THE INFLUENCE OF GENERALIZED FRATTINI SUBGROUPS ON THE SOLVABILITY OF A FINITE GROUP

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1. The Frattini and Fitting subgroups of a finite group G have been useful subgroups in establishing necessary and sufficient conditions for G to be solvable. In [**1**, pp. 657–658, Theorem 1], Baer used these subgroups to establish several very interesting equivalent conditions for G to be solvable. One of Baer's conditions is that $\phi(S)$, the Frattini subgroup of S, is a proper subgroup of $F(S)$, the Fitting subgroup of S, for each subgroup $S \neq 1$ of G. Using the Fitting subgroup and generalized Frattini subgroups of certain subgroups of G we provide certain equivalent conditions for G to be a solvable group. One such condition is that $F(S)$ is not a generalized Frattini subgroup of S for each subgroup $S \neq 1$ of G. Our results are given in Theorem 1. We note that Theorem 1 is mostly a rewording of Baer's theorem using generalized Frattini subgroups instead of the usual Frattini subgroups.

Many of the properties of the Frattini subgroup of a finite group G carry over to the wider class of subgroups of G known as generalized Frattini subgroups of G (see [**3**; **4**]). One such property is that $F(G/K) = F(G)/K$, where K is a generalized Frattini subgroup of G. We note that this property actually characterizes the generalized Frattini subgroups of G (see [**3**, Theorem 2.2]).

The last section of this note is devoted to self-normalizing maximal subgroups of finite groups. We prove that if the finite group G contains a solvable self-normalizing maximal subgroup K of core 1 and if every self-normalizing maximal subgroup of G of core 1 is conjugate to K in G, then G is solvable.

2. Preliminaries. The only groups considered here are finite. If K is a subgroup of a group G, then

$Z(K)$ is the centre of K,

$C_G(K)$ is the centralizer of K in G,

$N_G(K)$ is the normalizer of K in G,

$H(K)$ is the hypercentre of K (i.e. the terminal member of the upper central series of K).

$F(K)$ is the Fitting subgroup of K (i.e. the largest nilpotent normal subgroup of K),

$\phi(K)$ is the Frattini subgroup of K,

$[G:K]$ is the index of K in G.

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If x is an element of K, then $o(x)$ denotes the order of x.

A subgroup K of the group G is termed self-normalizing in G if $N_G(K) = K$.

In a group G, $L(G)$ denotes the intersection of all self-normalizing maximal subgroups of G, and in the case when G is nilpotent, then $L(G)$ is defined to be G.

A proper normal subgroup K of a group G is called a *generalized Frattini subgroup* of G if and only if $G = N_G(P)$ for each normal subgroup L of G and each Sylow p-subgroup P, p a prime, of L such that $G = KN_G(P)$ (see [**4**, p. 442]). Some of the elementary properties of generalized Frattini subgroups can be found in [**3**; **4**]. The Frattini subgroup of a finite group G is a generalized Frattini subgroup of G. Among the generalized Frattini subgroups of a non-nilpotent group G are the weakly hypercentral normal subgroups of G (see [**3**]) and the intersection of all self-normalizing maximal subgroups of G (see [**4**]). Generalized Frattini subgroups are characterized in [**3**, Theorem 2.2].

3. Some conditions for solvability of a finite group. We begin this section with a lemma which generalizes [**1**, p. 657, Lemma 3]. The proof of our result is very similar to that of Baer; however for the sake of completeness we prove the following lemma.

LEMMA 1. *Let K be a generalized Frattini subgroup of G. If K contains every proper normal subgroup of G and if G/K is non-nilpotent, then the centre and hypercentre are equal subgroups of $K = F(G)$.*

Proof. By (**3**, Theorem 2.1), it follows that $F(G) = K$.

If $G' \neq G$, then $G' \subset K$, hence G/K is abelian, a contradiction. By [**7**, 6.4.26, part (f)], $H(G) = Z(G)$. Hence, the proof is complete.

We note that the proof of the following theorem depends heavily on the proof of [**1**, pp. 657–658, Theorem 1].

THEOREM 1. *The following properties are equivalent for a group G:*

(i) *G is solvable;*

(ii) *If K is a generalized Frattini subgroup of G, then G/K is solvable;*

(iii) *If $L \neq 1$ is a homomorphic image of G, then $F(L)$ is not a generalized Frattini subgroup of L;*

(iv) *If $L \neq 1$ is a homomorphic image of G, then $F(L) \neq 1$;*

(v) *Every subgroup $S \neq 1$ of G has the following two properties:*

(a) *$Z[F(S)] = C_S[F(S)]$,*

(b) *If K is a generalized Frattini subgroup of S which contains every proper normal subgroup of S and if S/K is non-nilpotent, then K is the hypercentre of S;*

(vi) *If $S \neq 1$ is a subgroup of G and if K is a generalized Frattini subgroup of S which contains every proper normal subgroup of S, then S/K is nilpotent;*

(vii) *If $S \neq 1$ is a subgroup of G, then $F(S)$ is not a generalized Frattini subgroup of S.*

Proof. The equivalence of properties (i) and (ii) is a consequence of the following facts: a generalized Frattini subgroup of a group G is nilpotent (see [**4**, Theorem 3.1]) and therefore solvable; if N is a normal subgroup of G, then the solvability of N and G/N is necessary and sufficient for the solvability of G.

Assume that G is solvable and let $L \neq 1$ be a homomorphic image of G. Then L is also solvable. Because of [**4**, Corollary 3.6.1], $F(L)$ cannot be a generalized Frattini subgroup of L. Thus (iii) is a consequence of (i); and it is easy to see that (iv) is a consequence of (iii), since 1 is a generalized Frattini subgroup of each non-trivial finite group.

Assume next the validity of (iv). Let $S \neq 1$ be a subgroup of G. By [**1**, pp. 657–658, Theorem 1 (iv)], G is a solvable group. Therefore, it follows that $Z[F(S)] = C_S[F(S)]$ (**1**, pp. 657–658, Theorem 1 (viii)). Let K be a generalized Frattini subgroup of S which contains every proper normal subgroup of S. Since S is solvable, $F(S)$ is not a generalized Frattini subgroup of S [**4**, Corollary 3.6.1], hence $F(S) = S$. Therefore, S is nilpotent and thus (v) is a consequence of (iv).

Assume that (v) is valid. Let $S \neq 1$ be a subgroup of G and let K be a generalized Frattini subgroup of S which contains every proper normal subgroup of S. Suppose that S/K is non-nilpotent. Then by Lemma 1, it follows that $Z(S) = H(S) = K = F(S)$. However, $Z[F(S)] \subseteq F(S) = K < S = C_S(F(S))$, which contradicts (v)(a). Hence S/K is nilpotent, and (vi) is a consequence of (v).

Assume the validity of (vi). Since the Frattini subgroup of a finite group is a generalized Frattini subgroup, G is solvable [**1**, pp. 657–658, Theorem 1 (ix)]. Let $S \neq 1$ be a subgroup of G. Then S is solvable, hence $F(S)$ is not a generalized Frattini subgroup of S [**4**, Corollary 3.6.1]. Therefore, (vii) is a consequence of (vi).

Assume that property (vii) is valid, and let $S \neq 1$ be a subgroup of G. Since $\phi(S)$ is a generalized Frattini subgroup of S (see [**4**, Theorem 3.1]), $\phi(S)$ is properly contained in $F(S)$. By [**1**, pp. 657–658, Theorem 1 (x)], G is a solvable group. Therefore, (i) is a consequence of (vii); and this completes the proof of the equivalence of our seven properties.

It is well known that a (finite) group all of whose proper subgroups are solvable need not be solvable. However, we obtain the following corollary to Theorem 1.

COROLLARY 1. *Let G be a (non-trivial) group all of whose proper subgroups are solvable. Then G is solvable if and only if $F(G)$ is not a generalized Frattini subgroup of G.*

4. On self-normalizing maximal subgroups. We begin this section with the following theorem which is somewhat more general than Corollary 1.

THEOREM 2. *Let G be a non-nilpotent group all of whose self-normalizing* *maximal subgroups are solvable. Then G is solvable if and only if $L(G)$ is a proper subgroup of $F(G)$.*

Proof. Since G is non-nilpotent, $L(G)$ is a generalized Frattini subgroup of G [**4**, Theorem 3.5]. Because of [**4**, Theorem 3.1], $L(G)$ is a nilpotent normal subgroup of G, hence $F(G)$ contains $L(G)$.

Suppose that G is solvable. Then $L(G)$ is a proper subgroup of $F(G)$ by Theorem 1 (vii).

Conversely, let $L(G)$ be a proper subgroup of $F(G)$. Then there exists a self-normalizing maximal subgroup M such that $F(G)$ is not contained in M. Hence, $G = MF(G)$ and thus G is solvable, since $F(G)$ is nilpotent and M is solvable.

Huppert (**5**, *Satz* 22) showed that if all the proper subgroups of the finite group G are supersolvable, then G is solvable. Rose [**6**, p. 356] generalized Huppert's theorem to the following: *If the proper self-normalizing subgroups of G are supersolvable, then G is solvable.* We note that there exist finite groups G all of whose self-normalizing maximal subgroups are supersolvable, but G is not solvable. For let $H = \mathrm{GL}(3, 2)$, the general linear group of 3×3 matrices over the field of two elements. Then H is a simple group of order 168. Let f be the automorphism of H given by $f: x \to (x^{-1})^{\mathrm{T}}$, where y^{T} is the transpose of the matrix y of H and y^{-1} is the inverse of y in H. Let G denote the relative holomorph of H by $\{f\}$, $\{f\}$ is a cyclic group of order two. Then G is not solvable and splits over H. The self-normalizing maximal subgroups of G are supersolvable (see [**6**, pp. 351–352]). However, Rose [**6**, Theorem 4] proved the following: *If a finite group G has each of its self-normalizing maximal subgroups supersolvable and of prime power index in G, then G is solvable.* This result leads to the following theorem.

THEOREM 3. *Let G be a group all of whose self-normalizing maximal subgroups are supersolvable. Then the following statements are equivalent:*

(a) *G is solvable;*

(*b*) *The maximal subgroups of G are of prime power index;*

(c) *$F(G)$ is not a generalized Frattini subgroup of G;*

(d) *$G/F(G)$ is supersolvable.*

Proof. We can assume that G is non-nilpotent.

(a) implies (b). This is a well-known fact of solvable groups.

(b) implies (c). By [**6**, Theorem 4], G is a solvable group, hence $F(G)$ is not a generalized Frattini subgroup of G by Theorem 1 (vii).

(c) implies (d). Because of [**4**, Theorem 3.5], $L(G)$ is a generalized Frattini subgroup of G so that $L(G)$ is a proper subgroup of $F(G)$ [**4**, Theorem 3.1]. Hence, there exists a self-normalizing maximal subgroup M of G such that $G = MF(G)$. Therefore, $G/F(G)$ is supersolvable.

(d) implies (a). This follows immediately.

LEMMA 2. *Let G contain only one conjugate class of self-normalizing maximal subgroups. Then G contains a normal Sylow subgroup P and G/P is nilpotent.*

Proof. We can assume that G is a non-nilpotent group. Let M be a self-normalizing maximal subgroup of G and let p be a prime factor of $[G:M]$. Let P be a Sylow p-subgroup of G and assume that $N_G(P)$ is a proper subgroup of G. Let H be a maximal subgroup of G which contains $N_G(P)$. By [**7**, Theorem 6.2.2], H is a self-normalizing maximal subgroup of G. However, p does not divide $[G:H]$ which is impossible since H is conjugate to M. Hence, P is a normal subgroup of G.

Let K/P be a maximal subgroup of G/P. Then $[G:K] = [G/P:K/P]$ is not divisible by p so that K is a normal subgroup of G. Hence, each maximal subgroup of G/P is normal in G/P so that G/P is nilpotent [**7**, Theorem 6.4.14].

Remark 1. The proof of Lemma 2 is essentially that used in [**6**, proof of Theorem 1].

THEOREM 4. *Let the group G contain a solvable maximal subgroup K whose core is* 1. *Then G is solvable if and only if each maximal subgroup of core* 1 *is conjugate to K in G.*

Proof. Assume that G is a solvable group. Then each maximal subgroup of G of core 1 is conjugate to K in G [**2**, p. 138, Corollary 3 (iii)].

Conversely, suppose that each maximal subgroup of G of core 1 is conjugate to K in G. Assume that G is simple. Then G contains exactly one conjugate class of maximal subgroups, hence G is solvable by Lemma 2. Hence, we can assume that G is not simple. Let M be a minimal normal subgroup of G. Suppose that L is a second minimal normal subgroup of G. Because of [**2**, p. 120, Corollary 2(a)], it follows that $G = MK = LK$ and $M \cap K = L \cap K = 1$. Therefore, G/M and G/L are solvable groups so that $G/(M \cap L)$ is solvable. Since L and M are distinct minimal normal subgroups of G, it follows that $M \cap L = 1$ so that G is solvable.

Therefore, we can assume that G contains a unique minimal normal subgroup M. We also note that G/M is solvable, since $G = KM$ and K is solvable. Because of [**2**, p. 121, Lemma 3(ii)], G contains a solvable normal subgroup $N \neq 1$ of G. Since M is the unique minimal normal subgroup of G, N contains M so that M is solvable. Hence, G is solvable since both M and G/M are solvable. This completes the proof.

COROLLARY 2. *Let the group G contain a solvable maximal subgroup K whose core is K_G. Then G is solvable if and only if each maximal subgroup of G whose core is K_G is conjugate to K in G.*

Proof. The corollary follows from Theorem 4 and from [**2**, p. 138, Corollary 3(iii)].

Let the group G contain a nilpotent maximal subgroup K of core 1. Further, assume that each maximal subgroup of G of core 1 is nilpotent. Because of [**6**, p. 350, Corollary], it follows that each maximal subgroup of core 1 is conjugate to K in G. Hence, G is solvable by Theorem 4. Therefore we have established the following result.

COROLLARY 3 [**2**, p. 124, Lemma 5]. *Let the group G possess a maximal subgroup of core* 1. *If every maximal subgroup of G of core* 1 *is nilpotent, then G is solvable.*

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