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A SUBDIFFERENTIAL CHARACTERISATION OF BANACH SPACES WITH THE RADON–NIKODYM PROPERTY

J.R. GILES

A Banach space has the Radon–Nikodym Property if and only if every continuous weak* lower semi-continuous gauge on the dual space has a point of its domain where its subdifferential is contained in the natural embedding.

Collier [**3**, Theorem 1, p. 103] proved that a Banach space X has the Radon–Nikodym Property if and only if every continuous weak* lower semi-continuous convex function on an open convex subset A of the dual X^* is Fréchet differentiable at the points of a dense G_δ subset of A. Recently, Bachir and Daniilidis [**1**, Theorem 1, p. 379] extended this result to prove that X has the Radon–Nikodym Property if and only if every continuous weak* lower semi-continuous convex function defined on X^* is Gâteaux differentiable at some point of its domain with derivative in the natural embedding $\widehat{X}$. This result of Bachir and Daniilidis is related to a Gâteaux differentiability characterisation of Asplund spaces, [**6**, Theorem II.2, p. 9], [**4**, Theorem 2, p. 268]. However, Asplund spaces can also be characterised by a property of the subdifferential mapping of its continuous gauges, [**5**, Theorem 2, p. 155]. It is possible, following this approach, to provide a characterisation of Banach spaces with the Radon–Nikodym Property by subdifferentials of continuous weak* lower semicontinuous gauges, a simpler proof of the result of Bachir and Daniilidis and an extension of our characterisation of Asplund spaces.

For our proof we use the following characterisation, [**2**, Corollary 3.7.6, (1) $\Longleftrightarrow$ (3), p. 67].

PROPOSITION. *A Banach space X has the Radon–Nikodym Property if and only if every closed bounded convex subset of K of X contains an extreme point of* $\overline{\widehat{K}}^{w^*}$.

For a continuous gauge p on a Banach space X, the *subdifferential* of p at $x_0 \in X$ is a nonempty weak* compact convex subset of X^*,

$$\partial p(x_0) \equiv \{f \in X^* : f(x_0) = p(x_0) \text{ and } f(x) \le p(x) \text{ for all } x \in X\}$$

and p is Gâteaux differentiable at x_0 if and only if $\partial p(x_0)$ is singleton, ([**7**, p. 5]).

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Theorem 1. *For a Banach Space X, the following are equivalent.*

(i) *X has the Radon–Nikodym Property.*

(ii) *Every continuous weak* lower semicontinuous convex function on an open convex subset of $X^\star$ is Fréchet differentiable at the points of a dense G_δ subset of its domain.*

(iii) *Every continuous weak* lower semicontinuous convex function on an open convex subset of $X^\star$ is Gâteaux differentiable at a point of its domain with derivative in $\widehat{X}$.*

(iv) *Every continuous weak* lower semicontinuous gauge p on $X^\star$ has a point $f \in X^\star$ with $\partial p(f) \subseteq \widehat{X}$.*

Proof:

That (i) $\Longrightarrow$ (ii) is Collier's result, [**3**, Theorem 1, p. 103]. It is obvious that (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).

To prove (iv) $\Longrightarrow$ (i), consider K a closed bounded convex subset of X. We may assume that $0 \in K$. Consider the continuous positive sublinear functional p on $X^\star$ where

$$p(f) = \sup\{f(x) : x \in K\}.$$

As p is the gauge of $K^\circ \equiv \{f \in X^\star : p(f) \leq 1\}$, it is weak* lower semicontinuous on $X^\star$.

From (iv) there exists an $f_0 \in X^\star$ such that $\partial p(f_0) \subseteq \widehat{X}$. Now $\partial p(f_0)$ is an extreme subset of $\overline{\widehat{K}}^{w^\star}$. Since $\partial p(f_0)$ is weak* compact, by the Krein–Milman Theorem $\partial p(f_0)$ has an extreme point $\widehat{x}_0$ which is then an extreme point of $\overline{\widehat{K}}^{w^\star}$.

Suppose that $x_0 \notin K$. Then x_0 can be strongly separated from K by a weakly closed hyperplane and so $\widehat{x}_0$ can be strongly separated from $\widehat{K}$ by a weak* closed hyperplane. But this contradicts $\widehat{x}_0 \in \overline{\widehat{K}}^{w^\star}$. So we conclude that $\widehat{K}$ contains an extreme point of $\overline{\widehat{K}}^{w^\star}$ which, by the Proposition implies that X has the Radon–Nikodym Property. ∎

Bachir and Daniilidis point out that c_0, which does not have the Radon–Nikodym Property, does have the norm $\|\cdot\|_1$ on its dual ℓ_1 which is generically Gâteaux differentiable with derivatives in $\ell_\infty \backslash \widehat{c}_0$. But it should also be noticed that there is no point of ℓ_1 where the subdifferential of norm $\|\cdot\|_1$ lies in $\widehat{c}_0$.

It is worth expressing the equivalence (i) $\Longleftrightarrow$ (iv) of Theorem 1 in terms of set properties in the original space X.

Given a nonempty set K in a normed linear space X, a subset S of K is said to be *exposed* by $f_0 \in X^\star$ if for every $x \in S$,

$$f_0(x) = \sup f_0(K) > f_0(y) \quad \text{for all } y \in K \backslash S.$$

A *slice* of K by $f_0 \in X^\star$ is a subset of the form

$$S(K, f_0, \delta) \equiv \{x \in K : f_0(x) > \sup f_0(K) - \delta\} \quad \text{for } \delta > 0.$$

We say that S is *weakly exposed* by $f_0 \in X^\star$ if S is exposed by f_0 and given a weak open neighbourhood W of 0, there exists a $\delta > 0$ such that

$$S(K, f_0, \delta) \subseteq S + W.$$

LEMMA. *Given a nonempty closed bounded convex set K, $0 \in K$ in a normed linear space X, a subset S of K is weakly compact and weakly exposed by $f_0 \in X^\star$ if and only if $\widehat{S} = \partial p(f_0)$ where p is the continuous weak* lower semicontinuous gauge of the polar K^o.*

PROOF: Suppose $\widehat{S} = \partial p(f_0)$ but S is not weakly exposed by f_0. Then there exists a weak neighbourhood W of 0 and for each $n \in \mathbb{N}$, $x_n \in S(K, f_0, (1/n)) \backslash (S + W)$. Now $\{\widehat{x}_n\}$ has a weak* cluster point $F \in \overline{\widehat{K}}^{w^\star}$ and $F \in \partial p(f_0)$. But then $F \in \widehat{K}$ and $\{x_n\}$ has a weak cluster point in S which contradicts our supposition.

Conversely, suppose that S is weakly compact and weakly exposed by $f_0 \in X^\star$ but that there exists $F_0 \in \partial p(f_0) \backslash \widehat{S}$. Since $\widehat{S}$ is convex and weak* compact we can strongly separate F_0 from $\widehat{S}$ by some $g \in X^\star$.

Then there exists a weak* open set $N \equiv \{F \in X^{\star\star} : F(g) > \alpha\}$ such that $\widehat{S} \subseteq N$ and $F_0 \in \{F \in X^{\star\star} : F(g) < \alpha\}$. But also there exists a sequence of weak* open sets

$$M_n \equiv \left\{F \in X^{\star\star} : F(f_0) > \sup \widehat{f}_0\left(\overline{\widehat{K}}^{w^\star}\right) - \frac{1}{n}\right\} \cap \{F \in X^{\star\star} : F(g) < \alpha\}.$$

and $F_0 \in \partial p(f) \cap M_n$ for all $n \in \mathbb{N}$. Now the subdifferential mapping $f \mapsto \partial p(f)$ has the property that for any open subset U of $X^\star$ and weak* open half-space W in $X^{\star\star}$ where $\partial p(U) \cap W \neq \emptyset$ there exists a nonempty open subset V of U such that $\partial p(V) \subseteq W$. Using this and the extended Bishop–Phelps Theorem [**8**], we have for each $n \in \mathbb{N}$ there exists $\widehat{x}_n \in S\left(\overline{\widehat{K}}^{w^\star}, f_0, (1/n)\right) \cap M_n$. But then $\widehat{x}_n \in S(\widehat{K}, \widehat{f}_0, (1/n)) \backslash N$ which contradicts S being weakly exposed by f_0. ∎

Consequently we have a characterisation which generalises that of [**1**, Corollary 4 (i) $\iff$ (iii)].

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Theorem 2. *A Banach space X has the Radon–Nikodym Property if and only if every nonempty bounded closed convex set K in X has a subset which is weakly compact and weakly exposed.*

A Banach space X is an Asplund space if and only if its dual X^* has the Radon–Nikodym Property, [**7**, p. 82]. So our characterisations of the Radon–Nikodym Property imply extended characterisations of Asplund spaces.

Corollary. *For a Banach space X, the following are equivalent.*

(i) *X is an Asplund space.*

(ii) *Every continuous weak* lower semicontinuous gauge p on X^{**} has a point $F \in X^{**}$ with $\partial p(F) \subseteq \widehat{X}^*$.*

(iii) *Every nonempty weak* compact convex set K in X^* has a subset which is weakly compact and weakly exposed.*

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Department of Mathematics
The University of Newcastle
New South Wales 2308
Australia