SUPERDIAGONAL FORMS FOR COMPLETELY CONTINUOUS LINEAR OPERATORS

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Altman [**1**] showed that Riesz–Schauder theory remains valid for a completely continuous linear operator on a locally convex Hausdorff topological vector space over the complex field. In a later paper [**2**], he proved an analogue of the Aronszajn–Smith result; specifically, he showed that such an operator possesses a proper closed invariant subspace. The purpose of this paper is to show that Ringrose's theory of superdiagonal forms for compact linear operators [**3**] can be generalized to the case of a completely continuous linear operator on a locally convex Hausdorff topological vector space over the complex field. However, the proof given in [**3**] requires considerable modification.

Terminology pertaining to locally convex Hausdorff topological vector spaces in this paper is as in the book [**5**]. Throughout, X is a fixed locally convex Hausdorff topological vector space over the complex field and $X \neq \{0\}$.

DEFINITION. A linear operator T on X is called *completely continuous* if there is a neighbourhood U of O such that TU is a precompact set.

We observe that, by Proposition 7.2 of [**5**; p. 58], the topology of X has a basis of barrels. Since any subset of a precompact set is precompact, we may assume without loss of generality that U is a barrel.

Next, we describe Altman's generalization of Riesz–Schauder theory [**1**].

THEOREM A. *Let T be a completely continuous linear operator on X, and let λ be a non-zero complex number. There are two possibilities*:

(a) *$\lambda I - T$ is a homeomorphism of X onto itself*;

(b) *λ is an eigenvalue of T.*

The set of points which satisfy (b) is countable and it has no cluster point except possibly zero. Let λ be a non-zero eigenvalue of T. Then there is a positive integer $\nu(\lambda)$ with the following properties.

(i) *For each positive integer n, $(\lambda I - T)^n X$ is closed. Also*

$$(\lambda I - T)^{m+1} X = (\lambda I - T)^m X \qquad (m \geq \nu(\lambda))$$

and $\nu(\lambda)$ is the smallest positive integer with this property.

(ii) *For each positive integer n, $N((\lambda I - T)^n)$, the null-space of $(\lambda I - T)^n$, is finite-dimensional. Also*

$$N((\lambda I - T)^m) = N((\lambda I - T)^{m+1}) \qquad (m \geq \nu(\lambda))$$

and $\nu(\lambda)$ is the smallest positive integer with this property.

(iii) $(\lambda I - T)^m X \oplus N((\lambda I - T)^m) = X \qquad (m \geq \nu(\lambda)).$

(iv) *If $d(\lambda)$ is the dimension of the null-space of $(\lambda I - T)^{\nu(\lambda)}$, then*

$$1 \leq \nu(\lambda) \leq d(\lambda).$$

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NOTE. The integers $\nu(\lambda)$ and $d(\lambda)$ are called respectively the *index* and the *algebraic multiplicity* of the eigenvalue λ.

Altman [**2**] proved the following result.

THEOREM B. *Let T be a completely continuous linear operator on X. Then there is a proper closed subspace Y of X invariant under T, provided only that X has dimension at least two.*

Prior to proving the main results of this paper we require a preliminary result on quotient spaces.

THEOREM C. *Let M be a closed linear subspace of X. Let ϕ be the canonical mapping of X onto X/M. Then*

(i) *the topology of X/M is locally convex;*

(ii) *if P is a basis of continuous seminorms on X, we denote by P_M the family of seminorms on X/M consisting of the seminorms*

$$[x] \to \tilde{p}([x]) = \inf\{p(y) : y \in [x]\};$$

then P_M is a basis of continuous seminorms on X/M;

(iii) *ϕ is a continuous mapping of X onto X/M.*

For a proof of this result, the reader is referred to Proposition 7.9 of [**5**, p. 65]. Whenever, in this chapter, a quotient space is introduced it will be assumed that it has been topologised in the manner specified above.

Throughout the remainder of this paper, T denotes a fixed completely continuous linear operator on X. The term subspace will be used to describe a closed linear subspace of X. Clearly if Y is an invariant subspace for T, then $T \mid Y$, the restriction of T to Y, is also completely continuous. We have also the following result.

PROPOSITION 1. *Let Y be an invariant subspace for T. Then the operator T_Y defined on the quotent X/Y by*

$$T_Y[x] = [Tx] \qquad (x \in X)$$

is a completely continuous operator.

Proof. By definition there is a neighbourhood U of 0 such that the image TU is precompact. Let ϕ be the canonical mapping of X onto X/Y. Then $\phi(U)$ is a neighbourhood of the zero element in X/Y and, moreover, $T_Y\phi(U) = \phi(TU)$. By [**4**, p. 49], the continuous image of a precompact set is precompact. It follows that T_Y is completely continuous.

A family $\mathcal{F}$ of subspaces of X, which is totally ordered by the inclusion relation, will be termed a *nest* of subspaces. If in addition each subspace in $\mathcal{F}$ is invariant under T we shall describe $\mathcal{F}$ as an *invariant* nest. A trivial example of an invariant nest is the family consisting of the two subspaces $\{0\}$, X. Non-trivial invariant nests may be constructed using Altman's result, Theorem B.

We shall use the symbol $\subseteq$ to denote the inclusion relation and reserve $\subset$ for strict inclusion. The strong closure of a subset S of X will be denoted by cl S. Given a nest $\mathcal{F}$ of subspaces of X and $M \in \mathcal{F}$, we define

$$M_- = \mathrm{cl}[\bigcup\{L : L \in \mathcal{F}, L \subset M\}].$$

If there is no L in $\mathcal{F}$ such that $L \subset M$, we define $M_- = \{0\}$. It is clear that M_- is a subspace of X, and that it will be an invariant subspace if $\mathcal{F}$ is an invariant nest. Also $M_- \subseteq M$. It should be emphasized that the definition of M_- depends on the particular nest $\mathcal{F}$ under consideration and not merely on the subspace M. We shall say that $\mathcal{F}$ is *continuous* at M if $M = M_-$.

A nest $\mathcal{F}$ will be termed *simple* if

(i) $\{0\} \in \mathcal{F}$, $X \in \mathcal{F}$;

(ii) if $\mathcal{F}_0$ is any subfamily of $\mathcal{F}$, then the subspaces

$$\bigcap\{L : L \in \mathcal{F}_0\},\ \mathrm{cl}[\bigcup\{L : L \in \mathcal{F}_0\}]$$

are in $\mathcal{F}$;

(iii) if $M \in \mathcal{F}$, then the quotient space M/M_- is at most one-dimensional.

We note that condition (ii) implies that $M_- \in \mathcal{F}$ whenever $M \in \mathcal{F}$.

THEOREM 1. *There exists a simple nest $\mathcal{F}$, each of whose members is a subspace invariant under T.*

Proof. Let $\mathcal{N}_i$ denote the class of all invariant nests. Then $\mathcal{N}_i$ is not empty since it contains the trivial nest consisting of the subspaces $\{0\}$, X. The class $\mathcal{N}_i$ may be partially ordered by inclusion; if $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{N}_i$, we say $\mathcal{F}_1 < \mathcal{F}_2$ if every subspace in the family $\mathcal{F}_1$ is also member of $\mathcal{F}_2$. It is easily seen that, in this way, $\mathcal{N}_i$ is inductively ordered; for if $\mathcal{N}_0 \subseteq \mathcal{N}_i$ and $\mathcal{N}_0$ is totally ordered by the relation $<$, then

$$\mathcal{F}_0 = \bigcup\{\mathcal{F} : \mathcal{F} \in \mathcal{N}_0\}$$

is the least upper bound of $\mathcal{N}_0$ in $\mathcal{N}_i$. We may now deduce from Zorn's lemma the existence of at least one maximal nest of invariant subspaces, $\mathcal{F}$ say.

It is apparent that $\{0\}$, $X \in \mathcal{F}$, since otherwise $\mathcal{F}$ could be enlarged by the addition of these subspaces, contrary to the assumption that $\mathcal{F}$ is maximal. Moreover, let $\mathcal{F}_0$ be a sub-family of $\mathcal{F}$ and consider

$$M_0 = \bigcap\{L : L \in \mathcal{F}_0\}.$$

It is evident that that M_0 is a closed subspace of X. Let $M \in \mathcal{F}$. Since $\mathcal{F}$ is totally ordered by inclusion we have either (a) $M \subseteq L (L \in \mathcal{F}_0)$ and $M \subseteq M_0$, *or* (b) $L \subset M$ for some L in $\mathcal{F}_0$ and $M_0 \subset M$. It follows that the family obtained by adding M_0 to $\mathcal{F}$ remains totally ordered by inclusion and is therefore an invariant nest. Since $\mathcal{F}$ is maximal we deduce that $M_0 \in \mathcal{F}$. A similar argument shows that

$$\mathrm{cl}[\bigcup\{L : L \in \mathcal{F}_0\}] \in \mathcal{F}.$$

Hence, in order to prove that $\mathcal{F}$ is a simple invariant nest for T it remains to verify that, given any M in $\mathcal{F}$, the quotient space M/M_- is at most one-dimensional. Suppose that, for some M in $\mathcal{F}$, this is not the case. When $x \in M$ we denote by $[x]$ the coset

$x+M_-$. It follows from results stated earlier that M/M_- is a locally convex Hausdorff topological vector space. Since M and M_- are invariant under T, we may define a linear operator T_M from M/M_- into itself by the equation

$$T_M[x]=[Tx] \qquad (x\in M).$$

It follows from Proposition 1 that T_M is a completely continuous linear operator. Since M/M_- has dimension exceeding one, Theorem B implies the existence of a proper subspace L_M of M/M_- which is invariant under T_M. If we now put

$$L=\{x: x\in M, [x]\in L_M\},$$

then L is a subspace of X, being the inverse image under the continuous linear map $x\to[x]$ of the subspace L_M, such that $M_-\subset L\subset M$. Given any subspace N in $\mathcal{F}$ we have *either* (a) $M\subseteq N$ and $L\subset N$ *or* (b) $N\subset M$ and

$$N\subseteq \mathrm{cl}[\bigcup\{K\in\mathcal{F}: K\subset M\}]=M_-\subset L.$$

It follows that $L\notin\mathcal{F}$, and that the family obtained by adding L to $\mathcal{F}$ is a nest. This contradicts the assumption that $\mathcal{F}$ is maximal. Hence M/M_- is at most one-dimensional, for every M in $\mathcal{F}$, and so $\mathcal{F}$ is a simple nest.

Throughout the remainder of this paper we shall use the symbols T, $\mathcal{F}$ with the meanings attributed to them in the statement of Theorem 1. If $M\in\mathcal{F}$, then either $M=M_-$ or M/M has dimension one. In the latter case let $z_M\in M\backslash M_-$. Then, since M is invariant under T, we have $Tz_M\in M$, and hence Tz_M can be expressed uniquely in the form

$$Tz_M=\alpha_M z_M+y_M, \tag{1}$$

where α_M is a scalar and $y_M\in M_-$. It is easily verified that α_M does not depend on the particular choice of z_M. When $M=M_-$ we define $\alpha_M=0$. In this way we associate with each M in $\mathcal{F}$ a scalar α_M which we shall call the *diagonal coefficient* of T at M.

Let α be a scalar. We define the *diagonal multiplicity* of α to be the number (possibly infinite) of distinct subspaces M in $\mathcal{F}$ for which $\alpha_M=\alpha$.

PROPOSITION 2. *Let $M\in\mathcal{F}$ and let $\delta>0$ be given. Then there exists a subspace L in $\mathcal{F}$ such that $L\subset M$ and for every p in the basis P of all continuous seminorms on X we have*

$$\tilde{p}_L([Tx])\leq\delta \qquad (x\in M_-)$$

where $[y]$ denotes the coset $y+L(y\in X)$ and

$$\tilde{p}_L([x])=\inf\{p(y): y\in[x]\}.$$

REMARK. The interest of this lemma lies in the case in which $M=M_-$. When $M\neq M_-$, the result is trivial since we may take $L=M_-$.

Proof. Suppose that the lemma is false, and denote by $\mathcal{F}_0$ the class of all L in $\mathcal{F}$ such that $L\subset M$. Since we are going to vary L we shall not use the notation $[y]$ for cosets, but throughout the proof will write $y+L$. If $L\in\mathcal{F}_0$, the set

$$S_L=\{x\in U\cap M_-: \tilde{p}_L(Tx+L)>\delta, \text{ for some } p \text{ in } P\}$$

is not empty. Since $S_L \subseteq S_N$ if $N \subseteq L$ the family $\{S_L : L \in \mathscr{F}_0\}$ forms a filter base on the set U. Hence the family $\{TS_L : L \in \mathscr{F}_0\}$ forms a filter base on the precompact set TU. This filter has at least one accumulation point in the compact set K formed by taking the closure of TU in the completion $\hat{X}$ of X. Since

$$\tilde{p}_L(y+L) > \delta \qquad (y \in TS_L)$$

it follows that

$$\tilde{p}_L(x_0+L) \geqslant \delta, \tag{2}$$

where we use the same notation for the extension of p to $\hat{X}$. Furthermore we have $S_L \subseteq M$, $TS_L \subseteq M_-$ and hence

$$x_0 \in \mathrm{cl}[\bigcup\{L : L \in \mathscr{F}_0\}],$$

the closure being taken in the completion of X. Thus for some L in $\mathscr{F}_0$, we may choose an element y in L such that $p(x_0 - y) < \delta$. This contradicts (2) and the lemma is proved.

Proposition 3. *Let ρ be a non-zero eigenvalue of T and x a corresponding eigenvector. Let*

$$M = \bigcap\{L : L \in \mathscr{F},\ x \in L\}.$$

Then $M \in \mathscr{F}$ and $\rho = \alpha_M$..

Proof. The property (ii) of simple nests immediately implies that $M \in \mathscr{F}$. In proving that $\rho = \alpha_M$ we shall consider separately the two cases in which (respectively) $M = M_-$ and $M \neq M_-$.

(a) Suppose that $M = M_-$. Choose δ so that

$$0 < \delta < |\rho|, \tag{3}$$

and let L be chosen to satisfy the conclusions of Proposition 2. Since $L \subset M$ and $L \in \mathscr{F}$ it is an immediate consequence of the definition of M that $x \notin L$. We may choose, by a corollary to the Hahn–Banach theorem, a continuous linear functional f on M such that $f(x) = 1$ and $f(y) = 0 (y \in L)$. Let p be the seminorm defined by

$$p(z) = |f(z)| \qquad (z \in M).$$

Define

$$d(u, L) = \inf_{z \in L} p(u - z) \qquad (u \in M).$$

Observe that if $y \in L$, then $Ty \in L$ and hence

$$\begin{aligned} |\rho| = |\rho|\, d(x, L) &= d(\rho x, L) = d(Tx, L) \\ &= d(Tx + Ty, L) \leqslant \delta. \end{aligned}$$

The last inequality follows from Proposition 2. This contradicts (3). Hence case (a) cannot occur.

(b) We may now suppose that $M \neq M_-$. Then $x \in M$, but $x \notin M$, since M is, by definition, the smallest member of $\mathcal{F}$ containing x. Let $z_M \in M \backslash M_-$, and let y_M in M_- be chosen so that $Tz_M = \alpha_M z_M + y_M$. We may put

$$x = \beta z_M + y,$$

where $y \in M_-$ and $\beta \neq 0$. Then

$$\begin{aligned} 0 = Tx - \rho x &= T(\beta z_M + y) - \rho(\beta z_M + y) \\ &= \beta(\alpha_M z_M + y_M) + Ty - \rho(\beta z_M + y) \\ &= \beta(\alpha_M - \rho) z_M + \beta y_M + Ty - \rho y. \end{aligned}$$

Now y, y_M and (since M_- is invariant under T) Ty are all elements of M_- but $z_M \notin M_-$. Hence $\beta(\alpha_m - \rho) = 0$, and since $\beta \neq 0$, it follows that $\alpha_M = \rho$.

Proposition 3 asserts that a non-zero eigenvalue of T is a diagonal coefficient of T. We now prove a result in the opposite direction.

PROPOSITION 4. *Let $M \in \mathcal{F}$ and suppose that $\alpha_M \neq 0$. Then α_M is an eigenvalue of T.*

Proof. It is sufficient to show that α_M is an eigenvalue of the operator T obtained by restricting T to the space M. Since $\alpha_M \neq 0$ we have $M \neq M_-$. Now T_M is completely continuous. From equation (1) it follows that the range of the operator $T_M - \alpha_M I$ is contained in M_-, and is therefore not the whole space M. It follows from Theorem A that α_M is an eigenvalue of T_M and hence of T.

PROPOSITION 5. *Let ρ be a non-zero eigenvalue of T. Then the diagonal multiplicity of ρ is equal to its algebraic multiplicity as an eigenvalue of T.*

Proof. Let d denote the diagonal multiplicity, m the algebraic multiplicity, and ν the index of ρ relative to T. Then

(a) ν is the least integer such that $(T - \rho I)^{\nu+1} x = 0$ only if

$$(T - \rho I)^{\nu} x = 0 \qquad (x \in X);$$

(b) ν is the least integer such that

$$(T - \rho I)^{\nu+1} X = (T - \rho I)^{\nu} X;$$

(c) the null-space of the operator $(T - \rho I)^{\nu}$ has dimension m.

Let S be the completely continuous linear operator defined by

$$S - \lambda I = (T - \rho I)^{\nu},$$

where $\lambda = -(-\rho)^{\nu}$. Then λ is an eigenvalue of S which has index unity and algebraic multiplicity m. Since S is a polynomial in T, each subspace M in $\mathcal{F}$ is invariant under S. We may therefore consider the diagonal coefficients of S with respect to the nest $\mathcal{F}$.

Let $M \in \mathcal{F}$ and let α_M, σ_M denote the diagonal coefficient at M of T, S respectively. If $M = M_-$, we have $\alpha_M = \sigma_M = 0$. If $M \neq M_-$, then with the usual notation we may deduce from the equation $Tz_M = \alpha_M z_M + y_M$ that

$$(T-\rho I)z_M = (\alpha_M - \rho)z_M + y_M.$$

It easily follows that, for $n = 1, 2, \ldots$, we have

$$(T-\rho I)^n z_M = (\alpha_M - \rho)^n z_M + y^{(n)},$$

where $y^{(n)} \in M_-$. In particular, by taking $n = \nu$, we obtain

$$Sz_M = \lambda z_M + (\alpha_M - \rho)^\nu z_M + y^{(\nu)}.$$

Thus $\sigma_M = \lambda + (\alpha_M - \rho)^\nu$. We deduce that $\sigma_M = \lambda$ if and only if $\alpha_M = \rho$. Hence the diagonal multiplicity of λ relative to S is d. It is now sufficient to prove the lemma under the additional hypothesis that ρ has index unity relative to T, since in the general case we may reduce to this situation by replacing T, ρ by S, λ respectively.

Suppose therefore that ρ has index unity relative to T, and let N be the null-space of the operator $T - \rho I$. Given $x \in N$, define

$$M(x) = \bigcap \{L : L \in \mathcal{F}, x \in L\}.$$

From Proposition 3 and its proof we deduce that $M(x) \in \mathcal{F}$, $x \in M(x) \backslash M_-(x)$ and $\alpha_{M(x)} = \rho (x \in N, x \neq 0)$. The remainder of the proof is divided into three stages.

First, we show conversely that if $M \in \mathcal{F}$ and $\alpha_M = \rho$ then $M = M(x)$ for some non-zero x in N. For this purpose, let T_M denote the restriction of T to M, and let W_M, N_M be the range and null-space respectively of the operator $T_M - \rho I_M$. Then T_M is a completely continuous linear operator on M, and it is immediate from the definition of index in terms of null-spaces that ρ has index unity relative to T_M. Hence, by Theorem A,

$$W_M \oplus N_M = M.$$

Since, as in the proof of Proposition 4, $W_M \subseteq M$, it follows that N_M meets $M \backslash M_-$. If $x \in N_M \cap (M \backslash M_-)$, it is easily verified that $x \in N$, $x \neq 0$ and $M(x) = M$.

Secondly, let $M_1 \subset M_2 \subset \ldots \subset M_d$ be distinct members of the nest $\mathcal{F}$ at which T has diagonal coefficient ρ. We may choose non-zero vectors $x_1, \ldots, x_d \in N$ such that $M_i = M(x_i)(i = 1, \ldots, d)$. For each $i = 1, \ldots, d$, x_i is not a linear combination of $x_1, \ldots, x_{i-1}$; for this would imply that $x_i \in M(x_{i-1}) \subseteq M_-(x_i)$, which is not so. Hence, $x_1, \ldots, x_d$ are linearly independent elements of N, and since $\dim N = m$ we have $m \geq d$.

Thirdly, suppose that $m > d$. By the Hahn–Banach theorem, we can find linear functionals continuous on X, such that $\phi_i(x_i) \neq 0$, but $\phi_i(x) = 0 (x \in M_-(x_i))$. Then if $x \in M(x_i)$ and $\phi_i(x) = 0$, we have $x \in M_-(x_i)$. Now since $\dim N > d$, we may choose a non-zero vector x in N such that $\phi_i(x) = 0 \, (i = 1, \ldots, d)$. Then $\alpha_{M(x)} = \rho$, and therefore $M(x) = M(x_i)$ for some i. Thus $x \in M(x_i)$, $\phi_i(x) = 0$, and we have $x \in M_-(x_i) = M_-(x)$. However, this is impossible. Hence $m \leq d$. Since the reverse inequality has already been established we have $m = d$, and Proposition 4 is proved.

We now state a theorem which summarizes the principal results obtained in the preceding lemmas.

THEOREM 2. *Let T be a completely continuous linear operator on a locally convex Hausdorff topological vector space X over the complex field and let $\mathcal{F}$ be a simple nest of subspaces of X, each of which is invariant under T. Then*

(i) *a non-zero scalar ρ is an eigenvalue of T if and only if it is a diagonal coefficient of T;*

(ii) *the diagonal multiplicity of ρ is equal to its algebraic multiplicity as an eigenvalue of T;*

(iii) *the operator T has no non-zero eigenvalue if and only if $\alpha_M = 0(M \in \mathcal{F})$, or equivalently if and only if $TM \subseteq M_-(M \in \mathcal{F})$.*

Proof. The only statement not already proved is (iii). From (i), it follows that T has no non-zero eigenvalue if and only if $\alpha_M = 0(M \in \mathcal{F})$.

COROLLARY. *If there is a continuous simple nest of subspaces of X, each of which is invariant under T, then T has no non-zero eigenvalue.*

Proof. This follows from part (iii) of the preceding theorem.

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