z|=e^{\mu-\sigma/2}$ by $f(z)$ lie outside at least one of three discs $\mathcal{C}(0; \delta)$, $\mathcal{C}(1; \delta)$

and $C(\infty; \delta)$. In fact, let z_1 and z_2 be points on $|z| = e^{\sigma/2}$ and $|z| = e^{\mu - \sigma/2}$ respectively. Then $f(z_1)$ and $f(z_2)$ lie outside at least one of three discs $C(0; \delta')$, $C(1; \delta')$ and $C(\infty; \delta')$, where δ' is a positive number such that these three discs are mutually disjoint and hence can be taken independently of μ and $f(z)$. Suppose that $f(z_1)$ and $f(z_2)$ lie outside $C(\infty; \delta')$. Then by the fact mentioned above, we can find a positive δ_∞ such that the image curves of $|z| = e^{\sigma/2}$ and $|z| = e^{\mu - \sigma/2}$ by $f(z)$ lie outside $C(\infty; \delta_\infty)$. Next suppose that $f(z_1)$ and $f(z_2)$ lie outside $C(1; \delta')$. Then we see using the linear transformation which transforms points $w = 0$, $w = 1$ and $w = \infty$ to points $w = 1$, $w = \infty$ and $w = 0$ respectively that there is a positive δ_1 such that the images of $|z| = e^{\sigma/2}$ and $|z| = e^{\mu - \sigma/2}$ by $f(z)$ lie outside $C(1; \delta_1)$. Similarly we can find a positive δ_0 and set

$$\delta = \min \{ \delta_0, \delta_1, \delta_\infty \} > 0.$$

Obviously this δ satisfies our conditions. Now by the maximum principle we see that the image of $e^{\sigma/2} \leq |z| \leq e^{\mu - \sigma/2}$ by $f(z)$ lies outside at least one of $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$. Hence by Lemma 1 we can conclude that the diameter of the image of $|z| = e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $A_\delta e^{-(\mu - \sigma)/2}$, so that $A = A_\delta e^{\sigma/2}$ satisfies our condition.

3. Let E be a Cantor set on the closed interval $I_0: [-1/2, 1/2]$ on the real axis of the z -plane with successive ratios ξ_n , $0 < \xi_n = 2\ell_n < 2/3$. Defining the Cantor set E , we repeat successively to exclude an open segment from the middle of another segment and there remain 2^n segments of equal length $\prod_{k=1}^n \ell_k$ after we repeat n times, beginning with the interval I_0 . We denote these segments by $I_{n,k}$ ($n = 1, 2, \dots$; $k = 1, 2, \dots, 2^n$) and denote by $S_{n,k}$ ($n = 1, 2, \dots$; $k = 1, 2, \dots, 2^n$) the following annuli on the complementary domain Ω of E :

$$S_{n,k} = \{ z; (\prod_{k=1}^n \ell_k)(1 - \ell_{n+1}) < |z - z_{n,k}| < (\prod_{k=1}^{n-1} \ell_k)(1 - \ell_n)/2 \},$$

where $z_{n,k}$ is the middle point of $I_{n,k}$. The harmonic modulus μ_n of $S_{n,k}$ is greater than $\log(2/3\xi_n)$. We map $S_{n,k}$ conformally onto the annulus $1 < |\eta| < e^{\mu_n}$ and consider the inverse image $\Gamma_{n,k}$ of the circle $|\eta| = e^{\mu_n/2}$ on $S_{n,k}$. Supposing that $S_{n,k}$ encloses $S_{n+1,2k-1}$ and $S_{n+1,2k}$, we denote by $A_{n,k}$ the triply connected domain bounded by three curves $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. We now prove the following

LEMMA 2. *Let the successive ratios ξ_n satisfy the condition*

$$\lim_{n \rightarrow \infty} \xi_n = 0$$

and let $w = f(z)$ be a single-valued regular function in Ω which omits two values 0 and 1 and has E as the set of essential singularities. Then for any sufficiently small $\delta < 0$, there exists an infinite number of $\Delta_{n,k}$ such that the images of the three boundary components $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are contained completely in the three discs $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ one by one, where we assume that these three discs are mutually disjoint, and $f(z)$ takes each value outside the union of these three discs once and only once in $\Delta_{n,k}$.

Proof. Contrary suppose that there exists only a finite number of $\Delta_{n,k}$ such that their three boundary components are mapped into the three discs $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ one by one, and denote by n_0 the maximum of n taken over all such $\Delta_{n,k}$. Since

$$\mu_n > \log(2/3 \xi_n) \quad \text{and} \quad \lim_{n \rightarrow \infty} \xi_n = 0,$$

we can take $n_1 \geq n_0$, for a fixed $\sigma > 0$, so large that for each $n \geq n_1$

$$\mu_n > 2 + \sigma, \quad Ae^{-\mu_n/2} < K = \min\{1/24, \delta/3\} \quad \text{and} \quad Be^{-\mu_n/2} < 1/12,$$

where A and B are the constants in Lemma 1 and its corollary. The diameter of the image of $\Gamma_{n,k}$ with respect to the chordal distance is dominated by $Ae^{-\mu_n/2}$, consequently by K , if $n \geq n_1$, and hence there exists a spherical disc $C_{n,k}$ with chordal radius K which contains completely the image $f(\Gamma_{n,k})$. For $n \geq n_1$ take $\Delta_{n,k}$ with boundary curves $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. Then, since $K < \delta/3$, at least one of $C(0; \delta/3)$, $C(1; \delta/3)$ and $C(\infty; \delta/3)$, say $C(\infty; \delta/3)$, is disjoint from the union of $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ and hence each one of $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ cannot be disjoint from the union of the other two, for, if so for some one, there is $z_0 \in \Delta_{n,k}$ such that $f(z_0)$ lies outside the union of $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ and can be joined to the point at infinity with a curve A lying outside this union, and we are led to a contradiction that the element of the inverse function f^{-1} corresponding to z_0 can be continued analytically along A up to a point arbitrarily near the point at infinity so that $f(z)$ takes the value ∞ in $\Delta_{n,k}$. Therefore we can conclude that

(1°) for every $\Delta_{n,k}$, $n \geq n_1$, there is a spherical disc with chordal radius

$3K$ containing completely the image $f(\Delta_{n,k})$.

Next we shall consider $\Gamma_{n,k}$ for $n \geq n_1 + 1$. Then $\Delta_{n,k}$ and some $\Delta_{n-1,k'}$ have $\Gamma_{n,k}$ as the common boundary and $\Delta_{n-1,k'} \cup \Gamma_{n,k} \cup \Delta_{n,k} \supset S_{n,k}$. From (1°), the image of $\Delta_{n-1,k'} \cup \Gamma_{n,k} \cup \Delta_{n,k}$, consequently the image of $S_{n,k}$, are contained in a spherical disc with chordal radius $6K < 1/2$, so that, applying Lemma 1 in $S_{n,k}$ for $d = 6K$, we see that the diameter of $f(\Gamma_{n,k}) < 6KBe^{-\mu n/2} < K/2$. Hence for $n \geq n_1 + 1$, each boundary component of $\Delta_{n,k}$ has the image with diameter less than $K/2$. From the same reason as above we now conclude that

(2°) for $n \geq n_1 + 1$, the image of any $\Delta_{n,k}$ is contained in a spherical disc with chordal radius $3K/2$.

By induction we also see for every $p \geq 1$ that

(p°) for $n \geq n_1 + p - 1$, the image of any $\Delta_{n,k}$ is contained in a spherical disc with chordal radius $3K/2^{p-1}$.

Let $\Omega_{n,k}$ be the part of Ω bounded by the simple closed curve $\Gamma_{n,k}$ and let z_0 be a point of $\Gamma_{n,k}$. Then for any $z \in \Omega_{n,k}$, there is a $\Delta_{n+p,q}$ whose closure contains z and we can find a chain $\{\Delta_{n+i,j(i)}\}$ ($i = 0, 1, \dots, p$; $j(0) = k, j(p) = q$) joining $\Delta_{n,k}$ and $\Delta_{n+p,q}$. Supposing that $n \geq n_1$, we have by (p°) obtained above

$$\begin{aligned} [f(z_0), f(z)] &\leq \sum_{i=0}^p \text{diam. of } f(\Delta_{n+i,j(i)}) \text{ w.r.t. the chordal distance} \\ &\leq 2 \sum_{i=0}^p 3K/2^i \leq 12K < 1/2. \end{aligned}$$

By means of a linear transformation we can consider from the above that $f(z)$ is bounded in $\Omega_{n,k}$. On the other hand, E is a linear set of linear measure zero, so that E is a null-set of the class \mathcal{W} (Kametani [4], Ahlfors and Beurling [1]). Hence each point of the part of E contained in the interior of $\Gamma_{n,k}$ must be a removable singularity of the bounded function $f(z)$; this contradicts our assumption that each point of E is an essential singularity of $f(z)$.

Next suppose that $f(z)$ takes a value w_0 outside the union of $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ at two points z' and z'' in $\Delta_{n,k}$, whose boundary components $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are mapped into the discs $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ one by one, and join w_0 to $C(0; \delta)$ and $C(1; \delta)$ with curves A' and A'' respectively, which lie outside the union, do not intersect each other except at w_0 and do not pass through any projection of branch points of the

Riemannian image of $\Delta_{n,k}$ by $f(z)$. The elements of the inverse function f^{-1} corresponding to z' and z'' can be continued analytically along these curves to their end points and further from them along radii of $C(0; \delta)$ and $C(1; \delta)$ so that the curves in $\Delta_{n,k}$ corresponding to these continuations join each of z' and z'' to $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, where we assume that the images $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are contained in $C(0; \delta)$ and $C(1; \delta)$ respectively, and bound with parts of $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ a domain not containing $\Gamma_{n,k}$. Since $\Delta_{n,k}$ has no boundary other than $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, this domain must be a sub-domain of $\Delta_{n,k}$ and $f(z)$ must take the value ∞ there; this is a contradiction. Our proof is now complete.

4. We note that for each n and k ($n = 1, 2, \dots; k = 1, 2, \dots, 2^n$) $\Delta_{n,k}$ is bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ and lies on the right-side of $\Delta_{n,k'}$, $1 \leq k' < k$. We now estimate the harmonic modulus of any doubly connected domain contained in $\Delta_{n,k} \cup \Gamma_{n+1,2k-1} \cup \Delta_{n+1,2k-1}$ such that one connected component of its complement contains the circles $\Gamma_{n,k}$ and $\Gamma_{n+1,2k}$ and the other contains the circles $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$. Moving this domain in parallel so that the right end point of $I_{n+1,2k-1}$ comes to the origin, we see that the harmonic modulus of our domain is dominated by

$$\log \Psi\left(\left(\prod_{k=1}^n \ell_k\right)(1 - \xi_{n+1}) / \prod_{k=1}^{n+1} \ell_k\right) = \log \Psi(2(1 - \xi_{n+1}) / \xi_{n+1}),$$

where $\log \Psi(P/\rho)$ ($P, \rho > 0$) denotes the harmonic modulus of the normal domain of Teichmüller, the complement of the union of the two segments, $-\rho \leq x \leq 0$, $y = 0$ and $P \leq x \leq +\infty$, $y = 0$ in the z -plane ($z = x + iy$). It is well-known that

$$\Psi(P/\rho) < 16 \frac{P}{\rho} + 8,$$

and we obtain thus the following

LEMMA 3. *The harmonic modulus of any doubly connected domain considered above is dominated by $\log(32/\xi_{n+1})$.*

5. Now we shall show the existence of perfect Picard sets. We shall prove the following

THEOREM. *Let E be a Cantor set on the closed interval $I_0; [-1/2, 1/2]$ on the real axis of the z -plane with successive ratios $\xi_n, 0 < \xi_n < 2/3$. If the ratios ξ_n satisfy the condition*

$$\xi_{n+1} = o(\xi_n^2),$$

then E is a Picard set.

Proof. We take $\delta > 0$ so small that the discs $C(0; 2\delta)$, $C(1; 2\delta)$ and $C(\infty; 2\delta)$ are mutually disjoint. Contrary to our assertion, let us suppose that there exists a single-valued meromorphic function $f(z)$ in the complementary domain Ω of E which has E as the set of essential singularities and has three exceptional values at an essential singularity $\zeta \in E$, where we may assume that these values are 0, 1 and ∞ . Since our argument given in the below is applicable locally, it will not bring any loss of generality if we shall give a contradiction under the stronger assumption that $f(z)$ omits the values 0, 1 and ∞ in Ω .

Let n_0 be so large that $Ae^{-\mu n/2} < \delta/2$ for any $n \geq n_0$. By Lemma 2 there is a $\Delta_{n,k}$ ($n \geq n_0$) whose three boundary components are mapped into $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ one by one, where we may assume that the boundary curve $\Gamma_{n+1, 2k-1}$ of $\Delta_{n,k}$ is mapped into $C(\infty; \delta)$. Now we consider the quadruply connected domain $D = \Delta_{n,k} \cup \Gamma_{n+1, 2k-1} \cup \Delta_{n+1, 2k-1}$. The images of the boundary curves $\Gamma_{n+2, 4k-3}$ and $\Gamma_{n+2, 4k-2}$ of D are contained in some spherical discs C and C' with radius $Ae^{-\mu n+2/2} < \delta/2$ respectively, and we see that C and C' are contained in $C(\infty; 2\delta)$. In fact, one of them, say C , must contain the point at infinity, for otherwise, $f(z)$ must take the value ∞ in D , and hence is contained in $C(\infty; \delta)$. Suppose that C' is not contained in $C(\infty; 2\delta)$. Then there is a point $z \in \Delta_{n+1, 2k-1} \subset D$ whose image $f(z)$ lies in $C(\infty; 2\delta) - C(\infty; \delta) \cup C'$ and can be joined the origin or the point $w = 1$ with a path not intersecting the image of the boundary of $\Delta_{n+1, 2k-1}$, so that $f(z)$ takes the value 0 or 1 in $\Delta_{n+1, 2k-1}$. Contradiction. Thus we can find a positive d such that $d < 2\delta$ and the disc $C(\infty; d)$ contains C and C' .

Next we shall prove that $f(z)$ takes each value outside the union of the three discs $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; d)$ once and only once in D . By Lemma 2, $f(z)$ takes each value outside the union of $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ once and only once in $\Delta_{n,k}$, so that the inverse image Γ of the circle $[w, \infty] = 2\delta$ on $\Delta_{n,k}$ is a simple closed curve and separates $\Gamma_{n,k}$ and $\Gamma_{n+1, 2k}$ from $\Gamma_{n+2, 4k-3}$ and $\Gamma_{n+2, 4k-2}$. Now suppose that $f(z)$ takes a value w_0 outside the union of $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; d)$ at two points z' and z'' in D , and join w_0 with $C(0; \delta)$ with a curve λ which lies outside the union and does not pass through any projection of branch points of the Riemannian image of D by $f(z)$. The

elements of the inverse function f^{-1} corresponding to z' and z'' can be continued analytically along A to its end point and hence we see that every value on A is taken by $f(z)$ at least two times in D . Therefore we can assume that w_0 lies outside $C(\infty; 2\delta)$. Then one of z' and z'' must lie in the domain D' bounded by Γ , $\Gamma_{n+2, 4k-3}$ and $\Gamma_{n+2, 4k-2}$ and the corresponding element of f^{-1} can be continued analytically to the origin along a curve outside $C(\infty; 2\delta)$, so that $f(z)$ takes the value 0 in D' ; this contradicts our assumption.

Now we estimate d from below. To this purpose we consider the annulus $R: 2 < |w| < \sqrt{1-d^2}/d$ corresponding to the annulus $1/\sqrt{5} > [w, \infty] > d$ on the Riemann sphere Σ , which separates $C(0; \delta)$ and $C(1; \delta)$ from $C(\infty; d)$. As we have seen above, the Riemannian image of D covers R univalently, the ring domain on D corresponding to R , which has the same harmonic modulus as R , separates the boundary curves $\Gamma_{n,k}$ and $\Gamma_{n+1, 2k}$ of D from the boundary curves $\Gamma_{n+2, 4k-3}$ and $\Gamma_{n+2, 4k-2}$ of D . By Lemma 3 we have thus

$$\text{har. mod. of } R = \log(\sqrt{1-d^2}/2d) \leq \log(32/\xi_{n+1}).$$

Since $d < 2\delta < \pi/6$, we have the estimate

$$d \geq (\sqrt{1-(\pi/6)^2}/64)\xi_{n+1} = \frac{1}{M}\xi_{n+1}.$$

This implies that C' must intersect the disc $[w, \infty] \geq \xi_{n+1}/M = m$. Consider the domain $\Delta_{n+2, 4k-2}$ which, with $\Delta_{n+1, 2k-1}$, has $\Gamma_{n+2, 4k-2}$ as the common boundary. The images of three boundary curves $\Gamma_{n+2, 4k-2}$, $\Gamma_{n+3, 8k-5}$ and $\Gamma_{n+3, 8k-4}$ of $\Delta_{n+2, 4k-2}$ are contained in some three spherical discs $C' = C_{n+2, 4k-2}$, $C_{n+3, 8k-5}$ and $C_{n+3, 8k-4}$ with radii less than $Ae^{-\mu_{n+2}/2} < A'\sqrt{\xi_{n+2}}$, respectively, where $A' = \sqrt{3}/2 A$. We may suppose that n is so large that for each $p \geq 1$,

$$(12 A' M)^2 \xi_{n+p+1} \leq \xi_{n+p}^2.$$

Then $C_{n+2, 4k-2}$ does not contain the point at infinity, since $A'\sqrt{\xi_{n+2}} < m/12$, so that any one of $C_{n+2, 4k-2}$, $C_{n+3, 8k-5}$ and $C_{n+3, 8k-4}$ cannot be disjoint from the union of the other two. By the same reasoning in the proof of Lemma 2 we see that the image of $\Delta_{n+2, 4k-2}$ is contained in a spherical disc with radius less than $3A'\sqrt{\xi_{n+2}} < m/4$. Since this disc intersects the disc $[w, \infty] \geq m$, it must lie outside the disc $C(\infty; m/2)$. Next we consider the domain $\Delta_{n+3, 8k-5}$. Each image of its three boundary curves is contained in a spherical disc with radius less than

$$A'\sqrt{\xi_{n+3}} \leq \xi_{n+2}/12M < A'\sqrt{\xi_{n+2}}/12A'M < m/24,$$

where we suppose that $A' > 1/6M$; this does not bring any loss of generality. The same argument shows that the image of $\Delta_{n+3, 8k-5}$ is contained in a spherical disc with radius less than $m/8$ and lying outside the disc $C(\infty; m/4)$. The same holds for $\Delta_{n+3, 8k-4}$. Since, for each $p \geq 2$,

$$A'\sqrt{\xi_{n+p+1}} \leq \xi_{n+p}/12M < A'\sqrt{\xi_{n+p}}/2,$$

we can conclude by induction that the image of the domain $\Delta_{n+p+1, q}$ lying in the interior of the simple closed curve $\Gamma_{n+2, 4k-2}$ is contained in a spherical disc with radius less than $m/2^{p+1}$ and lying outside the disc $C(\infty; m/2^p)$. It follows that, in the interior of $\Gamma_{n+2, 4k-2}$, $f(z)$ takes values only in a spherical disc with radius less than

$$\sum_{p=1}^{\infty} m/2^{p+1} < m/2.$$

By means of a linear transformation we can consider that $f(z)$ is bounded in the interior of $\Gamma_{n+2, 4k-2}$; this contradicts our assumption that $f(z)$ has an essential singularity at every point of E . Our theorem is now established.

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