Ergod. Th. & Dynam. Sys., (2023), **43**, 1382–1432
doi:10.1017/etds.2021.162

Decay of correlations for certain isometric extensions of Anosov flows

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(*Received* 1 *April* 2020 *and accepted in revised form* 22 *November* 2021)

Abstract. We establish exponential decay of correlations of all orders for locally G-accessible isometric extensions of transitive Anosov flows, under the assumption that the strong stable and strong unstable distributions of the base Anosov flow are C^1. This is accomplished by translating accessibility properties of the extension into local non-integrability estimates measured by infinitesimal transitivity groups used by Dolgopyat, from which we obtain contraction properties for a class of 'twisted' symbolic transfer operators.

Key words: decay of correlations, frame flows, exponential mixing
2020 Mathematics Subject Classification: 37A25 (Primary)

1. *Introduction*

One of the strongest characteristics of chaotic behaviour in dynamical systems is the exponential decay of correlations, or exponential mixing; in addition to being of intrinsic interest, this is typically accompanied by other strong statistical properties for regular observables. Naturally, there has been substantial interest in understanding and characterizing the dynamical systems with this property.

Hyperbolicity and the joint structure of the strong stable and strong unstable foliations are among the most well-understood mechanisms driving chaotic behaviour in both discrete-time and continuous-time dynamical systems. For continuous-time systems in particular, the extent to which these foliations fail to be integrable is known to be especially important.

Unfortunately, even among Anosov flows, there is no complete characterization of those which are exponentially mixing. There have been many significant advances, however, and we note only a few results that are relevant to our main result: we refer the reader to [9, §1] for an excellent narrative.

Perhaps the most notable result was due to Dolgopyat, who showed in [11] that smooth Anosov flows with C^1 stable and unstable foliations were exponentially mixing, via a quantitative non-integrability estimate. Recently, [9]—following related work by [3–5]—established exponential mixing for Anosov flows in the absence of any regularity hypotheses on the unstable foliation.

Our attention will be restricted to the class of Anosov flows studied in [9, 11], which are known to be exponentially mixing. We consider compact isometric extensions of these flows, and give criteria for these extensions to be exponentially mixing. We prove the following result.

THEOREM A. *Let M, N and F be closed Riemannian manifolds, where $\pi : M \to N$ is a fibre bundle with fibres isometric to F. Suppose that $g_t : N \to N$ is a $C^{2+\epsilon}$ transitive Anosov flow preserving an equilibrium measure ν with a Hölder potential $\varsigma : N \to \mathbb{R}$. Moreover, suppose that either:*

(i) *the strong stable and strong unstable foliations are C^1, and ν has unstable conditional measures ν^u that are diametrically regular; or*

(ii) *the strong stable foliation is $C^{1+\epsilon'}$ for some $\epsilon' > 0$, and ν is the g_t-invariant Riemannian volume.*

Let G be a closed, connected, normal subgroup of the isometry group $\mathrm{Isom}(F)$ *that acts transitively on F, and equip F with the pushforward ω of the normalized Haar measure on G. Let $f_t : M \to M$ be a G-extension of g_t. If f_t is locally G-accessible, then it enjoys exponential decay of correlations of all orders for the invariant product measure $\nu \times \omega$. (Here, the product measure $\nu \times \omega$ refers to the pushforward of the locally defined measure $\nu \times \omega$ on trivialization charts, where the fibrewise measures are obviously compatible so long as the trivializations are G-equivariant.)*

In fact, the main portion of our argument, obtaining quantitative non-integrability estimates for the extension in terms of the infinitesimal transitivity group described in Definition 4.3, only requires that the strong stable distribution is C^1. The additional regularity hypotheses are merely required to obtain the necessary spectral bounds for the symbolic transfer operator associated to the dynamical system on the base manifold. In particular, (i) would allow us to invoke [11, Theorem 2], whereas (ii) would allow us to instead invoke [9, Proposition 3.16].

Note that qualitative mixing in this context can be deduced quickly from very broad theorems for partially hyperbolic systems; specifically, we note the general result of Burns and Wilkinson in [8] that essentially accessible centre-bunched volume-preserving systems are mixing of all orders. Quantitative results, on the other hand, seem to require a more specialized approach.

Theorem A is analogous to a result of Dolgopyat, who showed in [12] that accessible compact group extensions of discrete-time expanding dynamical systems are exponentially mixing, and whose techniques we make heavy use of in our proof. Our approach follows the strategy Winter used in [17] to establish exponential mixing for frame flows over convex cocompact hyperbolic manifolds with Dolgopyat's techniques. We begin by constructing a symbolic model for the extension flow, establishing uniform local non-integrability estimates for this symbolic model and using arguments employed by Dolgopyat in both

[11, 12] to obtain bounds for the spectrum of certain 'twisted' transfer operators. Our main technical result is the following bound.

THEOREM B. *Fix notation as in Theorem A, and suppose that the potential ς is C^1. Then there are constants $C > 0$ and $r < 1$ so that we have*

$$\|\mathcal{L}_{z,\rho}^n \varphi\|_{L^2(\nu^u)} \leq C\|\varphi\|_{C^1} r^n$$

for all $\varphi \in C^1(U, V^\rho)$, all non-trivial irreducible representations ρ of G, and any $z \in \mathbb{C}$ with $|\mathrm{Re}(z) - P(\varsigma)| < 1$.

The principal novelty here, and our main point of divergence with [17], is that we use the local G-accessibility (see Definition 2.4) of the extension flow f_t to obtain the necessary local non-integrability estimates. As in [17], we measure the local non-integrability of the extension using Dolgopyat's infinitesimal transitivity group.

Although we rely heavily on the techniques used in [12], translating these into our setting presents several difficulties. Most notably, the non-integrability of the strong stable and unstable foliations of the base flow g_t and the non-triviality of the fibre bundle $\pi : M \to N$ require some additional care to properly deal with, though we are ultimately able to adapt many of the same arguments.

One of the most striking applications of quantitative mixing results in this setting has been the recent work of Kahn and Markovic, using the exponential mixing of frame flows on hyperbolic manifolds in their resolution of the surface subgroup conjecture in [14]. In the specific case of the frame flow, Theorem A can be stated more succinctly.

COROLLARY C. *Let N be the unit tangent bundle of a closed $C^{2+\epsilon}$ n-manifold of quarter-pinched negative curvature equipped with an invariant Riemannian volume ν, and let M be the oriented full frame bundle over N equipped with the extension $\nu \times \omega$ by the Haar measure on* SO$(n-1)$. *If the frame flow f_t is locally accessible, then it enjoys exponential decay of correlations of all orders.*

2. *Preliminaries*

We fix some notation that we use throughout this paper: N will be a closed Riemannian manifold equipped with a probability measure ν and $g_t : N \to N$ a C^∞ Anosov flow preserving ν. Let M be a compact Riemannian fibre bundle $\pi : M \to N$ whose fibres $\pi^{-1}(x)$ are each isometric to a fixed compact, Riemannian manifold F, and let $f_t : M \to M$ be an extension of g_t satisfying $\pi \circ f_t = g_t \circ \pi$. We equip M with the product measure μ of ν and a probability measure on F.

The motivating example for everything that follows is when N is the unit tangent bundle for a closed n-manifold of quarter-pinched negative curvature, M is the oriented orthonormal frame bundle, g_t is the geodesic flow and f_t is the frame flow. In this case, we take ν to be the Liouville measure on N, and μ to be the local product of the Liouville measure and the normalized Haar measure on SO$(n-1, \mathbb{R})$.

2.1. *Dynamical preliminaries.*

Our goal is to show that, with appropriate hypotheses, f_t enjoys *exponential decay of correlations* or, equivalently, is *exponentially mixing*.

Definition 2.1. A flow f_t is said to be *exponentially mixing of order k for C^α functions* if there are constants $C > 0$ and $r < 1$ so that

$$\bigg| \int_M \varphi_0 \cdot (\varphi_1 \circ f_{t_1}) \cdot \cdots \cdot (\varphi_k \circ f_{t_k})\, d\nu - \bigg(\int_M \varphi_i \, d\nu \bigg) \cdot \cdots \cdot \bigg(\int_M \varphi_k \, d\nu \bigg) \bigg|$$
$$< C r^{\min_{i \neq j} |t_i - t_j|} \cdot \|\varphi_0\|_{C^\alpha} \cdot \cdots \cdot \|\varphi_k\|_{C^\alpha}$$

for all $\varphi_i \in C^\alpha(M, \mathbb{C})$. Here, $\| \cdot \|_{C^\alpha}$ denotes the usual α-Hölder norm

$$\|\varphi\|_{C^\alpha} := \sup_x |\varphi(x)| + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}$$

on the space $C^\alpha(M, \mathbb{C})$ of α-Hölder complex-valued functions.

Actually, we prove that there are constants $C > 0$ and $r <$ so that

$$\bigg| \int_M \varphi_0 \cdot (\varphi_1 \circ f_{t_1}) \cdot \cdots \cdot (\varphi_k \circ f_{t_k})\, d\nu \bigg| < C r^{\max t_i} \cdot \|\varphi_0\|_{C^\alpha} \cdot \cdots \cdot \|\varphi_k\|_{C^\alpha}$$

for all $\varphi_i \in C^\alpha(M, \mathbb{C})$ with $\int_M \varphi_i \, d\nu = 0$. It is an elementary exercise to show that this is equivalent to Definition 2.1.

Although having some degree of regularity is critical, exponential mixing for Hölder functions and exponential mixing for more regular functions are typically equivalent by a standard approximation argument. We provide a brief outline of the proof here.

LEMMA 2.1. *Suppose f_t is exponentially mixing of order k for C^1 functions. Then, f_t is exponentially mixing of order k for C^α functions, for any $\alpha > 0$.*

Proof. We perform the argument in the case $k = 1$. The general case can be obtained by repeating this inductively.

It is well known that, given $\varphi \in C^\alpha(M, \mathbb{C})$, we can find smooth approximations φ_ϵ to φ with $\int_M \varphi_\epsilon \, d\mu = 0$ satisfying

$$\|\varphi_\epsilon - \varphi\|_{C^0} \leq \epsilon^\alpha \|\varphi\|_{C^\alpha} \quad \text{and} \quad \|\varphi_\epsilon\|_{C^1} \leq \epsilon^{-\dim(M)-1} \|\varphi\|_{C^0}$$

for any $\epsilon > 0$; a proof of this can be found, for instance, in [13, Lemma 2.4]. As f_t is exponentially mixing (at rate, say, r^t) for C^1 functions by hypothesis, we can write

$$\begin{aligned}
&\bigg| \int_M \varphi \cdot (\psi \circ f_t)\, d\mu \bigg| \\
&\quad \leq \bigg| \int_M \varphi \cdot (\psi \circ f_t)\, d\mu - \int_M \varphi_\epsilon \cdot (\psi \circ f_t)\, d\mu \bigg| + \bigg| \int_M \varphi_\epsilon \cdot (\psi \circ f_t)\, d\mu \bigg| \\
&\quad \leq \bigg| \int_M (\varphi - \varphi_\epsilon) \cdot (\psi \circ f_t)\, d\mu \bigg| + C r^t \|\varphi_\epsilon\|_{C^1} \|\psi\|_{C^1} \\
&\quad \leq \|\varphi - \varphi_\epsilon\|_{C^0} \|\psi\|_{C^0} + C r^t \epsilon^{-(\dim(M)+1)} \|\varphi\|_{C^0} \|\psi\|_{C^1} \\
&\quad \leq \epsilon^\alpha \|\varphi\|_{C^\alpha} \|\psi\|_{C^0} + C r^t \epsilon^{-(\dim(M)+1)} \|\varphi\|_{C^\alpha} \|\psi\|_{C^1} \\
&\quad = (\epsilon^\alpha + C r^t \epsilon^{-(\dim(M)+1)}) \|\varphi\|_{C^\alpha} \|\psi\|_{C^1}
\end{aligned}$$

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for any fixed $t > 0$. Note that the last line does not involve φ_ϵ, so we can simply set $\epsilon = r^{kt}$ with $k = 1/2(\dim(M) + 1)$. This leaves us with the inequality

$$\left| \int_M \varphi \cdot (\psi \circ f_t)\, d\mu \right| \le (r^{\alpha(\dim(M)+1)^{-1}t} + Cr^t r^{-(1/2)t}) \|\varphi\|_{C^\alpha} \|\psi\|_{C^1}$$
$$\le (1 + C)(r^{\min(1/2, \alpha(\dim(M)+1)^{-1})})^t \|\varphi\|_{C^\alpha} \|\psi\|_{C^1}$$

and, noting that the base of the exponential term is at most 1, we see that f_t is exponentially mixing for any $\varphi \in C^\alpha(M)$ and $\psi \in C^1(M)$. Now, set $D := C + 1$ and $s := r^{\min(1/2, \alpha(\dim(M)+1)^{-1})}$ and consider $\varphi, \xi \in C^\alpha(M)$. Once again, we can find a smooth approximation ξ_ϵ to ξ with $\int_M \xi\, d\mu = 0$ satisfying

$$\|\xi_\epsilon - \xi\|_{C^0} \le \epsilon^\alpha \|\xi\|_{C^\alpha} \quad \text{and} \quad \|\xi_\epsilon\|_{C^1} \le \epsilon^{-\dim(M)-1} \|\xi\|_{C^0}$$

for all $\epsilon > 0$. By what we have just shown, we can now write

$$\begin{aligned}
&\left| \int_M \varphi \cdot (\xi \circ f_t)\, d\mu \right| \\
&\quad \le \left| \int_M \varphi \cdot (\xi \circ f_t)\, d\mu - \int_M \varphi \cdot (\xi_\epsilon \circ f_t)\, d\mu \right| + \left| \int_M \varphi \cdot (\xi_\epsilon \circ f_t)\, d\mu \right| \\
&\quad \le \left| \int_M \varphi \cdot ((\xi - \xi_\epsilon) \circ f_t)\, d\mu \right| + Ds^t \|\varphi\|_{C^\alpha} \|\xi_\epsilon\|_{C^1} \\
&\quad \le \|\xi - \xi_\epsilon\|_{C^0} \|\varphi\|_{C^0} + Ds^t \epsilon^{-(\dim(M)+1)} \|\varphi\|_{C^\alpha} \|\xi\|_{C^0} \\
&\quad \le \epsilon^\alpha \|\xi\|_{C^\alpha} \|\varphi\|_{C^\alpha} + Ds^t \epsilon^{-(\dim(M)+1)} \|\varphi\|_{C^\alpha} \|\xi\|_{C^\alpha} \\
&\quad \le (\epsilon^\alpha + Ds^t \epsilon^{-(\dim(M)+1)}) \|\varphi\|_{C^\alpha} \|\xi\|_{C^\alpha}
\end{aligned}$$

for any fixed $t > 0$. Once again, we can set $\epsilon = s^{kt}$ with $k = 1/2(\dim(M) + 1)$, leaving us with the bound

$$\left| \int_M \varphi \cdot (\xi \circ f_t)\, d\mu \right| \le (s^{\alpha(\dim(M)+1)^{-1}t} + Ds^t s^{-(1/2)t}) \|\varphi\|_{C^\alpha} \|\xi\|_{C^\alpha}$$
$$\le (1 + D)(s^{\min(1/2, \alpha(\dim(M)+1)^{-1})})^t \|\varphi\|_{C^\alpha} \|\xi\|_{C^\alpha},$$

which is independent of ϵ. Once again, we note that the base of the exponential term is at most one and does not depend on φ or ξ, completing our proof. □

Henceforth, we restrict our focus to establishing exponential mixing for C^1 functions; exponential mixing for Hölder functions then follows by Lemma 2.1. Of course, whether a system is exponentially mixing depends on the measure under consideration: we are interested in equilibrium measures for Hölder potentials.

Definition 2.2. For a continuous function $\varsigma : N \to \mathbb{R}$, we call a measure ν *an equilibrium state for g_t with potential ς* if ν maximizes

$$\int_N \varsigma\, d\nu + h_\nu(g_1)$$

among all g_t-invariant probability measures on N. To emphasize the potential, we write ν_ς for the equilibrium state corresponding to ς when it exists and is unique.

Of particular importance to us is the fact that equilibrium states admit a local product structure with respect to the strong stable and unstable foliations, and that they are invariant under the appropriate transfer operators; we expand on both of these properties in due course.

When g_t is an Anosov flow on a compact manifold, it is a classical result of Bowen and Ruelle [7] that equilibrium states for Hölder potentials exist and are unique. Of course, the measure of maximal entropy is always an equilibrium state for the trivial potential $\varsigma = 0$. In the case of the geodesic flow in negative curvature, the Liouville measure is the equilibrium state for the geometric potential

$$\varsigma(x) = -\frac{d}{dt}\bigg|_{t=0} (\log \|dg_t|_{W^u(x)}\|)$$

on the unit tangent bundle.

We are interested in extensions of Anosov flows that act fibrewise by isometries. Throughout, we assume that g_t is a $C^{2+\epsilon}$ flow whose strong stable and unstable distributions are C^1, so that the results of [11] apply.

Definition 2.3. We call a smooth flow $f_t\colon (M, \mu) \to (M, \mu)$ on a closed Riemannian manifold M a *G-extension* of $g_t\colon (N, \nu) \to (N, \nu)$ if $\pi : M \to N$ is a smooth fibre bundle where:

- the fibres $\pi^{-1}(x)$, with the induced metric, are all isometric to a closed Riemannian manifold F;
- G is a closed, connected normal subgroup of the isometry group $\mathrm{Isom}(F)$;
- G acts transitively on F and has no proper transitive normal subgroups;
- $\pi \circ f_t = g_t \circ \pi$;
- there is an atlas of trivializations of $\pi : M \to N$ for which all transition functions lie in G;
- with respect to these trivializations, the isometries of F induced by the flow f_t all lie in G;
- f_t preserves a measure μ satisfying $\pi_*(\mu) = \nu$; and
- the fibrewise disintegration of μ along the fibres of $\pi : M \to N$ is the pushforward of the normalized Haar measure on G to each fibre.

The primary driver of exponential mixing in our case will be a stronger variant of local accessibility.

Definition 2.4. Let $f_t\colon M \to M$ be a G-extension of $g_t\colon N \to N$. We call f_t *locally G-accessible* if, for every $\epsilon > 0$, any trivialization $\phi\colon \pi^{-1}(B_\epsilon(x)) \to B_\epsilon(x) \times F$ defined near $x \in N$ and any isometry $h \in G$, there is a sequence of points $x_0, \ldots, x_k \in N$ for which:

- $x_0 = x_k = x$;
- $x_0, \ldots, x_k \in B_\epsilon(x)$;
- we either have $x_{i+1} \in W^{su}_{g_t}(x_i)$ or $x_{i+1} \in W^{ss}_{g_t}(x_i)$ for each i; and

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- we have $h = h_k^0 \circ \cdots \circ h_0^1$, where $h_i^{i+1} : F \to F$ is given (via ϕ) by the isometry $\pi^{-1}(x_i) \to \pi^{-1}(x_{i+1})$ induced by leaves of the strong stable or strong unstable foliation of f_t.

2.2. *Symbolic dynamics.* In this section, we build a discrete, symbolic model for f_t: we follow [17], and accomplish this by artificially extending a standard Markov partition for the base flow. The results of [6, 16] on the existence of Markov partitions for hyperbolic dynamical systems are classical and well-understood; as such, we recall some of the important points but refrain from delving into the details.

THEOREM 2.2. (Bowen and Ratner) *If g_t is Anosov, then g_t has a Markov partition of size ϵ for any sufficiently small $\epsilon > 0$.*

Specifically, Bowen and Ratner constructed a Markov partition by taking local strong stable and unstable segments and forming a *Markov rectangle*.

Definition 2.5. Given $x \in N$ and $\epsilon > 0$, consider the local strong and weak stable and unstable manifolds of size ϵ through x given by

$$W_\epsilon^{ss}(x) := (W^{ss}(x) \cap B_\epsilon(x))^\circ, \qquad W_\epsilon^{su}(x) := (W^{su}(x) \cap B_\epsilon(x))^\circ,$$
$$W_\epsilon^{ws}(x) := (W^{ws}(x) \cap B_\epsilon(x))^\circ, \qquad W_\epsilon^{wu}(x) := (W^{wu}(x) \cap B_\epsilon(x))^\circ,$$

where in each case we have taken the connected component through x. For $u \in W_\epsilon^{su}(x)$ and $s \in W_\epsilon^{ss}(x)$, we define the *bracket of u and s* to be the point of intersection

$$[u, s] = W_\epsilon^{ss}(u) \cap W_\epsilon^{wu}(s),$$

which is necessarily unique when $\epsilon > 0$ is sufficiently small. We define the *Markov rectangle* $[W_\epsilon^{su}(x), W_\epsilon^{ss}(x)]$ to be the set

$$[W_\epsilon^{su}(x), W_\epsilon^{ss}(x)] := \{[u, s] \mid u \in W_\epsilon^{su}(x) \text{ and } s \in W_\epsilon^{ss}(x)\},$$

assuming $\epsilon > 0$ is sufficiently small.

For now, fix $\epsilon > 0$ chosen to be small enough that a Markov partition exists; we will likely need to adjust our choice of ϵ as we proceed. We let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a Markov partition of size ϵ for g_t, where each rectangle $R_i := [U_i, S_i]$ is generated by local strong stable and unstable manifolds $S_i := W_\epsilon^{ss}(z_i)$ and $U_i := W_\epsilon^{su}(z_i)$ through points $z_i \in N$. Set $R := \bigcup R_i$, $U := \bigcup U_i$ and $S := \bigcup S_i$.

Remark 2.3. As we assumed that the strong stable and unstable foliations for g_t were C^1, each R_i is naturally an open C^1 submanifold of N. It is important to note that both the Poincaré return map $\mathcal{P}$ and the return time map τ defined previously are the restrictions of locally C^1 functions on R, but both of these functions are discontinuous at points whose image would lie on the boundary of a Markov rectangle.

To fix this, we remove any point of discontinuity for any iterate $\mathcal{P}^n$ and $\tau^{(n)}$, and replace R with the corresponding full-measure residual subset R^*. Note that $\mathcal{P}$ and τ are both well defined on R^*, though we continue to reference R, U and S to make use of their C^1 structure.

A more extensive discussion of this can be found in [10, pp. 380–382]. We write $(\Sigma, \mathcal{P})$ for the Markov shift corresponding to the partition $\mathcal{R}$ of (N, g_t).

Definition 2.6. The *suspension* of $(\Sigma, \mathcal{P})$ with roof function τ is the flow $g'_t \colon \Sigma \times \mathbb{R}/\sim \to \Sigma \times \mathbb{R}/\sim$ defined by $g'_t(x, s) = (x, s + t)$, where we have declared $(x, \tau(x)) \sim (\mathcal{P}(x), 0)$.

Theorem 2.2 states exactly that the natural inclusion of R into N induces a Hölder-continuous semi-conjugacy between g_t and g'_t: this is precisely the result we wish to extend to f_t.

Fix a finite cover $\{V_i\}$ that trivializes the fibre bundle $\pi : M \to N$, with isometries $\phi_i : \pi^{-1}(V_i) \to V_i \times F$. At this point, we may need to revisit our choice of ϵ: let us assume that we have taken ϵ to be smaller than the Lebesgue number of the cover $\{V_i\}$. By definition, this means that each R_j lies entirely in some $V_{k(j)}$ and, hence, $\phi_{k(j)}$ induces an isometry $\pi^{-1}(R_j) \to R_j \times F$. We can put these together to form an isometry $\phi\colon \pi^{-1}(R) \to R \times F$.

It is worth noting that we have considerable freedom in our choice of isometries ϕ_i, and this is something we want to exploit to simplify our arguments in §4. As the centre-stable foliation of f_t is C^1, we can modify a given ϕ_i so that it is constant along the (local) leaves of this foliation. Specifically, we assume that the projection of each $\phi_i : \pi^{-1}(V_i) \to V_i \times F$ onto F is constant on each connected component $(W^{su}_{f_t}(x) \cap \pi^{-1}(V_i))^\circ$ for each $x \in V_i$.

We realize f_t as a suspension flow on $\Sigma \times F$ with roof function τ. To do this, we need information on the fibrewise action of g_t.

Definition 2.7. For $x \in R_{j_1} \cap R^*$ with $\mathcal{P}(x) \in R_{j_2}$, we define the *temporal holonomy* $\mathrm{Hol}(x)$ *at* x to be the isometry between $\phi_{k(j_1)}(\pi^{-1}(x))$ and $\phi_{k(j_2)}(\pi^{-1}(\mathcal{P}(x)))$ induced by f_t. This defines a function $\mathrm{Hol}\colon R^* \to G$.

We often write $\mathrm{Hol}^{(n)}(x)$ for $\mathrm{Hol}(\mathcal{P}^{n-1}(x)) \circ \cdots \circ \mathrm{Hol}(x)$. Note that function composition is the multiplication operation in G.

Given the temporal holonomy function $\mathrm{Hol}\colon R^* \to G$, we can, of course, recover f_t as a suspension flow on $(\Sigma \times F, \mathcal{P} \times \mathrm{Hol})$ with roof function τ. However, we push this a step further.

We can define a projection along the leaves of the strong stable foliation $\mathrm{proj}_S\colon R \to U$ by setting $\mathrm{proj}_S([u, s]) = u$. This allows us to construct a uniformly expanding model U for g_t, where the Poincaré return map $\mathcal{P}\colon R^* \to R^*$ descends to a map $\sigma : U^* \to U^*$ given by $\sigma := \mathrm{proj}_S \circ \mathcal{P}$.

Remark 2.4. By construction, the return time map τ and the temporal holonomy function Hol are both constant along the leaves of the strong stable foliation of g_t and, hence, descend to functions $\tau : U \to \mathbb{R}$ and $\mathrm{Hol}\colon U \to G$.

Set $U_\tau := U^* \times \mathbb{R}/\sim$ and $U_{\tau,\mathrm{Hol}} := U^* \times \mathbb{R} \times F/\sim$, where we declare $(u, \tau(u)) \sim (\mathcal{P}(u), 0)$ and $(u, \tau(u), k) \sim (\mathcal{P}(u), 0, (\mathrm{Hol}(u))(k))$, respectively. We write f'_t for the suspension flow $f'_t(u, s, k) = (u, s + t, k)$ on $U_{\tau,\mathrm{Hol}}$. It is not too difficult to show that

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the exponential mixing of f_t is equivalent to exponential mixing for f'_t, though we must first fix a measure on $U_{\tau,\mathrm{Hol}}$ to make sense of this.

Remark 2.5. As ν is an equilibrium state for a Hölder potential ϕ, there are measures ν_i^s and ν_i^u on each of the strong stable and unstable segments S_i and U_i so that ν is absolutely continuous with respect to the product $\nu_i^u \times \nu_i^s \times dt$. Moreover, these measures can be chosen so that the Radon–Nikodym derivative of ν with respect to the product $\nu_i^u \times \nu_i^s \times dt$ is uniformly bounded above by a constant $K > 1$ and below by $K^{-1} < 1$. We write ν^u and ν^s for the corresponding measures on U and S, respectively, and suppose that they have been normalized so that $\nu_U(U) = \nu_S(S) = 1$.

Remark 2.6. Up to replacing ϕ with a cohomologous function on $(\Sigma, \mathcal{P})$, we can assume that ϕ is the extension of a Hölder potential ϕ^U on U. As a consequence, we may as well assume that ν^u is, in fact, an equilibrium state for a Hölder potential ϕ^U on (U, σ).

These are classical results in thermodynamic formalism: we refer the reader to [15] and [18, pp. 87–91], respectively, for more details. We require, in addition, the conditional measure ν^u to have a doubling property later on, in order to control the spectrum of our transfer operators using Dolgopyat's methods.

Definition 2.8. We say that a measure ν^u has the *doubling* or *Federer* property, or is *diametrically regular*, if for any $k > 1$ there is a uniform constant $C > 0$ so that

$$\nu^u(B_{kr}(x)) < C\nu^u(B_r(x))$$

for all $x \in U$ and $r > 0$.

It is important to note that $U_{\tau,\mathrm{Hol}}$ has no natural smooth structure, and U^* may well have infinitely many connected components. However, we understand $C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ to mean the class of functions on $U_{\tau,\mathrm{Hol}}$, identified as a quotient of $U^* \times \mathbb{R} \times F/\sim$, that arise as the restriction of a smooth function on $U \times \mathbb{R} \times F$. It is easy to see that the symbolic flow f'_t leaves invariant $C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ defined in this way.

With the following lemma, we restrict ourselves to establishing exponential mixing for the suspension flow f'_t on the expanding model $U_{\tau,\mathrm{Hol}}$.

LEMMA 2.7. *If f'_t is exponentially mixing of order k for functions in $C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$, then f_t is exponentially mixing of order k for functions in $C^1(M, \mathbb{C})$.*

Proof. Once again, we perform the argument in the case $k = 1$. The general case can be obtained by repeating this inductively. Given $\varphi, \psi \in C^1(M, \mathbb{C})$ with $\int_M \varphi \, d\mu = \int_M \psi \, d\mu = 0$ and a fixed $k \in F$, we consider corresponding functions $\varphi_t, \psi_t \in C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ given by

$$\varphi_t(u, h, r) := \int_S \varphi(f_{t+r}(\phi^{-1}([u, s], h(k)))) \, d\nu^s,$$
$$\psi_t(u, h, r) := \int_S \psi(f_{t+r}(\phi^{-1}([u, s], h(k)))) \, d\nu^s,$$

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for each t. Let $C_0 q^t$ be the rate of contraction of S under g_t. As φ is C^1,

$$|\varphi(f_{t+r}(\phi^{-1}([u,s],h(k))) - \varphi(f_{t+r}(\phi^{-1}([u,s_0],h(k)))| < C_0 q^t \|\varphi\|_{C^1}$$

for any $s_0 \in S$. Of course, this yields

$$\bigg| \int_S \varphi(f_{t+r}(\phi^{-1}([u,s],h(k))) \, d\nu^s - \int_S \varphi(f_{t+r}(\phi^{-1}([u,s_0],h(k))) \, d\nu^s \bigg| \\ < C_0 q^t \|\varphi\|_{C^1} \nu^s(S)$$

after simply integrating both sides with respect to s. This can be rewritten as

$$\bigg| \int_S \varphi(f_{t+r}(\phi^{-1}([u,s],h(k))) \, d\nu^s - \varphi(f_{t+r}(\phi^{-1}([u,s_0],h(k))) \cdot \nu^s(S) \bigg| \\ < C_0 q^t \|\varphi\|_{C^1} \nu^s(S)$$

because s_0 is fixed. We then see quickly that the difference in the integrals

$$\int_{U_{\tau,\mathrm{Hol}}} \bigg(\int_S \varphi(f_{t+r}(\phi^{-1}([u,s],h(k)))) \, d\nu^s(s) \bigg) \\ \cdot \bigg(\int_S \psi(f_r(\phi^{-1}([u,s'],h(k)))) \, d\nu^s(s') \bigg) \, d\omega \, dr \, d\nu^u \tag{2.1}$$

and

$$\nu^s(S) \int_{U_{\tau,\mathrm{Hol}}} \int_S \varphi(f_{t+r}(\phi^{-1}([u,s_0],h(k)))) \\ \cdot \psi(f_r(\phi^{-1}([u,s'],h(k)))) \, d\nu^s(s') \, d\omega \, dr \, d\nu^u \tag{2.2}$$

is at most $C_1 q^t \|\varphi\|_{C^1} \|\psi\|_{C^0}$. But now, the same argument shows that

$$\varphi(f_{t+r}(\phi^{-1}([u,s_0],h(k)))) \cdot \psi(f_r(\phi^{-1}([u,s'],h(k))))$$

and

$$\varphi(f_{t+r}(\phi^{-1}([u,s'],h(k)))) \cdot \psi(f_r(\phi^{-1}([u,s'],h(k))))$$

must be within $C_2 q^t \|\varphi\|_{C^1} \|\psi\|_{C^0}$ of each other. Hence,

$$\nu^s(S) \int_{U_{\tau,\mathrm{Hol}}} \int_S \varphi(f_{t+r}(\phi^{-1}([u,s'],h(k)))) \\ \cdot \psi(f_r(\phi^{-1}([u,s'],h(k)))) d\nu^s(s') \, d\omega \, dr \, d\nu^u$$

is within $C_3 q^t \|\varphi\|_{C^1} \|\psi\|_{C^0}$ of (2.2). From the local product structure of ν, we see that this is within a constant multiplicative factor of K of the integral

$$\int_M \varphi(f_t(x)) \cdot \psi(x) \, d\mu \tag{2.3}$$

for any φ and ψ. To conclude, we simply observe that if $\int_M \varphi \, d\mu = \int_M \psi \, d\mu = 0$, then we must have

$$\int_{U_{\tau,\mathrm{Hol}}} \varphi_t(u,h,r) \, d\omega \, dr \, d\nu^u = \int_{U_{\tau,\mathrm{Hol}}} \psi_t(u,h,r) \, d\omega \, dr \, d\nu^u = 0$$

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for any $t \in \mathbb{R}$. Moreover, the regularity of φ_t and ψ_t is determined by the regularity of the bracket operation $[\cdot, \cdot]$: because we assumed that the foliations were C^1, we see that φ_t and ψ_t must also be C^1. Hence, if f_t' is exponentially mixing for functions in $C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$, (2.1) must decay exponentially in t, from which we conclude that (2.3) must also decay exponentially. □

2.3. *Representation theory.* In this section, we recall some classical results from the representation theory and harmonic analysis of compact Lie groups; our primary references are [2, 19].

Following [17, §3.6], our strategy is to decompose functions on $U_{\tau,\mathrm{Hol}}$ into components corresponding to irreducible representations of G. A function $\varphi \in C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ can be viewed as a C^1 function $\tilde{\varphi}\colon U_\tau \to L^2(G)$ by setting $\tilde{\varphi}(u, r) := \varphi(u, \cdot, r)$.

As G is a compact, connected Lie group, we can decompose $\tilde{\varphi}(u, r) \in L^2(G)$ into isotypic components corresponding to irreducible representations of G: this is, of course, the classical Peter–Weyl theorem.

THEOREM 2.8. (Peter–Weyl) *If G is a compact, connected Lie group, then there is a decomposition*

$$L^2(G) = \bigoplus_\rho (V^\rho)^{\oplus \dim \rho},$$

where the sum is taken over pairwise non-isomorphic irreducible representations of G, and the $(V^\rho)\oplus \dim \rho$ associated with non-isomorphic irreducible representations are pairwise orthogonal with respect to the standard inner product on $L^2(G)$.

We fix such a decomposition and write $\tilde{\varphi}(u, r) = \sum_\rho \tilde{\varphi}^\rho(u, r)$ for the decomposition of $\tilde{\varphi}(u, r)$ obtained by projecting onto each $(V^\rho)\oplus \dim \rho$. Abusing notation, we use φ to refer interchangeably to a function $U_\tau \to L^2(G)$ or to the function $U_{\tau,\mathrm{Hol}} \to \mathbb{C}$; there should be little ambiguity in either case.

For our later analysis, it is helpful to consider the derived representation $d\rho$ of the Lie algebra $\mathfrak{g}$ of G acting on $L^2(G)$, induced by the representation ρ of G on $L^2(G)$: see [2, §2.5.1] for details. We always assume that we have a fixed Ad-invariant norm $\|\cdot\|_{\mathfrak{g}}$ on $\mathfrak{g}$.

Definition 2.9. Given an irreducible representation $\rho\colon G \to \mathrm{GL}(V^\rho)$ of G (where we view $V^\rho \subset L^2(G)$), we define the norm $\|\rho\|$ of ρ to be the supremum

$$\|\rho\| := \sup_{\|X\|_{\mathfrak{g}}=1} \|d\rho(X)\|_{L^2(G)}$$

where $\|d\rho(X)\|_{L^2(G)}$ is the operator norm of $d\rho(X)$ viewed as an automorphism of $L^2(G)$.

It is a classical fact that $\|\rho\|$ is finite, and can be bounded in terms of the highest weight associated with ρ. In the following propositions, let $|\lambda|$ denote the norm of a weight λ arising from a fixed $\mathrm{ad}_{\mathfrak{g}}$-invariant inner product $\langle\cdot, \cdot\rangle$ on the Lie algebra $\mathfrak{g}$.

PROPOSITION 2.9. *Let ρ be a non-trivial irreducible representation, and let $\lambda(\rho)$ be its highest weight. There are uniform constants $C > 0$ and $m > 0$ so that*

$$\|\rho\| \leq C|\lambda(\rho)|^m.$$

Proof. See [2, Theorem 3.4.1]; note that the Hilbert–Schmidt norm is an upper bound for the operator norm. □

We also require some particular results on the growth rate of $\|\rho\|$.

PROPOSITION 2.10. *There is a constant $N > 0$ so that the sum $\sum_\rho \|\rho\|^{-n}$ taken over non-trivial irreducible representations ρ of G converges for any $n \geq N$.*

Proof. By Proposition 2.9, we can bound

$$\sum_\rho \|\rho\|^{-n} \leq C^{-n} \sum_\rho |\lambda(\rho)|^{-(m/n)},$$

which, by [2, Theorem 3.2.1], converges so long as m/n exceeds the rank of G. □

More generally, we can obtain decay estimates for the Fourier coefficients φ^ρ associated with irreducible representations ρ.

THEOREM 2.11. *For every $n > \dim(G)/4$, there is a $C > 0$ so that*

$$\|\rho\|^{an} \|\varphi^\rho\|_{L^2(G)} \leq C\|\varphi\|_{C^n}$$

for all non-trivial irreducible representations ρ of G and all $\varphi \in C^{2n}(U_\tau, L^2(G))$.

Proof. By [19, equations (1.5) and (1.11)], we have the inequality

$$\|\lambda(\rho)\|^{2k} \|\varphi^\rho\|_{L^2(G)} \leq \|\varphi\|_{C^k}$$

for any $\varphi \in C^{2k}(U_\tau, L^2(G))$ and $2k > \dim(G)/2$. By Proposition 2.9, we therefore have

$$\|\rho\|^{2k/m} \|\varphi^\rho\|_{L^2(G)} \leq C^{2k/m} \|\varphi\|_{C^k}$$

for some uniform $m > 0$, from which the theorem follows. □

3. *Twisted transfer operators*

In this section, we define Dolgopyat's 'twisted' transfer operators, and show how the spectral bounds we intend to obtain for these operators lead to correlation decay estimates for the expanding suspension semi-flow f_t' constructed in the previous section.

Recall that we can view a smooth function $\psi \in C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ as a function $\psi \in C^1(U_\tau, L^2(G))$. We can integrate out the time variable to obtain

$$\tilde{\psi}(u) := \int_0^{\tau(u)} \psi(u, r)\, dr$$

in $C^1(U, L^2(G))$; this is the space on which we would like to define our operators. The advantages of working with smooth (as opposed to Hölder) functions will become clear in §5, but we need to assume that ς is C^1 for most of our arguments. We explain how to

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modify our proof to deal with the general case where ς is Hölder in Corollary 5.8, using a standard approximation argument.

Let ρ be an irreducible representation of G acting on an isotypic component $V^\rho \subset L^2(G)$, and fix $z \in \mathbb{C}$. We define the transfer operator $\tilde{\mathcal{L}}_{z,\rho} : C^1(U, V^\rho) \to C^1(U, V^\rho)$ by

$$(\tilde{\mathcal{L}}_{z,\rho}\varphi)(u) := \sum_{\sigma(u')=u} e^{\varsigma_z(u')}(\rho(\mathrm{Hol}(u')) \cdot \varphi(u')),$$

where ς_z is the potential on the one-sided Markov model (Σ^+, σ) obtained as the restriction of the potential

$$\int_0^{\tau(u)} (\varsigma \circ p)(u, s, t)\, dt - z \cdot \tau(u, s)$$

defined on $(\Sigma, \mathcal{P})$. Note that ς_z is well-defined in light of Remark 2.6, where we assumed that both α and τ are constant in s. It is also worth remarking that both τ and p are C^1, because we assumed that the strong stable and unstable foliations of g_t were C^1. As a consequence, ς_z and, hence, $\tilde{\mathcal{L}}_{z,\rho}\varphi$ are both C^1, because we have restricted ourselves to the case where ς is smooth.

Let us recall some classical results of thermodynamic formalism; for a slightly more detailed treatment, we refer the reader to [18, pp. 87–91]. Let $P(\varsigma)$ be the topological pressure of ς for the map g_1. By the Ruelle–Perron–Frobenius theorem, the operator $\tilde{\mathcal{L}}_{P(\varsigma),0}$ associated with the trivial representation $\rho = 0$ has a unique positive eigenvector $\varphi_\varsigma \in C^1(U, \mathbb{R})$ with eigenvalue $e^{P(\varsigma)}$.

Recall that ν^u is an equilibrium measure for the restriction of the potential ς to (Σ^+, σ), which by construction has the same topological pressure $P(\varsigma)$. By the Lanford–Ruelle variational principle, this means that

$$e^{P(\varsigma)} \int_U \varphi\, d\nu^u = \int_U \tilde{\mathcal{L}}_{P(\varsigma),0}\varphi\, d\nu^u$$

for all $\varphi \in C^1(U, V^\rho)$. It will be convenient to normalize $\tilde{\mathcal{L}}_{P(\varsigma),0}\varphi$ so that it preserves the measure ν^u: let $\mathcal{L}_{z,\rho}$ be the operator defined by

$$(\mathcal{L}_{z,\rho}\varphi)(u) := \varphi_\varsigma(u) \frac{(\tilde{\mathcal{L}}_{z,\rho}(\varphi \cdot \varphi_\varsigma^{-1}))(u)}{e^{P(\varsigma)}}$$

for all $\varphi \in C^1(U, V^\rho)$.

Remark 3.1. With these renormalizations, we have

$$\int_U \varphi\, d\nu^u = \int_U \mathcal{L}_{P(\varsigma),0}\varphi\, d\nu^u$$

for all $\varphi \in C^1(U, \mathbb{R})$.

We can alternatively write

$$(\mathcal{L}_{z,\rho}\varphi)(u) = \sum_{\sigma(u')=u} e^{\alpha_z(u')}(\rho(\mathrm{Hol}(u')) \cdot \varphi(u')),$$

where we set

$$\alpha_z(u) := \int_0^{\tau(u)} (\varsigma \circ p)(u, s, t)\, dt - z \cdot \tau(u, s) \\ - \log(\varphi_\varsigma(u)) + \log(\varphi_\varsigma(\sigma(u))) - \log P(\varsigma)$$

for all $u \in U$: note that the positivity of φ_ς is required to ensure that $\log(\varphi_\varsigma(u))$ is well-defined. It will be helpful to have a similar formulation for the iterates

$$(\mathcal{L}^n_{z,\rho}\varphi)(u) = \sum_{\sigma^n(u')=u} e^{\alpha_z^{(n)}(u')}(\rho(\mathrm{Hol}^{(n)}(u')) \cdot \varphi(u')),$$

with

$$\alpha_z^{(n)}(u) := \int_0^{\tau^{(n)}(u)} (\varsigma \circ p)(u, s, t)\, dt \\ - z \cdot \tau^{(n)}(u, s) - \log(\varphi_\varsigma^{(n)}(u)) + \log(\varphi_\varsigma^{(n)}(\sigma(u))) - n \log P(\varsigma)$$

for all $u \in U$. Here, we write

$$\varphi_\varsigma^{(n)}(u) := \prod_{i=0}^{n-1} \varphi_\varsigma(\varsigma^i(u))$$

for the nth ergodic product along σ.

Our overarching goal is to establish spectral bounds for these operators; in §5, we ultimately prove the following theorem.

THEOREM 3.2. *There are constants $C > 0$ and $r < 1$ so that*

$$\|\mathcal{L}^n_{z,\rho}\varphi\|_{L^2(\nu^u)} \leq C\|\varphi\|_{C^1} r^n$$

for all $\varphi \in C^1(U, V^\rho)$, all non-trivial irreducible representations ρ of G, and any $z \in \mathbb{C}$ with $|\mathrm{Re}(z) - P(\varsigma)| < 1$.

The remainder of this section is devoted to showing how we obtain Theorem A from Theorem 3.2.

Given $\varphi_0, \ldots, \varphi_k \in C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ with $\int_{U_{\tau,\mathrm{Hol}}} \varphi_i \, d\nu^u \, d\omega \, dr = 0$, let

$$\beta_k(t_1, \ldots, t_k) := \int_U \int_0^{\tau(u)} \int_G \varphi_0(u, h, r)\bigg(\prod_{i=1}^k \varphi_i(u, h, r + t_i)\bigg)\, d\omega\, dr\, d\nu^u$$

be the kth correlation function $\beta_k : \mathbb{R}^k_+ \to \mathbb{R}$. To show that β_k decays exponentially in $t_1, \ldots, t_k$, we show that the integral defining its Laplace transform $\hat{\beta}(\xi_1, \ldots, \xi_k)$ converges absolutely for some fixed values of $\xi_1, \ldots, \xi_k$. The following lemma expresses the Laplace transform in terms of the transfer operators we have just defined, the proof of which consists almost entirely of elementary integral manipulations.

We are somewhat cavalier in interchanging sums and integrals, though this is eventually justified as the final expression we obtain is absolutely convergent.

LEMMA 3.3. *Given $\varphi_0, \ldots, \varphi_k \in C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ as previously with $\int_{U_{\tau,\mathrm{Hol}}} \varphi_i \, d\nu^u \, d\omega \, dr =$ 0, we can bound the Laplace transform of the kth-order correlation by*

$$|\hat{\beta}_k(\xi_1, \ldots, \xi_k)| \le \sum_{\rho} \sum_{n_1,\ldots,n_k=1}^{\infty} \int_U (\mathcal{L}^{\max n_j}_{\rho,P(\varsigma)+\xi} |\hat{\varphi}^{\rho}_{0,-\xi}|)(u) \bigg(\prod_{i=1}^{k} \|\hat{\varphi}_{i,\xi_i}\|_{C^0} \bigg) \, d\nu^u$$

for $\mathrm{Re}(\xi_1), \ldots, \mathrm{Re}(\xi_k) < 0$. *Here, we use* $\hat{\varphi}_{i,\xi_i}$ *to denote the function*

$$\hat{\varphi}_{i,\xi_i}(u, h) := \int_0^{\tau(u)} \varphi_i(u, h, t_i) e^{-\xi_i t_i} \, dt_i$$

and $\hat{\beta}_k$ *to denote the Laplace transform*

$$\hat{\beta}_k(\xi_1, \ldots, \xi_k) := \int_0^{\infty} \cdots \int_0^{\infty} \beta_k(t_1, \ldots, t_k) e^{-(\xi_1 t_1 + \cdots + \xi_k t_k)} \, dt_1 \ \ldots \ dt_k$$

for $\xi_i \in \mathbb{R}$.

Proof. By definition, the Laplace transform $\hat{\beta}_k(\xi_1, \ldots, \xi_k)$ of $\beta_k(t_1, \ldots, t_k)$ is given by

$$\int_0^{\infty} \cdots \int_0^{\infty} \int_U \int_{\max(0,\tau(u)-t_i)}^{\tau(u)} \int_G \varphi_0(u, h, r) \cdot \bigg(\prod_{i=1}^{k} \varphi_i(u, h, r + t_i) e^{-\xi_i t_i} \bigg) \, d\omega \, dr \, d\nu^u \, dt_1 \ \cdots \ dt_k \tag{3.1}$$

for $\xi_1, \ldots, \xi_k \in \mathbb{C}$. As the systems of inequalities

$$\left\{ \begin{aligned} 0 < r &< \tau(u) \\ \tau(u) - t_i &< r \\ 0 &< t_i \end{aligned} \right\} \quad \text{and} \quad \left\{ \begin{aligned} 0 < r &< \tau(u) \\ \tau(u) &< r + t_i \\ 0 &< t_i \end{aligned} \right\}$$

are obviously equivalent, we can rewrite (3.1) as

$$\int_U \int_G \int_0^{\tau(u)} \int_{\tau(u)}^{\infty} \cdots \int_{\tau(u)}^{\infty} \varphi_0(u, h, r) \cdot \bigg(\prod_{i=1}^{k} \varphi_i(u, h, t_i) e^{-\xi_i (t_i - r)} \bigg) \, dt_1 \ \cdots \ dt_k \, dr \, d\omega \, d\nu^u$$

by reparametrizing the domain of integration. Let us focus on the k innermost integrals over $t_1, \ldots, t_k$, which can be broken up into a sum of integrals

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \int_{\tau^{(n_1)}(u)}^{\tau^{(n_1+1)}(u)} \cdots \int_{\tau^{(n_k)}(u)}^{\tau^{(n_k+1)}(u)} \varphi_0(u, h, r) \bigg(\prod_{i=1}^{k} \varphi_i(u, h, t_i) e^{-\xi_i (t_i - r)} \bigg) \, dt_1 \ \cdots \ dt_k$$

over intervals of the form $[\tau^{(n)}(u), \tau^{(n+1)}(u)]$. Of course, we can rewrite this as

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \int_0^{\tau(\sigma^{n_1}(u))} \cdots \int_0^{\tau(\sigma^{n_k}(u))} \bigg[\varphi_0(u, h, r) \cdot \bigg(\prod_{i=1}^{k} \varphi_i(u, h, t_i + \tau^{(n_i)}(u)) e^{-\xi_i (t_i - r + \tau^{(n_i)}(u))} \bigg) \bigg] \, dt_1 \ \cdots \ dt_k$$

by changing variables, replacing t_i with $t_i + \tau^{(n_i)}(u)$. We can rewrite the integrand here as

$$\varphi_0(u, h, r)\bigg(\prod_{i=1}^{k} \varphi_i(\sigma^{n_i}(u), (\mathrm{Hol}^{(n_i)}(u))^{-1} \circ h, t_i)e^{-\xi_i(t_i - r + \tau^{(n_i)}(u))}\bigg) \tag{3.2}$$

using the identifications we made in constructing $U_{\tau,\mathrm{Hol}}$. Evaluating the integral of (3.2) with respect to t_1 through t_k over the intervals $[0, \tau(\sigma^{n_1}(u))]$ through $[0, \tau(\sigma^{n_k}(u))]$ then yields

$$\varphi_0(u, h, r)e^{r\xi}\bigg(\prod_{i=1}^{k} \hat{\varphi}_{i,\xi_i}(\sigma^{n_i}(u), (\mathrm{Hol}^{(n_i)}(u))^{-1} \circ h)e^{-\xi_i\tau^{(n_i)}(u)}\bigg), \tag{3.3}$$

where we set $\xi := \sum_{i=1}^{k} \xi_i$. Similarly, evaluating the integral of (3.3) with respect to r on $[0, \tau(u)]$ yields

$$\hat{\varphi}_{0,-\xi}(u, h)\bigg(\prod_{i=1}^{k} \hat{\varphi}_{i,\xi_i}(\sigma^{n_i}(u), (\mathrm{Hol}^{(n_i)}(u))^{-1} \circ h)e^{-\xi_i\tau^{(n_i)}(u)}\bigg)$$

and so (3.1) becomes

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \int_U \int_G \hat{\varphi}_{0,-\xi}(u, h)\bigg(\prod_{i=1}^{k} \hat{\varphi}_{i,\xi_i}(\sigma^{n_i}(u), (\mathrm{Hol}^{(n_i)}(u))^{-1} \circ h)e^{-\xi_i\tau^{(n_i)}(u)}\bigg)\, d\omega\, d\nu^u \tag{3.4}$$

after interchanging the order of integration and summation.

As ω is bi-invariant, we can replace h with $(\mathrm{Hol}^{(\max n_j)}(u)) \circ h$. Of course, we have the identity $(\mathrm{Hol}^{(n_i)}(u))^{-1} \circ \mathrm{Hol}^{(\max n_j)}(u) = \mathrm{Hol}^{(n_i - (\max n_j))}(\sigma^{n_i}(u))$, and (3.4) becomes

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \int_U \int_G \bigg[\hat{\varphi}_{0,-\xi}(u, \mathrm{Hol}^{(\max n_j)}(u) \circ h) \cdot \bigg(\prod_{i=1}^{k} \hat{\varphi}_{i,\xi_i}(\sigma^{n_i}(u), \mathrm{Hol}^{(n_i - (\max n_j))}(u) \circ h)e^{-\xi_i\tau^{(n_i)}(u)}\bigg)\bigg]\, d\omega\, d\nu^u$$

with this change of variables. This is equivalent to

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \int_G \int_U \bigg[\hat{\varphi}_{0,-\xi}(u, \mathrm{Hol}^{(\max n_j)}(u) \circ h) \cdot \bigg(\prod_{i=1}^{k} \hat{\varphi}_{i,\xi_i}(\sigma^{n_i}(u), \mathrm{Hol}^{(n_i - (\max n_j))}(u) \circ h)e^{-\xi_i\tau^{(n_i)}(u)}\bigg)\bigg]\, d\nu^u\, d\omega$$

by simply reversing the order of integration. Applying $\mathcal{L}_{P(\varsigma),0}$ to the integrand a total of $\max n_j$ times yields

$$\sum_{\sigma^{\max n_j}(u')=u} \bigg[\hat{\varphi}_{0,-\xi}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h) \cdot \bigg(\prod_{i=1}^{k} \hat{\varphi}_{i,\xi_i}(\sigma^{n_i}(u'), \mathrm{Hol}^{(n_i - (\max n_j))}(u') \circ h)e^{-\xi_i\tau^{(n_i)}(u')}\bigg)e^{\alpha_{P(\varsigma)}^{(\max n_j)}(u')}\bigg]$$

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for any given values of $n_1, \ldots, n_k \in \mathbb{N}$. Now, observe that $e^{-\mathrm{Re}(\xi_i)\tau^{(n_i)}(u)}$ is at most $e^{-\mathrm{Re}(\xi_i)\tau^{(\max n_j)}(u)}$ so long as each $\mathrm{Re}(\xi_i)$ is negative. Hence, we can bound the magnitude of the integrand from above by

$$\sum_{\sigma^{\max n_j}(u')=u} |\hat{\varphi}_{0,-\xi}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h)| \bigg(\prod_{i=1}^{k} \|\hat{\varphi}_{i,\xi_i}\|_{C^0} \bigg) e^{-\mathrm{Re}(\xi)\tau^{(\max n_j)}(u') + \alpha_{P(\varsigma)}^{(\max n_j)}(u')}$$

using the triangle inequality. Rearranging this expression, we see that the magnitude of the integrand is bounded above by

$$\bigg(\prod_{i=1}^{k} \|\hat{\varphi}_{i,\xi_i}\|_{C^0} \bigg) \cdot \bigg(\sum_{\sigma^{\max n_j}(u')=u} |\hat{\varphi}_{0,-\xi}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h)| e^{-\mathrm{Re}(\xi)\tau^{(\max n_j)}(u') + \alpha_{P(\varsigma)}^{(\max n_j)}(u')} \bigg), \quad (3.5)$$

which is reminiscent of the expression defining the transfer operator. To make this concrete, recall that we have an $L^2(G)$-invariant decomposition of the function $\hat{\varphi}_{0,-\xi} \in C^1(U, L^2(G))$ in terms of its isotypic components

$$\hat{\varphi}_{0,-\xi}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h) = \sum_{\rho} \hat{\varphi}_{0,-\xi}^{\rho}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h),$$

where the sum is taken over irreducible representations ρ of G, including the trivial representation. Once again, by the triangle inequality, we can bound (3.5) above by

$$\bigg(\prod_{i=1}^{k} \|\hat{\varphi}_{i,\xi_i}\|_{C^0} \bigg) \cdot \bigg(\sum_{\rho} \sum_{\sigma^{\max n_j}(u')=u} |\hat{\varphi}_{0,-\xi}^{\rho}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h)| e^{-\xi\tau^{(\max n_j)}(u') + \alpha_{P(\varsigma)}^{(\max n_j)}(u')} \bigg)$$

where we quickly recognize the transfer operator $\mathcal{L}_{\rho,P(\varsigma)+\mathrm{Re}(\xi)}^{\max n_j}$ applied to $|\hat{\varphi}_{0,-\xi}^{\rho}|$, noting, of course, that

$$|\hat{\varphi}_{0,-\xi}^{\rho}(u', \mathrm{Hol}^{(\max n_j)}(u') \circ h)| = |\rho(\mathrm{Hol}^{(\max n_j)}(u')) \cdot \hat{\varphi}_{0,-\xi}^{\rho}(u', h)|$$

for each ρ. Putting this back together, we see that (3.1) is bounded above by

$$\sum_{\rho} \sum_{n_1,\ldots,n_k=1}^{\infty} \int_U \int_G (\mathcal{L}_{\rho,P(\varsigma)+\mathrm{Re}(\xi)}^{\max n_j} |\hat{\varphi}_{0,-\xi}^{\rho}|)(u, h) \bigg(\prod_{i=1}^{k} \|\hat{\varphi}_{i,\xi_i}\|_{C^0} \bigg) \, d\omega \, d\nu^u$$

in magnitude, as desired. □

We now simply use the bounds in Theorem 3.2 to conclude that the Laplace transform $\hat{\beta}$ in Lemma 3.3 converges.

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THEOREM 3.4. *With conditions as above, there are uniform constants $C > 0$ and $r < 1$ so that*

$$\bigg| \int_{U_{\tau,\mathrm{Hol}}} \varphi_0(u,h,r)\bigg(\prod_{i=1}^{k} \varphi_i(u,h,r+t_i)\bigg)\, d\nu^u\, d\omega\, dr \bigg| \leq C r^{\max t_j} (\|\varphi_0\|_{C^1} \cdot \dots \cdot \|\varphi_k\|_{C^1})$$

for all $\varphi_0, \dots, \varphi_k \in C^1(U_{\tau,\mathrm{Hol}}, \mathbb{C})$ with $\int_{U_{\tau,\mathrm{Hol}}} \varphi_i\, d\nu^u\, d\omega\, dr = 0$.

Proof. We assume that Theorem 3.2 holds, and so we have

$$\|\mathcal{L}_{z,\rho}^n \varphi\|_{L^2(\nu^u)} \leq C\|\varphi\|_{C^1} r^n \tag{3.6}$$

for all non-trivial irreducible ρ, all $\varphi \in C^1(U, V^\rho)$ and for each $z \in \mathbb{C}$ with $|\mathrm{Re}(z) - P(\varsigma)| < 1$. Up to choosing larger values of C and r, we can also assume that the same inequality holds when ρ is trivial: this is precisely the main result of either [11, Theorem 2] or [9, Proposition 3.16], depending on the regularity hypothesis we are using in Theorem A.

We show that the expression bounding $\hat{\beta}(\xi_1, \dots, \xi_k)$ in Lemma 3.3 converges whenever the real parts of ξ_i simultaneously lie in the interval $-1/k < \mathrm{Re}(\xi_i) < 0$. The decay desired follows immediately from applying the inverse Laplace transform with the specific bounds we obtain.

Fix $\xi_1, \dots, \xi_k$ with $-1/k < \mathrm{Re}(\xi_i) < 0$. As before, we consider the function

$$\hat{\varphi}_{i,\xi_i}(u,h) = \int_0^{\tau(u)} \varphi_i(u,h,t) e^{-\xi_i t_i}\, dt_i$$

and decompose $\hat{\varphi}_{0,-\xi}$ into its isotypic components

$$\hat{\varphi}_{0,-\xi} = \sum_\rho \hat{\varphi}_{0,-\xi}^\rho,$$

where $\xi = \xi_1 + \dots + \xi_k$ as before, noting that the decomposition of $L^2(G)$ into irreducible subspaces commutes with the Laplace transform. By (3.6), whenever $-(1/k) < \mathrm{Re}(\xi_i) < 0$, we have

$$\begin{aligned}\int_U \int_G (\mathcal{L}_{\rho,P(\varsigma)+\mathrm{Re}(\xi)}^{\max n_j} |\hat{\varphi}_{0,-\xi}^\rho|)(u,h)\, d\omega\, d\nu^u &\leq \|(\mathcal{L}_{\rho,P(\varsigma)+\mathrm{Re}(\xi)}^{\max n_j} |\hat{\varphi}_{0,-\xi}^\rho|)\|_{L^2(\nu^u)} \\ &\leq C\|\hat{\varphi}_{0,-\xi}^\rho\|_{C^1} r^{\max n_j}\end{aligned}$$

for each ρ. Combining this with Lemma 3.3, we are reduced to ensuring that

$$\begin{aligned}&\sum_\rho \sum_{n_1,\dots,n_k=1}^{\infty} C\|\hat{\varphi}_{0,-\xi}^\rho\|_{C^1} r^{\max n_j} \bigg(\prod_{i=1}^k \|\hat{\varphi}_{i,\xi_i}\|_{C^0}\bigg) \\ &\quad < \sum_\rho \sum_{n=1}^{\infty} k!\, n^{k-1} C\|\hat{\varphi}_{0,-\xi}^\rho\|_{C^1} r^n \bigg(\prod_{i=1}^k \|\hat{\varphi}_{i,\xi_i}\|_{C^0}\bigg)\end{aligned} \tag{3.7}$$

converges. Although this appears promising, note that we can only bound

$$\|\hat{\varphi}_{0,-\xi}^\rho\|_{C^1} \leq D\|\rho\|\Big(\sup_{u \in U} \|\hat{\varphi}_{0,-\xi}^\rho(u,\cdot)\|_{L^2(G)}\Big)$$

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for some constant $D > 0$. To salvage this, we assume for the moment that we have chosen $\varphi_0 \in C^{2N'}(U_{\tau,\mathrm{Hol}}, \mathbb{R})$ for a sufficiently large $N' > 0$ so that we can invoke Theorem 2.11 to bound

$$\|\rho\| \|\hat{\varphi}^{\rho}_{0,-\xi}(u, \cdot)\|_{L^2(G)} \leq \frac{D' \|\hat{\varphi}_{0,-\xi}(u, \cdot)\|_{C^{N'}}}{\|\rho\|^N}$$

pointwise with a fixed constant $D' > 0$, where $N > 0$ is the constant guaranteed by Proposition 2.10. With this, it is clear that the expression on the right-hand side of (3.7) converges absolutely, which says immediately that β_k must decay exponentially fast in $t_1, \ldots, t_k$; this is almost what we wanted to show, but we need an explicit bound on $\hat{\beta}_k$ to obtain uniform estimates with the desired constants.

Integrating the defining expression for $\hat{\varphi}_{i,\xi_i}(u, h)$ by parts, we have

$$\hat{\varphi}_{i,\xi_i}(u, h) = \frac{\varphi_i(u, h, t_i) e^{-\xi_i t_i}}{-\xi_i}\bigg|_0^{\tau(u)} + \frac{1}{\xi_i} \int_0^{\tau(u)} \left(\frac{\partial}{\partial t_i} \varphi_i(u, h, t_i)\right) e^{-\xi_i t_i}\, dt_i$$

for each u and h. Assuming that we choose $\varphi_i \in C^2(U_{\tau,\mathrm{Hol}}, \mathbb{R})$, we can crudely bound

$$\|\hat{\varphi}_{i,\xi_i}\|_{C^1} \leq \frac{D'' \|\varphi_i\|_{C^2}}{1 + |\mathrm{Im}(\xi_i)|}$$

for some constant D'' that depends only on $\|\tau\|_{C^1}$; note that it is essential here that $\mathrm{Re}(\xi_i)$ is confined to a bounded interval. We can similarly bound

$$\|\hat{\varphi}_{0,-\xi}\|_{C^{N'}} \leq \frac{D'' \|\varphi_0\|_{C^{N'+1}}}{1 + |\mathrm{Im}(\xi)|}$$

by choosing a larger value for D'' if necessary. Putting this all together, we have

$$|\hat{\beta}_k(\xi_1, \ldots, \xi_k)| \leq \frac{C' \|\varphi_0\|_{C^{N'+1}}}{1 + |\mathrm{Im}(\xi)|} \left(\prod_{i=1}^{k} \frac{\|\varphi_i\|_{C^2}}{1 + |\mathrm{Im}(\xi_i)|}\right)$$

for all $\xi_i \in \mathbb{C}$ with $-1/k < \mathrm{Re}(\xi_i) < 0$. Now, we simply take the inverse Laplace transform in the variables ξ_1 through ξ_k in succession to obtain

$$|\beta_k(t_1, \ldots, t_k)| \\ \leq \frac{e^{\mathrm{Re}(\xi_1)t_1 + \cdots + \mathrm{Re}(\xi_k)t_k}}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\|\varphi_0\|_{C^{N'+1}}}{1 + |s_1 + \cdots + s_k|} \left(\prod_{i=1}^{k} \frac{\|\varphi_i\|_{C^2}}{1 + |s_i|}\right) ds_1 \ \ldots \ ds_k,$$

where s_i is to be interpreted as the imaginary part of ξ_i. To complete our decay estimate, we simply need to show that the integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{1 + |s_1 + \cdots + s_k|} \left(\prod_{i=1}^{k} \frac{1}{1 + |s_i|}\right) ds_1 \ \ldots \ ds_k$$

converges. Once the convergence of this integral has been established, we will have shown that

$$|\beta_k(t_1, \ldots, t_k)| < C r^{\max t_j} \|\varphi_0\|_{C^{N'+1}} \left(\prod_{i=1}^{k} \|\varphi_i\|_{C^2}\right)$$

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for all $\varphi_0 \in C^{N'+1}(U_{\tau,\mathrm{Hol}}, \mathbb{R})$ and $\varphi_i \in C^2(U_{\tau,\mathrm{Hol}}, \mathbb{R})$. An identical argument to that given in Lemma 2.1 extends this to C^1 functions, using [13, Lemma 2.4] once again. □

We now indicate how to establish the convergence of the integral encountered in the preceding proof.

LEMMA 3.5. *The integral*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{1+|s_1+\cdots+s_k|} \left(\prod_{i=1}^{k} \frac{1}{1+|s_i|} \right) ds_1 \ \ldots \ ds_k$$

converges.

Proof. We show, in fact, that the function defined by the integral

$$f(x) := \int_{-\infty}^{\infty} \frac{1}{(1+|x+y|)^{0.5-\epsilon}} \left(\frac{1}{1+|y|} \right) dy \tag{3.8}$$

decays at a rate of

$$|f(x)| \le \frac{C}{(1+|x|)^{0.5-2\epsilon}}$$

for all sufficiently small $\epsilon > 0$; the statement of the lemma follows immediately by direct successive integration. We work in the case when $x > 0$, and split the domain of integration in (3.8) into regions where $x+y, y < 0$, where $x+y > 0$ but $y < 0$ and where $x+y, y > 0$. In the first case, where $x+y$ and y are both negative, we evaluate

$$\int_{-\infty}^{-x} \frac{1}{(1-(x+y))^{0.5-\epsilon}} \left(\frac{1}{1-y} \right) dy$$

for a fixed $x > 0$. One can verify that the antiderivative of this expression is given by

$$\frac{((1-(x+y))/(1-y))^{0.5-\epsilon}\, {}_2F_1(0.5-\epsilon, 0.5-\epsilon; 1.5-\epsilon; x/(1-y))}{(0.5-\epsilon)(1-(x+y))^{0.5-\epsilon}}$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the principal branch of the analytic continuation of the Gaussian hypergeometric function. Hence, the integral evaluates to

$$\int_{-\infty}^{-x} \frac{1}{(1-(x+y))^{0.5-\epsilon}} \left(\frac{1}{1-y} \right) dy = \frac{{}_2F_1(0.5-\epsilon, 0.5-\epsilon; 1.5-\epsilon; x/(1+x))}{(0.5-\epsilon)(1+x)^{0.5-\epsilon}},$$

which can be bounded above by $C/(1+x)^{0.5-\epsilon}$ for an appropriate choice of constant $C > 0$, because by [1, 15.1.1] the defining series for ${}_2F_1(a, b; c; z)$ converges on the unit disc in the complex plane when $c-(a+b) > 0$. Similarly, on the region where $x+y$ is positive and y is negative, we evaluate

$$\int_{-x}^{0} \frac{1}{(1+(x+y))^{0.5-\epsilon}} \left(\frac{1}{1-y} \right) dy$$
$$= \left(\frac{(1+(x+y))^{0.5+\epsilon}\, {}_2F_1(0.5+\epsilon, 1; 1.5+\epsilon; (1+(x+y))/(2+x))}{(0.5+\epsilon)(2+x)} \right) \Bigg|_{y=-x}^{y=0}$$

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$$= \frac{(1+x)^{0.5+\epsilon}{}_2F_1(0.5+\epsilon, 1; 1.5+\epsilon; (1+x)/(2+x))}{(0.5+\epsilon)(2+x)} - \frac{{}_2F_1(0.5+\epsilon, 1; 1.5+\epsilon; 1/(2+x))}{(0.5+\epsilon)(2+x)},$$

which can once again be bounded above by $C/(1+x)^{0.5-\epsilon}$ for the same reasons, though with a possibly larger choice of $C > 0$. Finally, when both $x + y$ and y are positive, we evaluate

$$\int_0^\infty \frac{1}{(1+(x+y))^{0.5-\epsilon}} \left(\frac{1}{1+y}\right) dy$$
$$= \left(\frac{((1+(x+y))/(1+y))^{0.5-\epsilon}{}_2F_1(0.5-\epsilon, 0.5-\epsilon; 1.5-\epsilon; -x/(1+y))}{(0.5-\epsilon)(1+(x+y))^{0.5-\epsilon}} \right)\Bigg|_{y=0}^{y=\infty}$$
$$= -\frac{{}_2F_1(0.5-\epsilon, 0.5-\epsilon; 1.5-\epsilon; -x)}{0.5-\epsilon}$$

and we simply need to understand the asymptotics of ${}_2F_1(0.5-\epsilon, 0.5-\epsilon; 1.5-\epsilon; -x)$ as $x \to \infty$. By [1, 15.3.1], we have an integral representation given by

$$ {}_2F_1(0.5-\epsilon, 0.5-\epsilon; 1.5-\epsilon; -x) = \frac{\Gamma(1.5-\epsilon)}{\Gamma(0.5-\epsilon)\Gamma(1)} \int_0^1 \frac{1}{t^{0.5+\epsilon}(1+tx)^{0.5-\epsilon}}\, dt \tag{3.9}$$

for $x > 1$. Setting $u = tx$, we can rewrite

$$\int_0^1 \frac{1}{t^{0.5+\epsilon}(1+tx)^{0.5-\epsilon}}\, dt$$
$$= \frac{1}{x^{0.5-\epsilon}} \int_0^x \frac{1}{u^{0.5+\epsilon}(1+u)^{0.5-\epsilon}}\, du$$
$$= \frac{1}{x^{0.5-\epsilon}} \left(\int_0^1 \frac{1}{u^{0.5+\epsilon}(1+u)^{0.5-\epsilon}}\, du + \int_1^x \frac{1}{u^{0.5+\epsilon}(1+u)^{0.5-\epsilon}}\, du \right),$$

at which point we note that the expression in parentheses can be bounded above by $C(1 + \log x)$. Hence, (3.9) can be bounded above by $C/(1+x)^{0.5-2\epsilon}$ with a possibly larger choice of C, as desired. The proof in the case when $x < 0$ is identical. □

4. *Uniform local non-integrability estimates*

In this section, we use the local accessibility of f_t to establish the uniform local non-integrability estimates necessary to prove Theorem 3.2, drawing on techniques introduced by Dolgopyat in [12] for group extensions of expanding maps. These arguments require some additional care to adapt to our setting, the principal difficulty being the non-triviality of the fibre bundle $\pi : M \to N$ which requires us to choose trivializations to construct the infinitesimal transitivity group as in [12].

We want to translate the local accessibility of f_t into an infinitesimal statement on the Markov model we constructed in §2; we accomplish this in two main steps. The first step is to define a subalgebra of the Lie algebra $\mathfrak{g}$ of G that measures the 'non-integrability' of the

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fibre bundle over the weak stable and strong unstable foliations; this can be accomplished before making any reference to our symbolic model. The second step is to translate this into the symbolic model.

For most of what follows, we need to be careful to specify which chart V_* of the trivialization we are working with at any given point. This is a necessary complication to many of our arguments, because many of the objects we are working with are highly sensitive to the choice of trivialization. Fortunately, however, this will also afford us with the flexibility later on to work with trivializations that are specially adapted to our needs.

To start, we want to measure and relate three different holonomies associated with f_t: namely, the holonomies induced by the leaves of the strong stable foliation, the leaves of the strong unstable foliation and the flow.

Definition 4.1. Fix $x, y \in N$ with $y \in W^{su}_{g_t}(x)$, along with trivializations ϕ_x, ϕ_y of $\pi : M \to N$ at x and y corresponding to subsets $V_x, V_y \subset N$, respectively. We define the *unstable holonomy*

$$\Theta^{+}_{V_x,V_y}(x, y)\colon F \to F$$

between x and y to be the isometry induced by the map $\pi^{-1}(x) \to \pi^{-1}(y)$ that takes $a \in \pi^{-1}(x)$ to the (necessarily unique) point $b \in \pi^{-1}(y) \cap W^{su}_{f_t}(a)$. The identifications of $\pi^{-1}(x)$ and $\pi^{-1}(y)$ with F are obtained via the trivializations ϕ_x, ϕ_y. The *stable holonomy* $\Theta^{-}_{V_x,V_y}(x, y)$ is defined analogously for $y \in W^{ss}_{g_t}(x)$.

Definition 4.2. Fix $x, y \in N$ with $g_t(x) = y$, along with trivializations ϕ_x, ϕ_y of $\pi : M \to N$ at x and y corresponding to subsets $V_x, V_y \subset N$, respectively. We define the *temporal holonomy*

$$\mathrm{Hol}^{\phi_y}_{\phi_x}(x, y, t)\colon F \to F$$

between x and y to be the isometry induced by the map $\pi^{-1}(x) \to \pi^{-1}(y)$ that takes $a \in \pi^{-1}(x)$ to $f_t(a) \in \pi^{-1}(y)$. The identifications of $\pi^{-1}(x)$ and $\pi^{-1}(y)$ with F are obtained via the trivializations ϕ_x, ϕ_y.

By the end of this section, we only need to work with a fixed, finite collection of trivializations that cover N. At this stage, however, the flexibility in these definitions is crucial. Our first observation is that the unstable holonomy can be expressed in terms of the temporal holonomies induced by the flow; this is made precise in the following proposition, the proof of which is largely summarized in Figure 1.

PROPOSITION 4.1. *Fix $x, y \in N$ with $y \in W^{su}_{g_t}(x)$, along with trivializations $\phi_{0,x}$ and $\phi_{0,y}$ defined at x and y, respectively. Let $T = (t_n)$ be a monotonic sequence of times with $t_n = 0$ and $t_n \to -\infty$, and let $I_x = (\phi_{n,x})$ and $I_y = (\phi_{n,y})$ be sequences of trivializations for which $\phi_{k,x} = \phi_{k,y}$ for all $k \geq N$. Then we can write*

$$\Theta^{+}_{\phi_{0,x},\phi_{0,y}}(x, y) = \lim_{n\to\infty} \mathrm{Hol}^{(n)}_{I_y,T}(y)(\mathrm{Hol}^{(n)}_{I_x,T}(x))^{-1}, \tag{4.1}$$

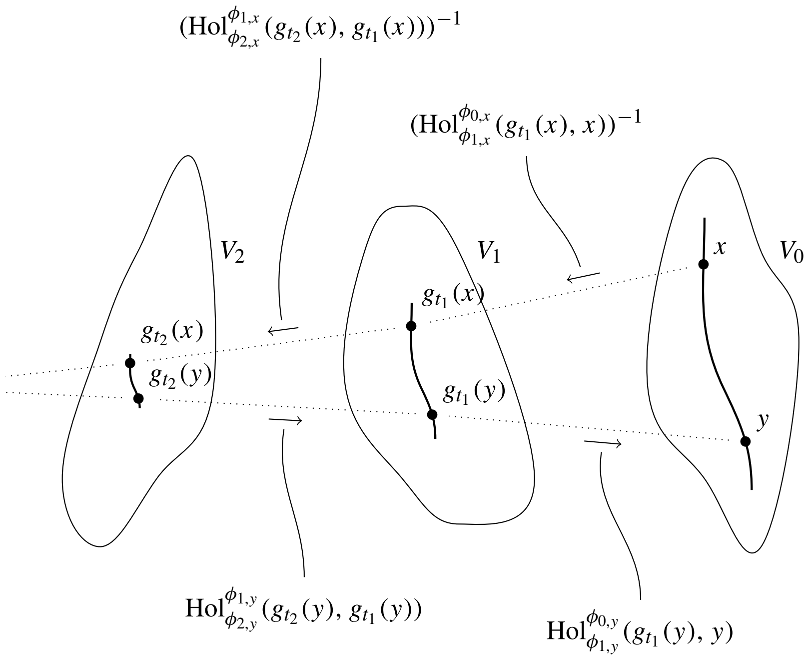

FIGURE 1. Measuring the unstable holonomy between x and y along a sequence of times $0 > t_1 > t_2 > \ldots$ with respect to trivializations $(\phi_{n,x})$ and $(\phi_{n,y})$ defined over charts V_n, illustrated in the case when the trivializations for x and y coincide. As the unstable leaf through x and y contracts under g_{t_n}, the remaining contribution to the unstable holonomy decreases.

where

$$\mathrm{Hol}^{(n)}_{I_*,T}(*) := \mathrm{Hol}_{\phi_{n,*}}^{\phi_{n-1,*}}(g_{t_n}(*), g_{t_{n-1}}(*), t_{n-1} - t_n) \circ \cdots \circ \mathrm{Hol}_{\phi_{1,*}}^{\phi_{0,*}}(g_{t_1}(*), *, -t_1)$$

is the n-step temporal holonomy measured with respect to the trivializations $I_.$ at times given by T.

Proof. The convergence of the limit is simply a consequence of the fact that $d(g_{t_n}(x), g_{t_n}(y)) \to 0$ as $n \to \infty$. More precisely, we can rewrite

$$\mathrm{Hol}^{(n+1)}_{I_y,T}(y)(\mathrm{Hol}^{(n+1)}_{I_x,T}(x))^{-1}$$

as

$$\begin{aligned}\mathrm{Hol}_{\phi_{n+1,y}}^{\phi_{n,y}}&(g_{t_{n+1}}(y), g_{t_n}(y), t_n - t_{n+1})\\ &\circ\ (\mathrm{Hol}^{(n)}_{I_y,T}(y)(\mathrm{Hol}^{(n)}_{I_x,T}(x))^{-1})\\ &\quad\circ (\mathrm{Hol}_{\phi_{n+1,x}}^{\phi_{n,x}}(g_{t_{n+1}}(x), g_{t_n}(x), t_n - t_{n+1}))^{-1},\end{aligned}$$

and because f_t is C^1, we see that $\mathrm{Hol}_{\phi_{n+1,x}}^{\phi_{n,x}}(g_{t_{n+1}}(x), g_{t_n}(x))$ must also be locally C^1 in $g_{t_{n+1}}(x)$. Since $d_N(g_{t_{n+1}}(y), g_{t_{n+1}}(x))$ decay exponentially fast as $n \to \infty$ and the

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trivializations I_x and I_y eventually agree, we see that

$$d_G(\mathrm{Hol}^{\phi_{n,y}}_{\phi_{n+1,y}}(g_{t_{n+1}}(y), g_{t_n}(y), t_n - t_{n+1}), \mathrm{Hol}^{\phi_{n,x}}_{\phi_{n+1,x}}(g_{t_{n+1}}(x), g_{t_n}(x), t_n - t_{n+1}))$$

must also decay exponentially fast. In particular, for any $h \in G$, this means that

$$d_G(\mathrm{Hol}^{\phi_{n,y}}_{\phi_{n+1,y}}(g_{t_{n+1}}(y), g_{t_n}(y), t_n - t_{n+1}) \circ h \\ \circ (\mathrm{Hol}^{\phi_{n,x}}_{\phi_{n+1,x}}(g_{t_{n+1}}(x), g_{t_n}(x), t_n - t_{n+1}))^{-1}, h)$$

decays exponentially fast and, hence,

$$d_G(\mathrm{Hol}^{(n+1)}_{I_y,T}(y)(\mathrm{Hol}^{(n+1)}_{I_x,T}(x))^{-1}, \mathrm{Hol}^{(n)}_{I_y,T}(y)(\mathrm{Hol}^{(n)}_{I_x,T}(x))^{-1})$$

decays exponentially fast as $n \to \infty$. As G is complete, the limit must exist. A similar argument shows that this limit is, in fact, equal to $\Theta^+_{\phi_{0,x},\phi_{0,y}}(x, y)$: because the unstable foliation of f_t is invariant under the flow, we can rewrite

$$\Theta^+_{\phi_{0,x},\phi_{0,y}}(x, y)$$

as

$$\mathrm{Hol}^{(n)}_{I_y,T}(y) \circ \Theta^+_{\phi_{n,x},\phi_{n,y}}(g_{t_n}(x), g_{t_n}(y)) \circ (\mathrm{Hol}^{(n)}_{I_x,T}(x))^{-1} \tag{4.2}$$

for any $n > 0$ and any sequences I_x, I_y and T as previously. As $t_n \to -\infty$, $d_N(g_{t_n}(x), g_{t_n}(y))$ decreases exponentially fast, and so $\Theta^+_{\phi_{n,x},\phi_{n,y}}(g_{t_n}(x), g_{t_n}(y))$ converges to the identity in G. Of course, this means that, as $n \to \infty$, (4.2) converges to the limit in (4.1). As the expression in (4.2) is constant at $\Theta^+_{\phi_{0,x},\phi_{0,y}}(x, y)$, this proves the proposition. □

We need to understand the infinitesimal behaviour of the stable and unstable foliations: rather than working with the unstable holonomy as defined, we instead consider its derivative along a leaf of the unstable foliation.

PROPOSITION 4.2. *The unstable holonomy $\Theta^+_{\phi_1,\phi_2}(x, y)$ is simultaneously C^1 in x and y, as x and y vary in a fixed leaf of the strong unstable foliation of g_t, and within charts associated with fixed C^1 trivializations ϕ_1 and ϕ_2.*

Proof. This follows immediately from our hypotheses on the regularity of f_t, which imply that the leaves of the strong unstable foliation of f_t are C^1. □

Definition 4.3. Fix $x \in N$, a trivialization ϕ defined near x and a vector $w \in T^1_x W^{su}_{g_t}(x)$. We define the *infinitesimal holonomy* at x in the direction of w to be the element

$$X^\phi_w(x) := \bigg(\frac{d}{du}\bigg|_{u=x}(\Theta^+_{V_x,V_x}(x, u))\bigg)(w)$$

of the Lie algebra $\mathfrak{g}$ of G. Let $\epsilon > 0$ be small enough that ϕ is defined over $B_\epsilon(x)$. The ϵ*-infinitesimal transitivity group* at x is defined to be the linear span

$$\mathfrak{h}^\phi_\epsilon(x) := \operatorname*{span}_{y,w}(X_{w'}(y) - X_w(x))$$

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taken over all $y \in W^{ss}_\epsilon(x)$ and $w \in T^1_x W^{su}_{g_t}(x)$. Here, w' denotes the pushforward of w to $T^1_y W^{su}_{g_t}(y)$ along the leaves of the centre stable foliation of g_t.

We soon verify that $\mathfrak{h}^\phi(x)$ is largely independent of the choice of trivialization ϕ, but it is worth making a few comments first.

Remark 4.3. Under our hypotheses, the foliation $W^{ws}_{g_t}$ is C^1, and so the holonomy it induces between the leaves of the foliation $W^{su}_{g_t}$ is also C^1. This is necessary for the pushforward of $w \in T^1_y W^{su}_{g_t}(y)$ in Definition 4.3 to make sense.

Remark 4.4. It is necessary to consider the *relative* infinitesimal holonomy, as we did in Definition 4.3. As we show in the course of proving Proposition 4.6, the vector $X^\phi_w(x)$ in Definition 4.3 is extremely sensitive to the choice of trivialization ϕ. For instance, it is certainly possible for $X^\phi_w(x)$ to be zero for all $w \in T^1_x W^{su}_{g_t}(x)$ if the trivialization ϕ is built to be constant along the leaves of the strong unstable foliation, and the existence of such trivializations will be extremely helpful in the course of proving Theorem 4.11.

Remark 4.5. The vectors $X^\phi_w(x)$ and $X^\phi_{w'}(y)$ vary continuously in x and w, by Proposition 4.2. However, because $\mathfrak{h}_\epsilon(x)$ is defined as the linear span of continuously varying vectors, it is only lower semi-continuous. In particular, there can be singular sets where the dimension of $\mathfrak{h}_\epsilon(x)$ jumps down.

It turns out that $\mathfrak{h}^\phi_\epsilon(x)$ will typically be independent of ϵ, though we do not prove this directly. We show instead that, if f_t is locally G-accessible, then $\mathfrak{h}_\epsilon(x)$ is generically equal to $\mathfrak{g}$. For most of what follows, we treat $\epsilon > 0$ as a fixed constant with no particular restrictions. Our first important calculation is that the conjugacy class of $\mathfrak{h}^\phi_\epsilon(x)$ does not depend on the trivialization ϕ, if the trivializations are chosen appropriately.

PROPOSITION 4.6. *Fix $\epsilon > 0$, $x \in N$ and trivializations $\phi_i \colon \pi^{-1}(V_i) \to V_i \times F$ for $i = 1, 2$. If $B_\epsilon(x) \subset V_i$, then*

$$\mathfrak{h}^{\phi_2}_\epsilon(x) = \mathrm{Ad}_{\mathrm{id}^{\phi_2}_{\phi_1}(x)}(\mathfrak{h}^{\phi_1}_\epsilon(x))$$

so long as ϕ_1 and ϕ_2 have constant projection to F along each leaf of the strong stable foliation of f_t and each flowline of f_t.

Here, $\mathrm{id}^{\phi_2}_{\phi_1}(x) \colon F \to F$ is used to denote the isometry induced by the identity map $\pi^{-1}(x) \to \pi^{-1}(x)$ with the domain and target identified with F via ϕ_1 and ϕ_2, respectively.

Proof. We can relate the unstable holonomies between x and $u \in W^{su}_{g_t}(x)$ with respect to ϕ_1 and ϕ_2 by

$$\Theta^+_{\phi_2,\phi_2}(x, u) = \mathrm{id}^{\phi_2}_{\phi_1}(u) \circ \Theta^+_{\phi_1,\phi_1}(x, u) \circ (\mathrm{id}^{\phi_2}_{\phi_1}(x))^{-1} \tag{4.3}$$

by definition. We now simply take the derivative of each side of (4.3) with respect to u at $u = x$; in the notation of Definition 4.3, this becomes

$$X^{\phi_2}_w(x) = \mathrm{Ad}_{\mathrm{id}^{\phi_2}_{\phi_1}(x)}(X^{\phi_1}_w(x)) + ((dR)_{(\mathrm{id}^{\phi_2}_{\phi_1}(x))^{-1}} \circ d(\mathrm{id}^{\phi_2}_{\phi_1})_x)(w) \tag{4.4}$$

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for any $w \in T_x^1 W_{g_t}^{su}(x)$, where dR denotes the derivative of right multiplication in G. Given any $y \in W_\epsilon^{ss}(x)$ and w' corresponding to w as in Definition 4.3, exactly the same calculation yields

$$X_{w'}^{\phi_2}(y) = \mathrm{Ad}_{\mathrm{id}_{\phi_1}^{\phi_2}(y)}(X_{w'}^{\phi_1}(y)) + ((dR)_{(\mathrm{id}_{\phi_1}^{\phi_2}(y))^{-1}} \circ d(\mathrm{id}_{\phi_1}^{\phi_2})_y)(w'), \tag{4.5}$$

assuming, of course, that y is sufficiently close to x that we are able to use the same trivializations ϕ_1, ϕ_2. As both trivializations are constant along the strong stable foliation of f_t and we chose $y \in W_\epsilon^{ss}(x)$, we clearly have $\mathrm{id}_{\phi_1}^{\phi_2}(x) = \mathrm{id}_{\phi_1}^{\phi_2}(y)$ and, hence,

$$(dR)_{(\mathrm{id}_{\phi_1}^{\phi_2}(x))^{-1}} = (dR)_{(\mathrm{id}_{\phi_1}^{\phi_2}(y))^{-1}}$$

as functions $T^1 G \to T^1 G$. Moreover, because the trivializations are also constant along the flowlines of f_t, we see that $\mathrm{id}_{\phi_1}^{\phi_2}$ must be constant along the leaves of the centre stable foliation of g_t. Hence, we must have

$$(d(\mathrm{id}_{\phi_1}^{\phi_2})_x)(w) = (d(\mathrm{id}_{\phi_1}^{\phi_2})_y)(w')$$

for all $w \in T_x^1 W_{g_t}^{su}(x)$. Subtracting (4.4) from (4.5) and using the fact that $\mathrm{id}_{\phi_1}^{\phi_2}(x) = \mathrm{id}_{\phi_1}^{\phi_2}(y)$, we then obtain

$$X_{w'}^{\phi_2}(y) - X_w^{\phi_2}(x) = \mathrm{Ad}_{\mathrm{id}_{\phi_1}^{\phi_2}(y)}(X_{w'}^{\phi_1}(y)) - \mathrm{Ad}_{\mathrm{id}_{\phi_1}^{\phi_2}(x)}(X_w^{\phi_1}(x))$$

as desired. □

There is an analogous relation between the ϵ-infinitesimal transitivity groups at points along a flowline of g_t, though the expansion of the unstable leaves prevent us from obtain an equality in this case.

PROPOSITION 4.7. *Fix $\epsilon > 0$, $x \in N$ and $t > 0$. Let ϕ_x and $\phi_{g_t(x)}$ be trivializations near x and $g_t(x)$ for which $B_\epsilon(x) \subset V_x$ and $B_\epsilon(g_t(x)) \subset V_{g_t(x)}$, and write h for the temporal holonomy*

$$h(x) := \mathrm{Hol}_{\phi_x}^{\phi_{g_t(x)}}(x, g_t(x), t)$$

measured with respect to ϕ_x and $\phi_{g_t}(x)$. We then have

$$\mathrm{Ad}_{h(x)}(\mathfrak{h}_\epsilon^{\phi_x}(x)) \subset \mathfrak{h}_\epsilon^{\phi_{g_t(x)}}(g_t(x))$$

so long as ϕ_x and $\phi_{g_t(x)}$ have constant projection to F along each leaf of the strong stable foliation of f_t and each flowline of f_t.

Proof. By Proposition 4.1, we have

$$\Theta_{\phi_2,\phi_2}^{+}(g_t(x), g_t(u)) = h(u) \circ \Theta_{\phi_1,\phi_1}^{+}(x, u) \circ (h(x))^{-1}$$

so long as u is sufficiently close to x. Noting the resemblance to (4.3), simply repeating our calculations in Proposition 4.6 yields

$$X_{w'}^{\phi_{g_t(x)}}(g_t(y)) - X_w^{\phi_{g_t(x)}}(g_t(x)) = \mathrm{Ad}_{h(x)}(X_{w'}^{\phi_x}(y)) - \mathrm{Ad}_{h(x)}(X_w^{\phi_x}(x))$$

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for all $y \in W^{ss}_\epsilon(x)$ and all $w \in T^1_x W^{su}_{g_t}(x)$. This completes the proof; note that we do not obtain equality this time because the strong stable leaves for g_t contract, and there will be $y' \in W^{ss}_\epsilon(g_t(x))$ that are not of the form $g_t(y)$ for $y \in W^{ss}_\epsilon(x)$. □

Note that Proposition 4.7 only yields an inclusion of the ϵ-infinitesimal transitivity groups, and only in forward time. Our goal is to show that $\mathfrak{h}^\phi_\epsilon(x)$ is exactly $\mathfrak{g}$ at every $x \in N$; unfortunately, the proof of Proposition 4.7 suggests that even the dimension of $\mathfrak{h}_\epsilon(x)$ may fail to be constant in general. Fortunately, given the topological transitivity of g_t, what we have proven so far is enough to show that the dimension is constant on a large set.

In light of Proposition 4.6, we can be somewhat cavalier in specifying the trivialization ϕ used in defining $\mathfrak{h}^\phi_\epsilon(x)$, if we are solely concerned with the dimension and restrict our attention to trivializations that satisfy the hypotheses of the proposition. We henceforth always assume that every trivialization we work with has constant projection to F along the strong stable leaves and flowlines of f_t.

COROLLARY 4.8. *Fix a collection of trivializations $\phi_1, \ldots, \phi_k$ defined over a cover $V_1, \ldots, V_k$ of N, and let $\epsilon > 0$ be the Lebesgue number of this cover. Then* $\dim \mathfrak{h}^*_\epsilon(\cdot)$ *attains its maximum value on an open, dense subset of full measure.*

Proof. Let $x \in N$ be a point at which $\mathfrak{h}^*_\epsilon(x)$ has maximal dimension. As $\mathfrak{h}^*_\epsilon(\cdot)$ is lower semi-continuous, it has maximal dimension on an open neighbourhood W containing x. By Proposition 4.7, $\mathfrak{h}^*_\epsilon(\cdot)$ therefore has maximal dimension on an open set containing the forward orbit of g_t. This is a dense set if g_t is topologically transitive.

As g_t is ergodic and $\dim \mathfrak{h}^*_\epsilon(\cdot)$ is measurable, it must be constant almost everywhere. The measure ν is an equilibrium measure with a Hölder potential and therefore has full support; hence, the open and dense set on which $\dim \mathfrak{h}^*_\epsilon(\cdot)$ has maximal dimension must also have full measure. □

Our next objective is to relate the ϵ-infinitesimal transitivity groups between points along a leaf of the strong unstable foliation of g_t. If we indeed had equality in Proposition 4.7, this would be a relatively straightforward application of Proposition 4.1. The lack of equality makes such an approach impossible, but we can still argue as in Corollary 4.8.

LEMMA 4.9. *Fix $\epsilon > 0$, $x \in N$ and $y \in W^{su}_{g_t}(x)$, along with trivializations ϕ_x and ϕ_y for which we have $B_{2\epsilon}(x) \subset V_x$ and $B_\epsilon(y) \subset V_y$. If x is backwards-recurrent under g_t and* $\dim \mathfrak{h}^*_\epsilon(x)$ *is maximal, then*

$$\mathfrak{h}^{\phi_y}_\epsilon(y) = \mathrm{Ad}_{\Theta^+_{\phi_x,\phi_y}(x,y)}(\mathfrak{h}^{\phi_x}_\epsilon(x))$$

and, in particular, $\dim \mathfrak{h}^*_\epsilon(\cdot)$ *is constant on $W^{su}_{g_t}(x)$.*

Proof. As $\dim \mathfrak{h}^*_\epsilon(\cdot)$ is lower semi-continuous, there is an open set $W \subset B_\epsilon(x)$ on which it is maximal. Because x is backwards recurrent, we can find a monotonic sequence of times $T = \{t_n\}$ with $t_n \to -\infty$ for which $g_{t_n}(x) \in W$ for each $n > 0$. Moreover, we can suppose that t_1 is large enough that we also have $g_{t_n}(y) \in W$ for each $n > 0$. Now, write

$$h^{(n)}(x) := \mathrm{Hol}^{\phi_x}_{\phi_x}(g_{t_n}(x), g_{t_{n-1}}(x), t_{n-1} - t_n) \circ \cdots \circ \mathrm{Hol}^{\phi_x}_{\phi_x}(g_{t_1}(x), x, -t_1)$$

and

$$h^{(n)}(y) := \mathrm{Hol}^{\phi_x}_{\phi_x}(g_{t_n}(y), g_{t_{n-1}}(y), t_{n-1} - t_n) \circ \cdots \circ \mathrm{Hol}^{\phi_y}_{\phi_x}(g_{t_1}(y), y, -t_1)$$

for the n-step holonomies at $g_{t_n}(x)$ and $g_{t_n}(y)$. As we have $g_{t_n}(x), g_{t_n}(y) \in W$ by construction, we have

$$\mathrm{Ad}_{h^{(n)}(x)}(\mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(x))) = \mathfrak{h}^{\phi_x}_{\epsilon}(x)$$

and

$$\mathrm{Ad}_{h^{(n)}(y)}(\mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(y))) = \mathfrak{h}^{\phi_y}_{\epsilon}(y)$$

by Proposition 4.7; note that we have implicitly used the fact that $B_{2\epsilon}(x) \subset V_x$ in writing $\mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(x))$ and $\mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(y))$, where V_x is the chart over which ϕ_x is defined. By rearranging these equations, we see that

$$d_{\mathrm{Gr}(\dim \mathfrak{h}^*_{\epsilon}(x), \mathfrak{g})}(\mathfrak{h}^{\phi_y}_{\epsilon}(y), \mathrm{Ad}_{h^{(n)}(y)(h^{(n)}(x))^{-1}}(\mathfrak{h}^{\phi_x}_{\epsilon}(x)))$$

is equal to

$$d_{\mathrm{Gr}(\dim \mathfrak{h}^*_{\epsilon}(x), \mathfrak{g})}(\mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(y)), \mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(x))),$$

where the distances are measured in the standard metric on the Grassmannian of $(\dim \mathfrak{h}^*_{\epsilon}(x))$-dimensional subspaces of $\mathfrak{g}$. As $\mathfrak{h}^{\phi_x}_{\epsilon}(\cdot)$ is lower semi-continuous and has maximal dimension on W, it must be continuous on W. Up to passage to the interior of a compact subset of W, we can assume that $\mathfrak{h}_{\epsilon}(\cdot)$ is uniformly continuous on W. Because $d_N(g_{t_n}(y), g_{t_n}(x)) \to 0$ as $n \to \infty$, we must then have

$$d_{\mathrm{Gr}(\dim \mathfrak{h}^*_{\epsilon}(x), \mathfrak{g})}(\mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(y)), \mathfrak{h}^{\phi_x}_{\epsilon}(g_{t_n}(x)))$$

as $n \to \infty$. This yields

$$\mathfrak{h}^{\phi_y}_{\epsilon}(y) = \lim_{n\to\infty} \mathrm{Ad}_{h^{(n)}(y)(h^{(n)}(x))^{-1}}(\mathfrak{h}^{\phi_x}_{\epsilon}(x)),$$

at which point we simply observe that $\mathrm{Ad}_g(\cdot)$ is continuous in g and that $h^{(n)}(y)(h^{(n)}(x))^{-1}$ converges to $\Theta^{+}_{\phi_x,\phi_y}(x, y)$ by Proposition 4.1. □

In addition to the preceding lemma, we require its analogue for the stable holonomies. The proof is identical, and we do not repeat it.

LEMMA 4.10. *Fix $\epsilon > 0$, $x \in N$ and $y \in W^{ss}_{g_t}(x)$, along with trivializations ϕ_x and ϕ_y for which we have $B_{2\epsilon}(x) \subset V_x$ and $B_{\epsilon}(y) \subset V_y$. If x is forwards-recurrent under g_t and* $\dim \mathfrak{h}^*_{\epsilon}(x)$ *is maximal, then*

$$\mathfrak{h}^{\phi_y}_{\epsilon}(y) = \mathrm{Ad}_{\Theta^{-}_{\phi_x,\phi_y}(x,y)}(\mathfrak{h}^{\phi_x}_{\epsilon}(x))$$

and, in particular, $\dim \mathfrak{h}^*_{\epsilon}(\cdot)$ *is constant on $W^{ss}_{g_t}(x)$.*

Now that we have Lemmas 4.9 and 4.10 to connect the ϵ-infinitesimal transitivity groups to the unstable and stable holonomies, respectively, we can achieve the first major goal of this section: translating the local accessibility of f_t into an infinitesimal statement about $\mathfrak{h}_\epsilon^*(\cdot)$. To begin, we show that if f_t is G-accessible, then $\mathfrak{h}_\epsilon^\phi(x)$ must be Ad_G-invariant for any bi-recurrent $x \in N$.

THEOREM 4.11. *Fix $\epsilon > 0$, $x \in N$ and a trivialization ϕ_x for which $B_\epsilon(x) \subset V_x$. Suppose that 2ϵ is smaller than the Lebesgue number of a finite cover $\{V_i\}$ of N corresponding to trivializations $\{\phi_i\}$. If x is bi-recurrent under g_t, $\dim \mathfrak{h}_\epsilon^{\phi_x}(x)$ is maximal and f_t is G-accessible, then $\mathfrak{h}_\epsilon^{\phi_x}(x)$ is Ad_G-invariant.*

Proof. Suppose that x is bi-recurrent. Fix an isometry $g \in G$ and consider a stable–unstable sequence $x_0, x_1, \ldots, x_k, x_{k+1} = x_0$ in N with $x_0 = x$ and whose total holonomy is g, where x_{i+1} is either on the strong stable or strong unstable leaf through x_i for g_t. As g is the total holonomy, we can write g as the composite

$$\mathrm{id}_{\phi_{x_k}}^{\phi_{x_0}}(x_0) \circ \Theta^{\pm}_{\phi_{x_k},\phi_{x_k}}(x_k, x_0) \circ \ldots \mathrm{id}_{\phi_{x_1}}^{\phi_{x_2}}(x_2) \circ \Theta^{\pm}_{\phi_{x_1},\phi_{x_1}}(x_1, x_2) \circ \mathrm{id}_{\phi_{x_0}}^{\phi_{x_1}}(x_1) \circ \Theta^{\pm}_{\phi_{x_0},\phi_{x_0}}(x_0, x_1),$$

where we can freely assume that each consecutive pair x_i, x_{i+1} has a common trivialization ϕ_{x_i} for which $B_\epsilon(x_i), B_\epsilon(x_{i+1}) \subset V_{x_i}$; this is true up to refining the sequence (Figure 2). Suppose, moreover, that we have chosen $\phi_{x_0} = \phi_x$.

We would like to now invoke Proposition 4.6 and Lemmas 4.9 and 4.10 to show that $\mathfrak{h}_\epsilon^{\phi_x}(x)$ is Ad_g-invariant for the g corresponding to the total holonomy along this sequence. The x_i we have chosen, however, may fail to be forwards- or backwards-recurrent as necessary. However, note that $\mathrm{id}_{\phi_{x_i}}^{\phi_{x_{i+1}}}(\cdot)$, $\Theta^{+}_{\phi_{x_i},\phi_{x_i}}(\cdot,\cdot)$ and $\Theta^{-}_{\phi_{x_i},\phi_{x_i}}(\cdot,\cdot)$ are all locally continuous in all of their arguments. As bi-recurrent points are dense in N, given any $\delta > 0$, we can find a sequence of bi-recurrent points $x_0', x_1', \ldots, x_k', x_{k+1}' = x_0'$ near $x_0, x_1, \ldots, x_k, x_{k+1} = x_0$ with $x_0' = x$ whose total holonomy g' is given by

$$\mathrm{id}_{\phi_{x_k}}^{\phi_{x_0}}(x_0') \circ \Theta^{\pm}_{\phi_{x_k},\phi_{x_k}}(x_k', x_0') \circ \cdots \circ \mathrm{id}_{\phi_{x_1}}^{\phi_{x_2}}(x_2') \circ \Theta^{\pm}_{\phi_{x_1},\phi_{x_1}}(x_1', x_2') \circ \mathrm{id}_{\phi_{x_0}}^{\phi_{x_1}}(x_1') \circ \Theta^{\pm}_{\phi_{x_0},\phi_{x_0}}(x_0', x_1')$$

so that we have $d_G(g', g) < \delta$. As each x_i' is bi-recurrent, successive applications of Proposition 4.6 and Lemmas 4.9 and 4.10 show that we have

$$\mathfrak{h}_\epsilon^{\phi_x}(x) = \mathrm{Ad}_{g'}(\mathfrak{h}_\epsilon^{\phi_x}(x))$$

for g' arbitrarily close to G. As $\mathrm{Ad}_{g'}$ is continuous in g', we then obtain

$$\mathfrak{h}_\epsilon^{\phi_x}(x) = \mathrm{Ad}_{g}(\mathfrak{h}_\epsilon^{\phi_x}(x))$$

as desired. □

It is worth remarking that, under our standing assumption that trivializations must have constant projection to F along strong stable leaves of f_t, the stable holonomies $\Theta^{-}_{\phi_{x_i},\phi_{x_i}}(x_i, x_{i+1})$ that appeared in the preceding proof must all be trivial.

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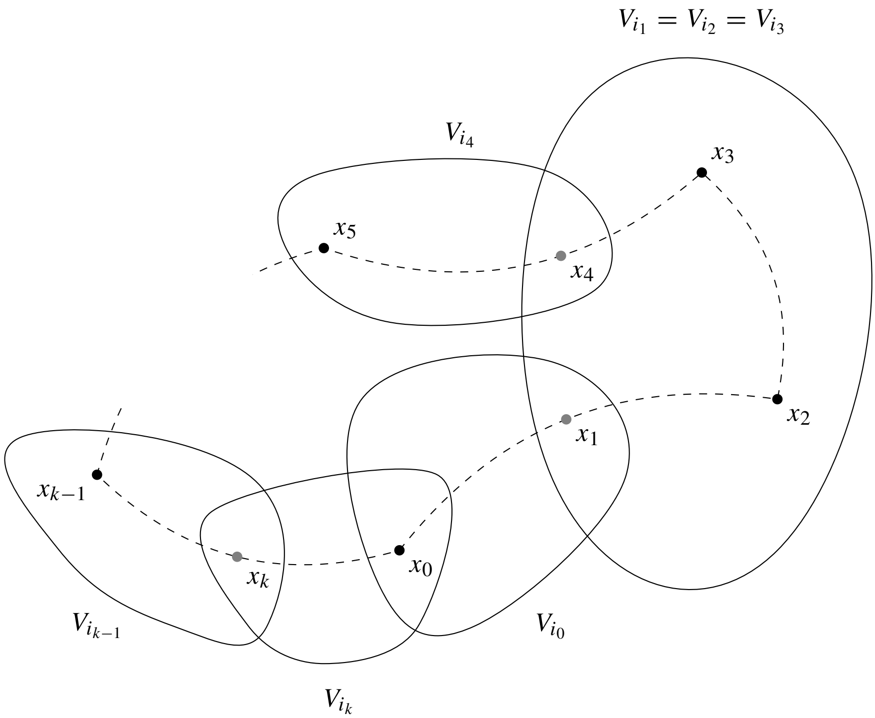

FIGURE 2. A refined stable–unstable sequence $x_0, x_1, \ldots, x_k, x_{k+1} = x_0$. Any stable–unstable sequence can be refined so that for each $0 \leq n \leq k$, there is a trivialization ϕ_{x_n} over a chart V_{x_n} containing both x_n and x_{n+1}. This refinement has the same total holonomy.

We want to show that $\mathfrak{h}_\epsilon^*(\cdot)$ is equal to the full Lie algebra $\mathfrak{g}$, and the local G-accessibility of the extension will be crucial at this point. In fact, with Theorem 4.11, we can show that local G-accessibility at a point $x \in N$ is roughly equivalent to having $\mathfrak{h}_\epsilon^\phi(x) = \mathfrak{g}$.

THEOREM 4.12. *Fix $\epsilon > 0$, $x \in N$ and a trivialization ϕ_x defined over $V_x \subset N$ for which we have $B_{3\epsilon}(x) \subset V_x$. Moreover, suppose that $\mathfrak{h}_\epsilon^{\phi_x}(\cdot)$ is continuous and has maximal dimension on $B_{3\epsilon}(x)$, and that the forwards orbit of x under g_t is dense in $B_{2\epsilon}(x)$. If f_t is locally G-accessible at x, then $\mathfrak{h}_\epsilon^{\phi_x}(x) = \mathfrak{g}$.*

Proof. Without loss of generality, suppose that we have chosen ϕ_x so that $\Theta^+_{\phi_x,\phi_x}(x, u)$ is trivial for all $u \in (W^{su}_{g_t}(x) \cap B_{2\epsilon}(x))^\circ$, and so that $\Theta^-_{\phi_x,\phi_x}(s_1, s_2)$ is trivial whenever s_1 and s_2 lie on the same (local) leaf of the strong stable foliation of g_t in $B_{2\epsilon}(x)$.

We write $\mathfrak{h} := \mathfrak{h}_\epsilon^{\phi_x}(x)$ and suppose for the sake of contradiction that $\mathfrak{h} \subsetneq \mathfrak{g}$ is a proper subalgebra. By Proposition 4.9 and our choice of ϕ_x, we must have $\mathfrak{h}_\epsilon^{\phi_x}(u) = \mathfrak{h}$ for all $u \in B_\epsilon(x)$ lying on the local leaf through x of the strong unstable foliation of g_t. Hence, for any $u_1 \in B_\epsilon(x)$, each vector $X_{w'}^{\phi_x}(u_1)$ used in the definition of $\mathfrak{h}_\epsilon^{\phi_x}(u)$ must lie in the Lie algebra $\mathfrak{h}$. Integrating this, we see that the unstable holonomies $\Theta^+_{\phi_x,\phi_x}(u_1, u_2)$ are constrained to $\exp(\mathfrak{h})$ for all $u_1, u_2 \in B_\epsilon(x)$ that lie on the same local leaf of the strong unstable foliation of g_t.

Now, consider $H := \exp(\mathfrak{h})$ and consider an element $g \in G \setminus H$ that lies in the complement. By our construction of ϕ_x, all unstable holonomies are constrained to H and all stable holonomies are trivial; hence, no local sequence of stable and unstable holonomies along a sequence of points $x, x_1, x_2, \ldots x_k, x$ lying in $B_\epsilon(x)$ can result in a total holonomy of g. Moreover, because $\mathfrak{h}$ is an ideal by Theorem 4.11, H is a normal subgroup of G; hence, we cannot obtain a total holonomy of g for *any* choice of trivialization. As f_t is locally G-accessible at x, this is a contradiction. □

It is worth noting that accessibility within a single, fixed trivialization ϕ_x at a given $x \in N$ is vital here. Indeed, if we only had *global* rather than *local* accessibility, the unstable holonomies could be constrained to $\exp(\mathfrak{h})$ as in the proof of Theorem 4.12 without any contradiction, because the transition functions between different trivialization charts need not act by $\exp(\mathfrak{h})$.

With Theorem 4.12, all that remains in this section is to verify that this translates properly into the symbolic model we are working with. In principle, the difficulty is that unstable leaves for the discrete dynamical system $(R, \mathcal{P})$ are typically not unstable leaves for g_t; fortunately, our choice of trivializations will circumvent almost all of these problems.

Recall that $(R, \mathcal{P})$ is a Markov partition associated with g_t, which descends to a C^1 expanding model (U, σ) by projecting along leaves of the strong stable foliation of g_t. We want to define unstable holonomies entirely within the symbolic model; a natural candidate for a definition comes from Proposition 4.1. For everything that follows, we assume that each $R_i \subset N$ has been assigned a fixed trivialization ϕ_i defined on a neighbourhood $B_\epsilon(R_i)$, with the property that ϕ_i has constant projection to F along each leaf of the strong stable foliation of f_t and each flowline of f_t. By choosing and fixing trivializations at each point in R, we no longer need to specify which trivialization we are using, at least when dealing with the symbolic model.

Definition 4.4. Fix $x \in R$ and $y \in W^{su}_{\mathcal{P}}(x)$. The *symbolic unstable holonomy* $\Theta^+_{\text{symb}}(x, y)$ is defined to be the limit

$$\Theta^+_{\text{symb}}(x, y) := \lim_{n\to\infty} \text{Hol}^{(n)}(\mathcal{P}^{-n}(y))(\text{Hol}^{(n)}(\mathcal{P}^{-n}(x)))^{-1},$$

where we write

$$\text{Hol}^{(n)}(u) := \text{Hol}(\mathcal{P}^{n-1}(u)) \circ \cdots \circ \text{Hol}(u)$$

for the nth holonomy under the Poincaré return map.

It is straightforward to verify that this limit exists. Moreover, our choice of trivializations ensures that the symbolic unstable holonomies agree with the appropriate (non-symbolic) unstable holonomies.

PROPOSITION 4.13. *Fix $x \in R$ and $y \in W^{su}_{\mathcal{P}}(x)$. There is a (possibly negative) t so that $g_t(y) \in W^{su}_{g_t}(x)$ for which we have*

$$\Theta^+_{\text{symb}}(x, y) = \Theta^+_{\phi_x,\phi_y}(x, g_t(y)),$$

assuming x and y are sufficiently close, where ϕ_x, ϕ_y are trivializations corresponding to the respective parts of the Markov partition.

Proof. The existence of such a t satisfying $|t| < \tau(y), \tau(\mathcal{P}^{-1}(y))$ follows immediately from the construction of the Markov partition $(R, \mathcal{P})$. We then clearly have

$$\Theta^{+}_{\text{symb}}(x, y) = \text{Hol}^{\phi_y}_{\phi_y}(g_t(y), y, -t) \circ \Theta^{+}_{\phi_x, \phi_y}(x, g_t(y))$$

by Proposition 4.1 and Definition 4.4. By our choice of trivialization and the fact that t does not exceed the return time of y, $\text{Hol}^{\phi_y}_{\phi_y}(g_t(y), y, -t)$ is the identity in G, as desired. □

Now, we can analogously define the *symbolic* infinitesimal transitivity group.

Definition 4.5. Fix $x \in U_i$ and a vector $w \in T^1_x U_i$. We define the *symbolic infinitesimal holonomy* at x in the direction of w to be the element

$$X^{\text{symb}}_w(x) := \bigg(\frac{d}{du}\bigg|_{u=x} (\Theta^{+}_{\text{symb}}(x, u))\bigg)(w)$$

of the Lie algebra $\mathfrak{g}$ of G. The *symbolic infinitesimal transitivity group* at x is defined to be the linear span

$$\mathfrak{h}_{\text{symb}}(x) := \operatorname*{span}_{s,s',w}(X_{w'}([x, s']) - X_w([x, s])),$$

where we take the span over $s, s' \in S_i$, $w \in T^1_x U_i$ and let w' be the projection of w to $[x, s']$ via centre-stable leaves followed by the flow.

With very little work, we can now prove the following result.

THEOREM 4.14. *Fix a bi-recurrent $x \in U_i$ at which* $\dim \mathfrak{h}_\epsilon(x)$ *is maximal, and suppose that f_t is G-accessible. Then* $\mathfrak{h}_{\text{symb}}(x) = \mathfrak{g}$.

Proof. By Proposition 4.13, the unstable holonomies used in Definitions 4.3 and 4.5 are the same. Hence, the regular and symbolic transitivity groups agree, and so by Theorem 4.11, we have $\mathfrak{h}_{\text{symb}}(x) = \mathfrak{g}$. □

This is almost the result we want, but we need to use some Lie theory to extract the explicit estimates that we use in the next section. We want to phrase this in terms of σ, which means minor notational changes in the preceding theorems. Recall that $\sigma : U \to U$ is not actually invertible; to make sense of the inverse, we must choose branches of σ^{-n} locally.

Definition 4.6. A *consistent past* for $u \in U_i$ is a sequence of maps $\{v^{(n)} : U_i \to U_{j(n)} \mid n \geq 0\}$ where $v^{(0)} = \text{id}\,|_{U_i}$ and $\sigma \circ v^{(n)} = v^{(n-1)}$.

Remark 4.15. A consistent past for $u \in U_i$ corresponds exactly to a choice of stable element $s \in S_i$: we can recover the maps $\{v^{(n)}\}$ by projecting the Poincaré return map $\mathcal{P}^{(-n)}$ along leaves of the strong stable foliation.

Finally, we can establish the main estimate; this is fundamentally the same estimate as [12, Lemma 3.17] and [17, Lemma 5.5], the primary difference being in the analysis of the infinitesimal transitivity group needed to arrive at it.

THEOREM 4.16. *Let ρ be an isotypic component of the representation of G on $L^2(G)$. There is an open subset $U_{lni} \subset U$ and constants $\epsilon, \delta, n_0 > 0$ so that, for any $x \in U_{lni}$ and any $\varphi \in C^1(U, V^\rho)$, there are consistent pasts $v_1 = \{v_1^{(n)}\}$ and $v_2 = \{v_2^{(n)}\}$ defined near x and a C^1 vector field $w\colon U_{lni} \to T^1U_{lni}$ satisfying*

$$\|(d\rho(X^{\text{symb}}_{w(u),v_1}(u) - X^{\text{symb}}_{w(u),v_2}(u)))(\varphi(u))\|_{L^2(G)} \geq \epsilon\|\rho\|$$

for all $u \in B_\delta(x)$.

Proof. By definition, we have

$$X^{\text{symb}}_{w,v_i}(x) = \frac{d}{du}\bigg|_{u=x}(\lim_{n\to\infty} \text{Hol}^{(n)}(v_i^{(n)}(u))(\text{Hol}^{(n)}(v_i^{(n)}(x))^{-1})$$

for all $u \in B_\delta(x)$. The derivatives of the terms in the sequence converge exponentially fast in n, so we can interchange the limit and the derivative to obtain

$$X^{\text{symb}}_{w,v_i}(x) = \lim_{n\to\infty}\left(\frac{d}{du}\bigg|_{u=x}(\text{Hol}^{(n)}(v_i^{(n)}(y))(\text{Hol}^{(n)}(v_i^{(n)}(x)))^{-1})\right)$$

for any consistent past v_i. As the $X^{\text{symb}}_{w(u),v_i} - X^{\text{symb}}_{w(u),v_j}$ taken over pasts v_i, v_j and vectors w form a basis of $\mathfrak{g}$, there is a finite n_0 so that the approximations

$$X^{\text{symb}}_{w,v_i^{(n_0)}}(x) := \frac{d}{du}\bigg|_{u=x}(\text{Hol}^{(n_0)}(v_i^{(n_0)}(y))(\text{Hol}^{(n_0)}(v_i^{(n_0)}(x)))^{-1})$$

can also be used to form a basis $X^{\text{symb}}_{w(u),v_i^{(n_0)}}(x) - X^{\text{symb}}_{w(u),v_j^{(n_0)}}(x)$, taken again over pasts v_i, v_j and vectors w. Hence, we have a Casimir

$$\Omega = \sum (g_{ij})^{-1}(X^{\text{symb}}_{w,v_i^{(n_0)}}(x) - X^{\text{symb}}_{w,v_j^{(n_0)}}(x))(X^{\text{symb}}_{w,v_i^{(n_0)}}(x) - X^{\text{symb}}_{w,v_j^{(n_0)}}(x))$$

for some finite collection of pasts v_i, v_j and a given vector w. This acts on V^ρ by scalar multiplication by $\|\rho\|^2$, and so we obtain

$$d\rho\left(\sum (g_{ij})^{-1}(X^{\text{symb}}_{w,v_i^{(n_0)}}(x) - X^{\text{symb}}_{w,v_j^{(n_0)}}(x))^2\right)(\varphi(x)) = \|\rho\|^2\varphi(x)$$

for this collection of pasts, and the same vector w. Now, there is a uniform constant $\epsilon > 0$, depending only on the g_{ij} and $\mathfrak{g}$, so that

$$\|d\rho(X^{\text{symb}}_{w,v_i^{(n_0)}}(x) - X^{\text{symb}}_{w,v_j^{(n_0)}}(x))(\varphi(x))\|_{L^2(G)} \geq \epsilon\|\rho\|\varphi(x)$$

for some choice of v_i, v_j and w. As $X^{\text{symb}}_{w,v}(u)$ varies continuously in w and u, there is a neighbourhood $B_\delta(x)$ of x in U and a C^1 vector field $w\colon B_\delta(x) \to T^1B_\delta(x)$ for which

$$\|d\rho(X^{\text{symb}}_{w(u),v_i^{(n_0)}}(u) - X^{\text{symb}}_{w(u),v_j^{(n_0)}}(u))(\varphi(u))\|_{L^2(G)} \geq \frac{\epsilon}{2}\|\rho\|\varphi(u)$$

holds for all $u \in B_\delta(x)$, as desired. □

5. *Spectral bounds for $\mathcal{L}_{z,\rho}^n$*

In this section, we establish bounds for the twisted transfer operators $\mathcal{L}_{z,\rho}^n$ acting on $C^1(U, V^\rho)$ with respect to the $L^2(\nu^u)$-norm. The key ingredients are the local non-integrability estimate in Theorem 4.16, the diametric regularity of the measure ν^u and the C^1 regularity of α_z from the definition of the transfer operators.

The strategy adopted in this section is by now classical, dating back to Dolgopyat in [11, 12]; an account of this almost completely adapted to our setting was given by Winter in [17]. The reader already familiar with these arguments should find few surprises in this section, but we feel it necessary to include them given their delicate nature and the minor differences in our contexts.

We begin by recalling the definition of the twisted transfer operator

$$(\mathcal{L}_{z,\rho}^n \varphi)(u) := \sum_{\sigma^n(u')=u} e^{\alpha_z^{(n)}(u')} \rho(\mathrm{Hol}^{(n)}(u'))\varphi(u')$$

associated with an irreducible representation ρ of G on $V^\rho \subset L^2(G)$. For the trivial representation $\rho = 0$, Dolgopyat [11] established contraction by using the uniform non-integrability of the strong stable and unstable foliations for g_t to establish cancellation in the terms defining the transfer operator. In principle, the difficulty in obtaining contraction for $\|\mathcal{L}_{z,\rho}^n \varphi\|_{L^2(\nu^u)}$ as $n \to \infty$ lies in the possibility that the rotation introduced by the action of ρ may 'undo' any cancellation between the vectors $\varphi(u') \in V^\rho$ that we might obtain from the dynamics on the base, and that this may happen on a set of large measure.

The local non-integrability estimate provided by Theorem 4.16, however, suggests that we should typically be able to find u_1', u_2' with $\sigma^n(u_1') = \sigma^n(u_2') \in U_{lni}$ so that

$$\rho(\mathrm{Hol}^{(n)}(u_1'))\varphi(u_1') \quad \text{and} \quad \rho(\mathrm{Hol}^{(n)}(u_2'))\varphi(u_2')$$

are 'uniformly' non-parallel. The main argument in the section boils down to verifying that this can be accomplished on an adequately large set, with explicit uniformity estimates.

Throughout this section, we work with a fixed isotypic component V^ρ of the regular representation of G on $L^2(G)$. However, it is worth noting that most of the intermediate constants will fundamentally depend on ρ, and keeping track of these dependencies is essential to obtaining a final bound in Theorem 5.7 that is independent of ρ.

Though we need to deal with $\mathcal{L}_{z,\rho}^n \varphi$ for any $\varphi \in C^1(U, V^\rho)$, it will be helpful to work instead with slightly more regular real-valued functions $\Phi \in C^1(U, \mathbb{R}^+)$ with bounded logarithmic derivative; in other words, we require

$$\sup_{w \in T_u^1 U} |(d\Phi)_u(w)| < C\Phi(u)$$

for all $u \in U$. We use $\mathcal{K}_C$ to denote the class of such functions

$$\mathcal{K}_C := \{\Phi \in C^1(U, \mathbb{R}^+) \mid \log \Phi \text{ is } C\text{-Lipschitz}\}$$

for any constant $C > 0$.

As g_t is Anosov, the expansion rates of g_t on U are bounded away from 1. As the return times $\tau : R \to \mathbb{R}$ are bounded away from zero, the slowest expansion rates of $d\sigma^n$ on U are

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therefore also bounded away from 1. For what follows, let $f\kappa^n$ and bK^n with $1 < \kappa < K$ be the slowest and fastest expansion rate of any unit vector in $T^1 U$ under $d\sigma^n$.

We are interested in functions $\Phi \in \mathcal{K}_C$ with bounded logarithmic derivative because they can be used to control less-regular functions $\varphi \in C^1(U, \mathbb{R})$. The following lemma makes this precise, and is analogous to [11, Lemma 14] and [17, Lemma 6.5].

LEMMA 5.1. *Fix $C > 0$, $\varphi \in C^1(U, V^\rho)$ and $\Phi \in \mathcal{K}_C$. There is a $\delta > 0$ so that, if*

$$\|\varphi(u)\|_{L^2(G)} < \Phi(u),$$
$$\sup_{w \in T^1_u U} \|(d\varphi)_u(w)\|_{L^2(G)} < C\Phi(u)$$

for all $u \in U$, then for any $u_0 \in U$,

$$\|\varphi(v^{(n)}(u))\|_{L^2(G)} \leq \tfrac{3}{4}\Phi(v^{(n)}(u))$$

or

$$\|\varphi(v^{(n)}(u))\|_{L^2(G)} \geq \tfrac{1}{4}\Phi(v^{(n)}(u))$$

for all $u \in B_\delta(u_0)$, each $n > 0$ and each consistent past $v = \{v^{(n)}\}$ defined near u_0. Moreover, for any $u_0 \in U$,

$$\Phi(v^{(n)}(y)) \leq 2\Phi(v^{(n)}(x))$$

for all $x, y \in B_\delta(u_0)$. The choice of constant $\delta > 0$ can be made so that we have $\delta C = A$ for some uniform constant A, which does not depend on ρ, Φ, φ, u_0, v or n.

Proof. As we chose $\Phi \in \mathcal{K}_C$, $\log \Phi$ is C-Lipschitz and we therefore have

$$\begin{aligned} |\log(\Phi(v^{(n)}(y))) - \log(\Phi(v^{(n)}(x)))| &\leq \frac{C}{f\kappa^n} d(x, y) \\ &\leq \frac{C}{f} d(x, y) \end{aligned}$$

for any $x, y \in U$. Exponentiating both sides, this means

$$\Phi(v^{(n)}(y)) \leq e^{(C/f)d(x,y)}\Phi(v^{(n)}(x)) \tag{5.1}$$

for any $x, y \in U$. Now, suppose that $d(x, y) \leq 2\delta$ for some $\delta > 0$, and fix a unit speed path $\gamma : [0, 2\delta] \to U_0$ with $\gamma(0) = y$ and $\gamma(2\delta) = x$. We then have

$$\begin{aligned} |\|\varphi(v^{(n)}(x))|_{L^2(G)} - \|\varphi(v^{(n)}(y))\|_{L^2(G)}| &\leq \int_0^{2\delta} |\langle \operatorname{grad}(\varphi \circ v^{(n)})(\gamma(t)), \gamma'(t)\rangle_{T^1 N}|\, dt \\ &\leq \frac{C}{f\kappa^n} \int_0^{2\delta} (\Phi \circ v^{(n)})(\gamma(t))\, dt \\ &\leq \frac{C}{f\kappa^n} 2\delta e^{(C/f\kappa^n)d(x,y)}\Phi(v^{(n)}(y)) \\ &\leq \frac{C}{f}\delta e^{(C/f)2\delta}\Phi(v^{(n)}(y)) \end{aligned}$$

by the fundamental theorem of calculus and our bounds on φ. Fix δ small enough to ensure

$$\frac{C}{f}2\delta e^{(C/f)2\delta} \leq \frac{1}{8}$$

and

$$e^{(C/f)2\delta} \leq 2$$

hold simultaneously; note that we really only require that $C\delta$ is sufficiently small, and so δ can be chosen inversely proportional to C. We therefore have

$$\Phi(v^{(n)}(y)) \leq 2\Phi(v^{(n)}(x)) \tag{5.2}$$

and

$$\|\varphi(v^{(n)}(x))\|_{L^2(G)} \leq \|\varphi(v^{(n)}(y))\|_{L^2(G)} + \tfrac{1}{8}\Phi(v^{(n)}(y)) \tag{5.3}$$

whenever $d(x, y) < 2\delta$. To conclude, suppose that we had

$$\|\varphi(v^{(n)}(y))\|_{L^2(G)} \leq \tfrac{1}{4}\Phi(v^{(n)}(y))$$

at some $y \in B_\delta(u_0)$, for a given $u_0 \in U$ and n. Then by (5.3) and (5.2), we must have

$$\begin{aligned}\|\varphi(v^{(n)}(x))\|_{L^2(G)} &\leq \tfrac{1}{4}\Phi(v^{(n)}(y)) + \tfrac{1}{8}\Phi(v^{(n)}(y)) \\ &\leq \tfrac{2}{4}\Phi(v^{(n)}(x)) + \tfrac{2}{8}\Phi(v^{(n)}(x)) \\ &\leq \tfrac{3}{4}\Phi(v^{(n)}(x))\end{aligned}$$

for any $x \in B_\delta(u_0)$, as desired. □

LEMMA 5.2. *Fix $C > 0$, $\varphi \in C^1(U, V^\rho)$ and $\Phi \in \mathcal{K}_C$. Take $\delta > 0$ as in Lemma 5.1 and suppose that we have*

$$\begin{aligned}\|\varphi(u)\|_{L^2(G)} &< \Phi(u), \\ \sup_{w \in T_u^1 U} \|(d\varphi)_u(w)\| &< C\Phi(u)\end{aligned}$$

for all $u \in U$. If $v^{(n)}$ is a past defined on $B_\delta(u_0)$ satisfying

$$\|\varphi(v^{(n)}(u))\|_{L^2(G)} \geq \tfrac{1}{4}\Phi(v^{(n)}(u))$$

for all $u \in B_\delta(u_0)$, and some given $u_0 \in U$, then we have

$$\sup_{w \in T_u^1 B_\delta(u_0)} \left\| \left(d\left(\frac{\varphi \circ v^{(n)}}{\|\varphi \circ v^{(n)}\|_{L^2(G)}} \right) \right)_u (w) \right\|_{L^2(G)} \leq \frac{8C}{f\kappa^n}$$

for all $u \in B_\delta(u_0)$.

Proof. Differentiating the fraction, we see that we need to bound the supremum of

$$\left\| \frac{(d(\varphi \circ v^{(n)}))_u(w)\|\varphi(v^{(n)}(u))\|_{L^2(G)} - \varphi(v^{(n)}(u))(d(\|\varphi \circ v^{(n)}\|_{L^2(G)}))_u(w)}{\|\varphi(v^{(n)}(u))\|_{L^2(G)}} \right\|_{L^2(G)} \tag{5.4}$$

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for any $u \in U$ and $w \in T^1_u B_\delta(u_0)$. Note that we always have

$$(d(\|\varphi \circ v^{(n)}\|_{L^2(G)}))_u(w) \leq \|(d(\varphi \circ v^{(n)}))_u(w)\|_{L^2(G)}$$

and so (5.4) is at most

$$\sup_{w \in T^1_u B_\delta(u_0)} \frac{2\|\varphi(v^{(n)}(u))\|_{L^2(G)}\|(d(\varphi \circ v^{(n)}))_u(w)\|_{L^2(G)}}{\|\varphi(v^{(n)}(u))\|^2_{L^2(G)}} \tag{5.5}$$

by the triangle inequality. Cancelling terms, we can reduce (5.5) to

$$\sup_{w \in T^1_u B_\delta(u_0)} \frac{2\|(d(\varphi \circ v^{(n)}))_u(w)\|_{L^2(G)}}{\|\varphi(v^{(n)}(u))\|_{L^2(G)}},$$

which is at most

$$\sup_{\substack{w \in T^1_u B_\delta(u_0) \\ w' \in T^1_{v^{(n)}(u)} B_\delta(u_0)}} \frac{2\|(d(\varphi))_{v^{(n)}(u)}(w')\|_{L^2(G)}\|(dv^{(n)})_u(w)\|_{T^1 N}}{\|\varphi(v^{(n)}(u))\|_{L^2(G)}} \tag{5.6}$$

by the chain rule. As $f\kappa^n$ is the slowest expansion rate of any vector in T^1U under $d\sigma^n$, we can bound

$$\sup_{w \in T^1_u U} \|(dv^{(n)})_u(w)\|_{T^1 N} \leq \frac{1}{f\kappa^n}$$

for all $u \in U$. By hypothesis, we also have

$$\sup_{w' \in T^1_{v^{(n)}(u)} U} \|(d\varphi)_{v^{(n)}(u)}(w')\|_{L^2(G)} \leq C\Phi(v^{(n)}(u))$$

for all $u \in U$, and so (5.6) can be bounded above by

$$\frac{2C\Phi(v^{(n)}(u))}{f\kappa^n\|\varphi(v^{(n)}(u))\|_{L^2(G)}},$$

which is, in turn, at most

$$\frac{8C\Phi(v^{(n)}(u))}{f\kappa^n\Phi(v^{(n)}(u))}$$

by hypothesis. Cancelling Φ, we obtain the result desired. □

We need a simple lemma before proceeding to the main argument of the section; the proof is a straightforward application of the polarization identity.

PROPOSITION 5.3. *Let $(V, \cdot)$ be an inner product space, and suppose that we have*

$$\|\hat{v} - \hat{w}\| \geq \epsilon$$

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for $v, w \in V$ and $\epsilon < 1$, where $\hat{v}, \hat{w}$ denote the unit vectors in the directions of v and w, respectively. If $\|v\| \le \|w\|$, then

$$\|v + w\| \le \left(1 - \frac{\epsilon^2}{4}\right)\|v\| + \|w\|.$$

We might be tempted to argue that, if $\varphi \in C^1(U, V^\rho)$ is controlled by $\Phi \in \mathcal{K}_C$ as in Lemma 5.1, then we can similarly bound $\|(\mathcal{L}^n_{z,\rho}\varphi)(u)\|_{L^2(G)}$ by $(\mathcal{L}^n_{\mathrm{Re}(z),0}\Phi)(u)$ pointwise; however, although this turns out to be true, it is not particularly helpful because $\mathcal{L}^n_{P(\varsigma),0}\Phi$ fails to contract as $n \to \infty$. Indeed, because Φ is strictly positive by definition, $\mathcal{L}^n_{P(\varsigma),0}\Phi$ will converge to $\int_U \Phi\, d\nu^u > 0$.

The solution is to artificially introduce contraction into the transfer operators, and Theorem 4.16 is precisely what ensures that we can do this while maintaining a pointwise bound. In the next lemma, we show that we can uniformly and explicitly bound $\|(\mathcal{L}^n_{z,\rho}\varphi)(u)\|_{L^2(G)}$ away from $(\mathcal{L}^n_{\mathrm{Re}(z),0}\Phi)(u)$ on a measurable portion of any sufficiently small set.

The following lemma establishes the key cancellation mechanism in our setting, and mirrors [17, Lemma 6.6] and, together with Lemma 5.5, is analogous to [11, §8].

LEMMA 5.4. *Fix $C > 0$, $\varphi \in C^1(U, V^\rho)$ and $\Phi \in \mathcal{K}_C$ with*

$$\|\varphi(u)\|_{L^2(G)} < \Phi(u),$$
$$\sup_{w \in T^1_u U} \|(d\varphi)_u(w)\|_{L^2(G)} < C\Phi(u)$$

for all $u \in U$. Let $U_{lni} \subset U$ be the open subset given by Theorem 4.16 and let $\delta > 0$ be the constant given by Lemma 5.1. There are constants $n_0 > 0$, $\epsilon > 0$ and $s < 1$ so that, for any $x \in U_{lni}$ with $B_\delta(x) \subset U_{lni}$, we can find

- *a point $y \in U_{lni}$ with $B_{s\delta}(y) \subset B_\delta(x)$; and*
- *a pair of pasts $v_1^{(n_0)}, v_2^{(n_0)}$ defined on $B_\delta(x)$;*

for which we can bound

$$\|e^{\alpha_z^{(n_0)}(v_1^{(n_0)}(u))}\rho(\mathrm{Hol}^{(n_0)}(v_1^{(n_0)}(u)))\varphi(v_1^{(n_0)}(u)) + e^{\alpha_z^{(n_0)}(v_2^{(n_0)}(u))}\rho(\mathrm{Hol}^{(n_0)}(v_2^{(n_0)}(u)))\varphi(v_2^{(n_0)}(u))\|_{L^2(G)}$$

above by

$$\left(1 - \frac{(\epsilon\delta(1 + |\mathrm{Im}(z)|)\|\rho\|)^2}{4096}\right)e^{\alpha^{(n_0)}_{\mathrm{Re}(z)}(v_1^{(n_0)}(u))}\Phi(v_1^{(n_0)}(u)) + e^{\alpha^{(n_0)}_{\mathrm{Re}(z)}(v_2^{(n_0)}(u))}\Phi(v_2^{(n_0)}(u))$$

for all $u \in B_{s\delta}(y)$ and all $z \in \mathbb{C}$ with $|\mathrm{Re}(z) - P(\varsigma)| < 1$. The constant ϵ can be chosen independently of ρ and C, whereas n_0 depends only on $C/\|\rho\|$. The constant s can be chosen uniformly in C and ρ.

Proof. We deal with a fixed $n > 0$ throughout the proof, and specify how large n needs to be as we proceed; it is important to note that we cannot deal with arbitrarily large n without foregoing the uniformity of the bounds we wish to obtain. We proceed in cases depending on which alternative of Lemma 5.1 holds. For any pasts $v_1^{(n)}$ and $v_2^{(n)}$, if we have

$$\|\varphi(v_1^{(n)}(u))\|_{L^2(G)} \leq \tfrac{3}{4}\Phi(v_1^{(n)}(u))$$

for all $u \in B_\delta(x)$, then because ρ is unitary we can clearly bound

$$\|e^{\alpha_z^{(n)}(v_1^{(n)}(u))}\rho(\mathrm{Hol}^{(n)}(v_1^{(n)}(u)))\varphi(v_1^{(n)}(u)) \\ + e^{\alpha_z^{(n)}(v_2^{(n)}(u))}\rho(\mathrm{Hol}^{(n)}(v_2^{(n)}(u)))\varphi(v_2^{(n)}(u))\|_{L^2(G)}$$

above by

$$\tfrac{3}{4}(e^{\alpha_{\mathrm{Re}(z)}^{(n)}(v_1^{(n)}(u))}\Phi(v_1^{(n)}(u)) + e^{\alpha_{\mathrm{Re}(z)}^{(n)}(v_2^{(n)}(u))}\Phi(v_2^{(n)}(u)))$$

for all $u \in B_\delta(x)$. In this case, we are done by simply setting $y := x$. Similarly, if we had

$$\|\varphi(v_2^{(n)}(u))\|_{L^2(G)} \leq \tfrac{3}{4}\Phi(v_2^{(n)}(u))$$

for all $u \in B_\delta(x)$, then we are again done by setting $y := x$, up to interchanging our choice of v_1 and v_2. So we may as well assume that the second alternative of Lemma 5.1 holds for both $v_1^{(n)}$ and $v_2^{(n)}$, and that we therefore have

$$\|\varphi(v_\ell^{(n)}(u))\|_{L^2(G)} \geq \tfrac{1}{4}\Phi(v_\ell^{(n)}(u)) \tag{5.7}$$

for all $u \in B_\delta(x)$. We temporarily abbreviate

$$g_\ell(u) := \mathrm{Hol}^{(n)}(v_\ell^{(n)}(u)),$$
$$\hat{\varphi}_\ell(u) := \frac{\varphi(v_\ell^{(n)}(u))}{\|\varphi(v_\ell^{(n)}(u))\|_{L^2(G)}}$$

and

$$\hat{\psi}_\ell(u) := e^{\mathrm{Im}(z)\tau^{(n)}(v_\ell^{(n)}(u))i}\,\frac{\varphi(v_\ell^{(n)}(u))}{\|\varphi(v_\ell^{(n)}(u))\|_{L^2(G)}}$$

for the sake of clarity. Note that $\hat{\varphi}_\ell$ and $\hat{\psi}_\ell$ are well-defined on $B_\delta(x)$ as an immediate consequence of (5.7). Now, by reversing the triangle inequality,

$$\|\rho(g_1(x))\hat{\psi}_1(x) - \rho(g_2(x))\hat{\psi}_2(x)\|_{L^2(G)} \tag{5.8}$$

is at least

$$\|\rho(g_1(x))\hat{\psi}_1(y) - \rho(g_2(x))\hat{\psi}_2(y)\|_{L^2(G)} \\ - \|\rho(g_1(x))\hat{\psi}_1(y) - \rho(g_1(x))\hat{\psi}_1(x)\|_{L^2(G)} - \|\rho(g_2(x))\hat{\psi}_2(x) - \rho(g_2(x))\hat{\psi}_2(y)\|_{L^2(G)} \tag{5.9}$$

for any $y \in B_\delta(x)$. As group elements act by isometries with respect to the $L^2(G)$ norm, (5.9) is equal to

$$\|\rho(g_2(y)g_2^{-1}(x)g_1(x))\hat{\psi}_1(y) - \rho(g_2(y))\hat{\psi}_2(y)\|_{L^2(G)} \\ -\|\hat{\psi}_1(y) - \hat{\psi}_1(x)\|_{L^2(G)} - \|\hat{\psi}_2(x) - \hat{\psi}_2(y)\|_{L^2(G)}$$

which we can bound below by

$$\begin{aligned}&\|\rho(g_2(y)g_2^{-1}(x)g_1(x))\hat{\psi}_1(y)-\rho(g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}\\&\quad-\|\hat{\psi}_1(y)-\hat{\psi}_1(x)\|_{L^2(G)}-\|\hat{\psi}_2(x)-\hat{\psi}_2(y)\|_{L^2(G)}\\&\quad-\|\rho(g_2(y))\hat{\psi}_2(y)-\rho(g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}\end{aligned}\tag{5.10}$$

using the reverse triangle inequality once again. We can rewrite (5.10) as

$$\begin{aligned}&\|\rho(g_1(x)g_1^{-1}(y)g_1(y))\hat{\psi}_1(y)-\rho(g_2(x)g_2^{-1}(y)g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}\\&\quad-\|\hat{\psi}_1(y)-\hat{\psi}_1(x)\|_{L^2(G)}-\|\hat{\psi}_2(x)-\hat{\psi}_2(y)\|_{L^2(G)}\\&\quad-\|\rho(g_2(y))\hat{\psi}_2(y)-\rho(g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}\end{aligned}\tag{5.11}$$

by replacing $\hat{\psi}_1(y)$ with $g_1^{-1}(y)g_1(y)\hat{\psi}_1(y)$ in the first term of the first line, and multiplying both terms on the first line by $g_2(x)g_2^{-1}(y)$. Hence, (5.8) is bounded below by (5.11), and we see that

$$\|\rho(g_1(x))\hat{\psi}_1(x)-\rho(g_2(x))\hat{\psi}_2(x)\|_{L^2(G)}+\|\rho(g_2(y))\hat{\psi}_2(y)-\rho(g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}$$

is at least

$$\begin{aligned}&\|\rho(g_1(x)g_1^{-1}(y)g_1(y))\hat{\psi}_1(y)-\rho(g_2(x)g_2^{-1}(y)g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}\\&\quad-\|\hat{\psi}_1(y)-\hat{\psi}_1(x)\|_{L^2(G)}-\|\hat{\psi}_2(x)-\hat{\psi}_2(y)\|_{L^2(G)}\end{aligned}\tag{5.12}$$

after rearranging. By Theorem 4.16, there is an $N>0$, $\epsilon>0$, a choice of pasts $v_1^{(n)}$ and $v_2^{(n)}$, and a y with $d(x,y)=\delta/2$ for which we have

$$\begin{aligned}&\|\rho(g_1(x)g_1^{-1}(y)g_1(y))\hat{\psi}_1(y)-\rho(g_2(x)g_2^{-1}(y)g_1(y))\hat{\psi}_1(y)\|_{L^2(G)}\\&\quad\geq\epsilon(1+|\mathrm{Im}(z)|)\|\rho\|\frac{\delta}{2}\end{aligned}\tag{5.13}$$

for all $n\geq N$; note that we have applied the theorem to

$$e^{\mathrm{Im}(z)\tau^{(n)}(u)i}\rho(\mathrm{Hol}^{(n)}(u))\frac{\varphi(u)}{\|\varphi(u)\|_{L^2(G)}},$$

which is certainly a smooth function in $C^1(U,V^\rho)$. On the other hand, by Lemma 5.2, $\hat{\varphi}_\ell(u)$ is at worst $8C/f\kappa^n$-Lipschitz in u on $B_\delta(x)$, and we can estimate

$$\begin{aligned}|(d(e^{\mathrm{Im}(z)(\tau^{(n)}\circ v_\ell^{(n)})i}))_u(w)|_{L^2(G)}&=(|\mathrm{Im}(z)e^{\mathrm{Im}(z)(\tau^{(n)}\circ v_\ell^{(n)})i}|)\bigg|\sum_{i=1}^{n}(d(\tau\circ v_\ell^i))_{v_\ell^{(n)}(u)}(w')\bigg|\\&\leq|\mathrm{Im}(z)|\bigg(\sum_{i=1}^{n}\frac{\|\tau\|_{C^1}}{f\kappa^i}\bigg)\\&\leq|\mathrm{Im}(z)|\frac{\|\tau\|_{C^1}}{f(\kappa-1)}\end{aligned}$$

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for all $u \in U$, $w \in T^1_u B_\delta(x)$ and $w' \in T^1_{v_\ell(u)} B_\delta(x)$. Hence, we can bound

$$\|\hat{\psi}_\ell(x) - \hat{\psi}_\ell(y)\| \le \left(\frac{8C}{f\kappa^n} + |\mathrm{Im}(z)|\frac{\|\tau\|_{C^1}}{f(\kappa-1)}\right) d(x,y)$$

because we chose $y \in B_\delta(x)$. Suppose that n and $\|\rho\|$ are large enough so that we have

$$\left(\frac{8C}{f\kappa^n} + |\mathrm{Im}(z)|\frac{\|\tau\|_{C^1}}{f(\kappa-1)}\right) < \frac{\epsilon}{8}(1+|\mathrm{Im}(z)|)\|\rho\|$$

for some constant $K > 0$. Note that we can make this choice of n so that it depends only on $C/\|\rho\|$ and not $\|\rho\|$ directly; moreover, the requirement that $\|\rho\|$ be sufficiently large can be made absolute, and in particular is independent of z. This ensures

$$\|\hat{\psi}_\ell(x) - \hat{\psi}_\ell(y)\|_{L^2(G)} \le \frac{\epsilon}{8}(1+|\mathrm{Im}(z)|)\|\rho\|\delta \tag{5.14}$$

because $d(x,y) \le \delta$. Note that our choice of n here depends only on $C/\|\rho\|$. Plugging (5.13) and (5.14) into (5.12), we conclude that

$$\begin{aligned}&\|\rho(g_1(x))\hat{\psi}_1(x) - \rho(g_2(x))\hat{\psi}_2(x)\|_{L^2(G)} + \|\rho(g_1(y))\hat{\psi}_1(y) - \rho(g_2(y))\hat{\psi}_2(y)\|_{L^2(G)}\\ &\quad \ge \frac{\epsilon}{4}\delta(1+|\mathrm{Im}(z)|)\|\rho\|\end{aligned}$$

and so

$$\|\rho(g_1(x))\hat{\psi}_1(x) - \rho(g_2(x))\hat{\psi}_2(x)\|_{L^2(G)} > \frac{\epsilon}{8}\delta(1+|\mathrm{Im}(z)|)\|\rho\| \tag{5.15}$$

or

$$\|\rho(g_1(y))\hat{\psi}_1(y) - \rho(g_2(y))\hat{\psi}_2(y)\|_{L^2(G)} > \frac{\epsilon}{8}\delta(1+|\mathrm{Im}(z)|)\|\rho\| \tag{5.16}$$

must hold. Without loss of generality, suppose (5.16) holds. Using the Lipschitz estimate on $\hat{\varphi}_\ell$, we can bound

$$\sup_{w\in T^1_u B_\delta(x)} \|(d(\rho(g_\ell)\hat{\psi}_\ell))_u(w)\|_{L^2(G)}$$

by

$$\sup_{w\in T^1_u B_\delta(x)} \|(((d\rho)_{g_\ell(u)} \circ (dg_\ell)_u)(w))\hat{\psi}_\ell(u) + \rho(g_\ell(u))(d\hat{\psi}_\ell)_u(w)\|_{L^2(G)} \tag{5.17}$$

for all $u \in B_\delta(x)$. A straightforward calculation shows that

$$\begin{aligned}\sup_{w\in T^1_u B_\delta(x)} \|(dg_\ell)_u(w)\| &\le \sup_{w\in T^1_u B_\delta(x)} \sum_{i=1}^{n} \|(d(\mathrm{Hol} \circ v^{(i)}))_u(w)\|_{T^1 G}\\ &\le \sum_{i=1}^{n} \frac{\|\mathrm{Hol}\|_{C^1}}{f\kappa^i}\\ &\le \frac{\|\mathrm{Hol}\|_{C^1}}{f(\kappa-1)}\end{aligned}$$

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and we have

$$\sup_{w'\in T^1_{g_\ell(u)}G} \|(d\rho)_{g_\ell(u)}(w')\|_{L^2(G)} \leq \|\rho\|$$

by Definition 2.9; note that because ρ is a homomorphism, the operator norm of $d\rho$ at $g_\ell(u)$ is equivalent to the norm at the identity. Hence, (5.17) is at most

$$\|\rho\|\frac{\|\ \mathrm{Hol}\ \|_{C^1}}{f(\kappa-1)}+\frac{\epsilon}{8}(1+|\mathrm{Im}(z)|)\|\rho\|$$

by our choice of δ, because ρ acts by $L^2(G)$-isometries. We can make a uniform choice of $s<1$ so that

$$s\left(\|\rho\|\frac{\|\ \mathrm{Hol}\ \|_{C^1}}{f(\kappa-1)}+\frac{\epsilon}{8}(1+|\mathrm{Im}(z)|)\|\rho\|\right)<\frac{\epsilon}{32}(1+|\mathrm{Im}(z)|)\|\rho\| \tag{5.18}$$

and we then have

$$\|\rho(g_1(y))\hat{\psi}_1(y)-\rho(g_1(u))\hat{\psi}_1(u)\|_{L^2(G)}<\frac{\epsilon}{32}\delta(1+|\mathrm{Im}(z)|)\|\rho\| \tag{5.19}$$

and

$$\|\rho(g_2(y))\hat{\psi}_2(y)-\rho(g_2(u))\hat{\psi}_2(u)\|_{L^2(G)}<\frac{\epsilon}{32}\delta(1+|\mathrm{Im}(z)|)\|\rho\| \tag{5.20}$$

for all $u\in B_{s\delta}(y)$. Combining (5.19) and (5.20) with (5.15), we now have

$$\|\rho(g_1(u))\hat{\psi}_1(u)-\rho(g_2(u))\hat{\psi}_2(u)\|_{L^2(G)}>\frac{\epsilon}{16}\delta(1+|\mathrm{Im}(z)|)\|\rho\|$$

for all $u\in B_{s\delta}(y)$. Now, fix $u\in B_{s\delta}(y)$. By Proposition 5.3, we can then bound

$$\begin{aligned}&\|e^{\alpha_z^{(n)}(v_1^{(n)}(u))}\rho(\mathrm{Hol}^{(n)}(v_1^{(n)}(u)))\varphi(v_1^{(n)}(u))\\&\quad+\ e^{\alpha_z^{(n)}(v_2^{(n)}(u))}\rho(\mathrm{Hol}^{(n)}(v_2^{(n)}(u)))\varphi(v_2^{(n)}(u))\|_{L^2(G)}\end{aligned}$$

above by

$$\left(1-\frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|\|)^2}{1024}\right)e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v_1^{(n)}(u))}\Phi(v_1^{(n)}(u))+e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v_2^{(n)}(u))}\Phi(v_2^{(n)}(u)) \tag{5.21}$$

assuming without loss of generality that

$$\|e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v_1^{(n)}(u))}\varphi(v_1^{(n)}(u))\|_{L^2(G)}\leq\|e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v_2^{(n)}(u))}\varphi(v_2^{(n)}(u))\|_{L^2(G)} \tag{5.22}$$

held for this particular u; this is true up to interchanging $v_1^{(n)}$ and $v_2^{(n)}$. It simply remains to ensure that a similar inequality extends to $B_{s\delta}(y)$ for this particular choice of $v_1^{(n)}$ and $v_2^{(n)}$: this is not immediate because (5.22) could certainly fail to hold on the entire ball $B_{s\delta}(y)$.

This requires some extra work. Note that we have

$$\begin{aligned}&\sup_{w\in T_u^1U}\left\|\left(d\left(e^{\alpha^{(n)}_{\mathrm{Re}(z)}}\Phi\right)\right)_u(w)\right\|_{L^2(G)}\\&\quad\leq\sup_{w\in T_u^1U}|(de^{\alpha^{(n)}_{\mathrm{Re}(z)}})_u(w)|\|\Phi(u)\|_{L^2(G)}+e^{\alpha^{(n)}_{\mathrm{Re}(z)}(u)}\|(d\Phi)_u(w)\|_{L^2(G)}\end{aligned}$$

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$$\leq \|\alpha^{(n)}_{\mathrm{Re}(z)}\|_{C^1} e^{\alpha^{(n)}_{\mathrm{Re}(z)}(u)} \|\Phi(u)\|_{L^2(G)} + Ce^{\alpha^{(n)}_{\mathrm{Re}(z)}(u)} \Phi(u)$$
$$\leq \bigg(\sum_{i=1}^{n} bK^i \|\alpha_{\mathrm{Re}(z)}\|_{C^1}\bigg) e^{\alpha^{(n)}_{\mathrm{Re}(z)}(u)} \|\Phi(u)\|_{L^2(G)} + Ce^{\alpha^{(n)}_{\mathrm{Re}(z)}(u)} \Phi(u)$$
$$\leq \bigg(b\|\alpha_{\mathrm{Re}(z)}\|_{C^1} \frac{K^{n+1}-K}{K-1} + C\bigg) e^{\alpha^{(n)}_{\mathrm{Re}(z)}(u)} \|\Phi(u)\|_{L^2(G)}$$

for all $u \in U$. By making s smaller if necessary, we can ensure that, in addition to (5.18), we also have

$$s\delta(b\|\alpha_{\mathrm{Re}(z)}\|_{C^1} \frac{K^{n+1}-K}{K-1} + C) < \delta C = A,$$

where A is the uniform constant guaranteed by Lemma 5.1. Note that this can be accomplished by a uniform choice of s, because n is fixed and z is required to satisfy $|\mathrm{Re}(z) - P(\varsigma)| < 1$. As a consequence, we see by Lemma 5.1 that

$$\tfrac{1}{2} e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_\ell(u'))} \Phi(v^{(n)}_\ell(u')) \leq e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_\ell(u))} \Phi(v^{(n)}_\ell(u)) \leq 2e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_\ell(u'))} \Phi(v^{(n)}_\ell(u')) \tag{5.23}$$

for all $u, u' \in B_{s\delta}(y)$. Now, suppose that

$$\|e^{\alpha^{(n)}_z(v^{(n)}_1(u'))} \rho(\mathrm{Hol}^{(n)}(v^{(n)}_1(u')))\varphi(v^{(n)}_1(u'))$$
$$+\ e^{\alpha^{(n)}_z(v^{(n)}_2(u'))} \rho(\mathrm{Hol}^{(n)}(v^{(n)}_2(u')))\varphi(v^{(n)}_2(u'))\|_{L^2(G)}$$

were bounded above by

$$e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_1(u'))} \Phi(v^{(n)}_1(u')) + \bigg(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{1024}\bigg) e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_2(u'))} \Phi(v^{(n)}_2(u')) \tag{5.24}$$

for some $u' \in B_{s\delta}(y)$; this is in contrast to the bound by (5.21) that we have at $u \in B_{s\delta}(y)$. If we had

$$e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_2(u'))} \Phi(v^{(n)}_2(u')) \leq e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_1(u'))} \Phi(v^{(n)}_1(u')),$$

then (5.24) is, in turn, bounded above by

$$\bigg(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{1024}\bigg) e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_1(u'))} \Phi(v^{(n)}_1(u')) + e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_2(u'))} \Phi(v^{(n)}_2(u')),$$

which is consistent with (5.21). If, on the other hand, we had

$$e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_1(u'))} \Phi(v^{(n)}_1(u')) \leq e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_2(u'))} \Phi(v^{(n)}_2(u')),$$

then we can invoke (5.23) twice to see that

$$e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_1(u))} \Phi(v^{(n)}_1(u)) \leq 2e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_1(u'))} \Phi(v^{(n)}_1(u'))$$
$$\leq 2e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_2(u'))} \Phi(v^{(n)}_2(u'))$$
$$\leq 4e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v^{(n)}_2(u))} \Phi(v^{(n)}_2(u))$$

for *any* other $u \in B_{s\delta}(y)$. Hence, (5.21) is at most

$$e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v_1^{(n)}(u))}\Phi(v_1^{(n)}(u)) + \left(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096}\right)e^{\alpha^{(n)}_{\mathrm{Re}(z)}(v_2^{(n)}(u))}\Phi(v_2^{(n)}(u))$$

and so we can make a consistent choice of $v_1^{(n)}$, $v_2^{(n)}$ on the entire ball $B_{s\delta}(y)$. Note that if (5.15) held instead, everything that followed would have been identical, with $B_{s\delta}(x)$ instead of $B_{s\delta}(y)$. □

The point of Lemma 5.4 is that, in any ball $B_\delta(x)$ of radius δ, we can always find a uniformly smaller ball $B_{s\delta}(y) \subset B_\delta(x)$ on which $\mathcal{L}^n_{z,\rho}\varphi$ is *strictly* and *uniformly* bounded away from $\mathcal{L}^n_{\mathrm{Re}(z),0}(\Phi)$. This means that we can 'bump' Φ down on any such ball $B_{\delta/4}(y)$ without affecting our inequality. Moreover, using the diametric regularity of the measure ν^u, we can ensure that we are able to do this on a set of uniformly large measure.

LEMMA 5.5. *Fix $C > 0$, $\varphi \in C^1(U, V^\rho)$ and $\Phi \in \mathcal{K}_C$. Suppose we have*

$$\|\varphi(u)\|_{L^2(G)} < \Phi(u),$$
$$\sup_{w\in T^1_u U} \|(d\varphi)_u(w)\|_{L^2(G)} < C\Phi(u)$$

for all $u \in U$, and let $\delta > 0$ be the constant guaranteed by Lemma 5.1 and $n_0 > 0$, $\epsilon > 0$ and $s < 1$ be the constants guaranteed by Lemma 5.4. For a given ρ and $z \in \mathbb{C}$ with $|\mathrm{Re}(z) - P(\varsigma)| < 1$, we can find a function $\beta \in C^1(U, \mathbb{R})$ for which we have

$$\|(\mathcal{L}^{n_0}_{z,\rho}\varphi)(u)\|_{L^2(G)} \leq (\mathcal{L}^{n_0}_{\mathrm{Re}(z),0}(\beta\Phi))(u)$$

for all $u \in U$, as well as

$$\|\mathcal{L}^{n_0}_{P(\varsigma),0}(\beta\Phi)\|_{L^2(\nu^u)} \leq \left(1 - r\frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096}\nu^u(U_{lni})\right)\|\Phi\|_{L^2(\nu^u)}$$

for some uniform constant $r < 1$. The constant r depends on n_0 and C, but not on ρ, z, φ or Φ; the function β may depend on all of these.

Proof. By the Vitali covering lemma, we can find a finite collection of points $x_1, \ldots, x_k \in U_{lni}$ so that the balls $B_\delta(x_1), \ldots, B_\delta(x_k) \subset U_{lni}$ of radius δ are pairwise disjoint, whereas the balls $B_{3\delta}(x_1), \ldots, B_{3\delta}(x_k)$ of radius 3δ cover U_{lni}. By Lemma 5.4, for each i we can find a ball $B_{s\delta}(y_i) \subset B_\delta(x_i)$ and pasts $v^{(n_0)}_{1,i}$, $v^{(n_0)}_{2,i}$ so that

$$\|e^{\alpha^{(n_0)}_z(v^{(n_0)}_{1,i}(u))}\rho(\mathrm{Hol}^{(n_0)}(v^{(n_0)}_{1,i}(u)))\varphi(v^{(n_0)}_{1,i}(u)) + e^{\alpha^{(n_0)}_z(v^{(n_0)}_{2,i}(u))}\rho(\mathrm{Hol}^{(n_0)}(v^{(n_0)}_{2,i}(u)))\varphi(v^{(n_0)}_{2,i}(u))\|_{L^2(G)}$$

is bounded above by

$$\left(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096}\right)e^{\alpha^{(n_0)}_{\mathrm{Re}(z)}(v^{(n_0)}_{1,i}(u))}\Phi(v^{(n_0)}_{1,i}(u)) + e^{\alpha^{(n_0)}_{\mathrm{Re}(z)}(v^{(n_0)}_{2,i}(u))}\Phi(v^{(n_0)}_{2,i}(u))$$

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for all $u \in B_{s\delta}(y_i)$. For each i, define a C^1 radially decreasing bump function η_i centred at $v_{1,i}^{(n_0)}(y_i)$ by

$$\eta_i(v_1^{(n_0)}(u)) = \begin{cases} \dfrac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096} & \text{if } d(u,y_i) \le \dfrac{s\delta}{2}, \\ \dfrac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096} \\ \quad \cdot \exp\left(1+\dfrac{1}{((1/s\delta)\, d(u,y_i)-(1/2))^2-1}\right) & \text{if } \dfrac{s\delta}{2} < d(u,y_i) < s\delta, \\ 0 & \text{if } s\delta \le d(u,y_i), \end{cases}$$

for all $u \in B_\delta(x_i)$. We can smoothly extend η_i to all of U by setting $\eta_i = 0$ outside $v_1^{(n_0)}(u)(B_\delta(x_i))$. To define β, we simply set

$$\beta(u) := 1 - \sum_i \eta_i(u)$$

and by Lemma 5.4 we clearly have

$$\|(\mathcal{L}_{z,\rho}^{n_0}\varphi)(u)\|_{L^2(G)} \le (\mathcal{L}_{\mathrm{Re}(z),0}^{n_0}(\beta\Phi))(u)$$

for all $u \in U$. It simply remains to estimate $\|\mathcal{L}_{P(\varsigma),0}^{n_0}(\beta\Phi)\|_{L^2(\nu^u)}$.

In principle, we have only introduced contraction into a single term in $\mathcal{L}_{P(\varsigma),0}^{n_0}(\beta\Phi)$ at any given point; we need to argue that this was significant enough for our purposes. The regularity of Φ is essential here. As in (5.1), note that the ratio between $\Phi(v_a^{(n_0)}(u))$ and $\Phi(v_b^{(n_0)}(u))$ cannot exceed

$$\frac{\Phi(v_a^{(n_0)}(u))}{\Phi(v_b^{(n_0)}(u))} \le e^{(C/f)\,\mathrm{diam}(N)}$$

for any pair of pasts v_a, v_b. As a consequence, we must have

$$\frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096(\ell_0 e^{(C/f)\,\mathrm{diam}(N)})} \sum_{v_a^{(n_0)}} \Phi(v_a^{(n_0)}(u)) \le \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096} \Phi(v_{1,i}^{(n_0)}(u)) \quad (5.25)$$

where ℓ_0 is the number of pasts $v_a^{(n_0)}$ of length n_0, and the sum is taken over all such pasts. Of course, we can find a uniform $D > 0$ so that $D^{-1} \le e^{\alpha_{\mathrm{Re}(z)}^{(n_0)}(v_a^{(n_0)}(u))} \le D$ for all pasts and all $u \in U$; note that D can be chosen independently of z so long as we require $|\mathrm{Re}(z) - P(\varsigma)| < 1$. From this, we can bound

$$\begin{aligned} &\frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096 D^2(\ell_0 e^{(C/f)\,\mathrm{diam}(N)})} \sum_{v_a^{(n_0)}} e^{\alpha_{\mathrm{Re}(z)}^{(n_0)}(v_a^{(n_0)}(u))} \Phi(v_a^{(n_0)}(u)) \\ &\quad \le \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096} e^{\alpha_{\mathrm{Re}(z)}^{(n_0)}(v_{1,i}^{(n_0)}(u))} \Phi(v_{1,i}^{(n_0)}(u)) \end{aligned}$$

following (5.25). This leads immediately to obtaining the pointwise bound

$$(\mathcal{L}^{n_0}_{P(\varsigma),0}(\beta\Phi))(u) \le \left(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096D^2(\ell_0 e^{(C/f)\,\mathrm{diam}(N)})}\right)(\mathcal{L}^{n_0}_{P(\varsigma),0}\Phi)(u)$$

for all $u \in \bigcup B_{s\delta/2}(y_i)$ by construction. On the other hand, because $\mathcal{L}^{n_0}_{P(\varsigma),0}$ is monotonic, we of course have

$$(\mathcal{L}^{n_0}_{P(\varsigma),0}(\beta\Phi))(u) \le (\mathcal{L}^{n_0}_{P(\varsigma),0}\Phi)(u)$$

for all $u \in U - \bigcup B_{s\delta/2}(y_i)$. Taken together, these inequalities mean that we can bound

$$\|(\mathcal{L}^{n_0}_{P(\varsigma),0}(\beta\Phi))\|_{L^2(\nu^u)}$$

from above by

$$\left(\int_{\bigcup B_{s\delta/2}(y_i)}\left(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096D^2(\ell_0 e^{(C/f)\,\mathrm{diam}(N)})}\right)^2((\mathcal{L}^{n_0}_{P(\varsigma),0}\Phi)(u))^2\,d\nu^u\right)^{1/2} + \left(\int_{U-\bigcup B_{s\delta/2}(y_i)}((\mathcal{L}^{n_0}_{P(\varsigma),0}\Phi)(u))^2\,d\nu^u\right)^{1/2}$$

for our particular choice of ρ and z. As the operator $\mathcal{L}^{n_0}_{P(\varsigma),0}$ preserves the measure ν^u (by our renormalization), this is exactly

$$\|\Phi\|_{L^2(\nu^u)}\left(1 - \frac{(\epsilon\delta(1+|\mathrm{Im}(z)|)\|\rho\|)^2}{4096D^2(\ell_0 e^{(C/f)\,\mathrm{diam}(N)})}\nu^u\left(\bigcup B_{s\delta/2}(y_i)\right)\right).$$

By the diametric regularity of the measure ν^u, there is a uniform constant $r < 1$ for which

$$\nu^u(B_{s\delta/2}(y_i)) \ge r\nu^u(B_{3\delta}(y_i))$$

for all i. Hence, we have

$$\nu^u\left(\bigcup B_{s\delta/2}(y_i)\right) \ge r\nu^u\left(\bigcup B_{3\delta}(y_i)\right) \ge r\nu^u(U_{lni}),$$

completing the proof. □

It is important to recognize that many of the estimates so far do, in fact, depend on ρ, $\mathrm{Im}(z)$ or C; this is problematic for the spectral bounds we want to obtain. To isolate some of these dependencies, we restrict our attention to control functions $\Phi \in \mathcal{K}_{C(1+|\mathrm{Im}(z)|)\|\rho\|}$, where we hope to be able to make a uniform choice of an appropriate C. In particular, we want to find a C so that $\mathcal{K}_{C(1+|\mathrm{Im}(z)|)\|\rho\|}$ is invariant under $\mathcal{L}^n_{z,\rho}$, at least for z with $\mathrm{Re}(z)$ sufficiently close to $P(\varsigma)$.

PROPOSITION 5.6. *There is a uniform choice of constant $C > 0$ so that, for all $\varphi \in C^1(U, V^\rho)$ and $\Phi \in \mathcal{K}_{C(1+|\mathrm{Im}(z)|\|\rho\|}$, if we have*

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$$\|\varphi(u)\|_{L^2(G)} \leq \Phi(u),$$

$$\sup_{w \in T_u^1 U} \|(d\varphi)_u(w)\|_{L^2(G)} \leq C(1 + |\mathrm{Im}(z)|)\Phi(u)$$

for all $u \in U$, then we can bound

$$\sup_{w \in T_u^1 U} \|(d(\mathcal{L}_{z,\rho}^n \varphi))_u(w)\|_{L^2(G)} \leq C(1 + |\mathrm{Im}(z)|)\|\rho\|(\mathcal{L}_{\mathrm{Re}(z),0}^n \Phi)(u)$$

for all $u \in U$, and all $n > 0$.

Proof. Fix $u \in U$. By definition, the transfer operator can be expressed as the sum

$$(\mathcal{L}_{z,\rho}^n \varphi)(u) = \sum_i e^{\alpha_z^{(n)}(v_i^{(n)}(u))} \rho(\mathrm{Hol}^{(n)}(v_i^{(n)}(u)))\varphi(v_i^{(n)}(u)) \tag{5.26}$$

over pasts $v_i^{(n)}$. We need to control

$$\sup_{w \in T_u^1 U} \|d(\mathcal{L}_{z,\rho}^n(\varphi))_u(w)\|_{L^2(G)},$$

which we will accomplish by differentiating (5.26) term-by-term. For a fixed i, the derivative

$$d(e^{\alpha_z^{(n)}(v_i^{(n)}(u))} \rho(\mathrm{Hol}^{(n)}(v_i^{(n)}(u)))\varphi(v_i^{(n)}(u)))_u(w)$$

can be bounded by the sum

$$\|e^{\alpha_z^{(n)}(v_i^{(n)}(u))}(\rho(\mathrm{Hol}^{(n)}(v_i^{(n)}(u)))\varphi(v_i^{(n)}(u)))(d(\alpha_z^{(n)} \circ v_i^{(n)}))_u(w)\|_{L^2(G)} \tag{5.27}$$

$$+ \|e^{\alpha_z^{(n)}(v_i^{(n)}(u))}((d(\rho(\mathrm{Hol}^{(n)} \circ v_i^{(n)})))_u(w))\varphi(v_i^{(n)}(u))\|_{L^2(G)} \tag{5.28}$$

$$+ \|e^{\alpha_z^{(n)}(v_i^{(n)}(u))}\rho(\mathrm{Hol}^{(n)}(v_i^{(n)}(u)))(d(\varphi \circ v_i^{(n)}))_u(w)\|_{L^2(G)} \tag{5.29}$$

for all $w \in T_u^1 U$. We bound each of these terms individually, beginning with (5.27). Observe that

$$\begin{aligned}(d(\alpha_z^{(n)} \circ v_i^{(n)}))_u(w) &= \sum_{j=1}^{n} |(d(\alpha_z \circ v_i^{(j)}))_u(w)| \\ &\leq \sum_{j=1}^{n} \frac{\|\alpha_z\|_{C^1}}{f\kappa^j} \\ &\leq \frac{\|\alpha_z\|_{C^1}}{f(\kappa - 1)}\end{aligned}$$

and so (5.27) is at most

$$\frac{\|\alpha_z\|_{C^1}}{f(\kappa - 1)} e^{\alpha_{\mathrm{Re}(z)}^{(n)}(v_i^{(n)}(u))} \|\varphi(v_i^{(n)}(u))\|_{L^2(G)}$$

for all $u \in U$ and $w \in T_u^1 U$. Exactly the same calculations show us that

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$$\begin{aligned}\|(d(\mathrm{Hol}^{(n)} \circ v_i^{(n)}))_u(w)\|_{T^1G} &\leq \sum_{j=1}^{n} \|(d(\mathrm{Hol} \circ v_i^{(j)}))_u(w)\|_{T^1G} \\ &\leq \sum_{j=1}^{n} \frac{\|\ \mathrm{Hol}\ \|_{C^1}}{f\kappa^j} \\ &\leq \frac{\|\ \mathrm{Hol}\ \|_{C^1}}{f(\kappa-1)}\end{aligned}$$

and, hence, that (5.28) is at most

$$\bigg(\|\rho\| \frac{\|\ \mathrm{Hol}\ \|_{C^1}}{f(\kappa-1)}\bigg) e^{\alpha_{\mathrm{Re}(z)}(v_i^{(n)}(u))} \|\varphi(v_i^{(n)}(u))\|_{L^2(G)}$$

by the definition of $\|\rho\|$ and the chain rule. Finally, we can bound (5.29) by

$$\frac{C(1+|\mathrm{Im}(z)|)\|\rho\|}{f\kappa^n} e^{\alpha_{\mathrm{Re}(z)}(v_i^{(n)}(u))} \Phi(v_i^{(n)}(u))$$

by hypothesis. Combining all of these with the pointwise bound on the norm of φ, we see that so long as we have

$$\frac{\|\alpha_z\|_{C^1}}{f(\kappa-1)} + \|\rho\| \frac{\|\ \mathrm{Hol}\ \|_{C^1}}{f(\kappa-1)} + \frac{C(1+|\mathrm{Im}(z)|)\|\rho\|}{f\kappa^n} < C(1+|\mathrm{Im}(z)|)\|\rho\|, \tag{5.30}$$

we obtain the bounds desired. Note that $\|\ \mathrm{Hol}\ \|_{C^1}$ is a uniform constant and $\|\alpha_z\|_{C^1}$ is at most $\|\alpha_{\mathrm{Re}(z)}\|_{C^1}(1+|\mathrm{Im}(z)|)$. As $\|\alpha_{\mathrm{Re}(z)}\|_{C^1}$ is uniformly bounded when $\mathrm{Re}(z)$ is confined to the bounded interval $|\mathrm{Re}(z) - P(\varsigma)| < 1$, we can clearly choose a $C > 0$ so that (5.30) holds for all $n > 0$. □

We are now in a position to prove Theorem 3.2; the strategy is entirely analogous to that outlined in [11, §6] or [17, Theorem 6.8].

THEOREM 5.7. *Fix a non-trivial isotypic representation $\rho > 0$ and $z \in \mathbb{C}$ with $|\mathrm{Re}(z) - P(\varsigma)| < 1$. There are uniform constants $D > 0$ and $r_0 < 1$ so that*

$$\|\mathcal{L}_{z,\rho}^n \varphi\|_{L^2(\nu^u)} \leq D r_0^n \|\varphi\|_{C^1}$$

for all $\varphi \in C^1(U, V^\rho)$. Neither D nor r_0 depends on ρ, φ or z.

Proof. Fix $C > 0$, $N > 0$ as in Lemma 5.6 and $n_0 > N$ as in Lemma 5.5. We begin by setting

$$\begin{aligned}\varphi_0(u) &:= \varphi(u), \\ \Phi_0(u) &:= \|\varphi\|_{C^1}\end{aligned}$$

for which we clearly have $\Phi_0 \in \mathcal{K}_{C(1+|\mathrm{Im}(z)|)\|\rho\|}$ as well as the bounds

$$\begin{aligned}\|\varphi_0(u)\|_{L^2(G)} &\leq \Phi_0(u), \\ \sup_{w \in T_u^1 U} \|(d\varphi_0)_u(w)\|_{L^2(G)} &\leq C(1+|\mathrm{Im}(z)|)\|\rho\|\Phi_0(u)\end{aligned}$$

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assuming that $C > 0$ is large enough so that $C\|\rho\| > 1$ for all ρ. By Lemma 5.5, we can find a function $\beta_0 \in C^1(U, \mathbb{R})$ for which we have

$$\|(\mathcal{L}_{z,\rho}^{n_0}\varphi_0)(u)\|_{L^2(G)} \leq (\mathcal{L}_{\mathrm{Re}(z),0}^{n_0}(\beta_0\Phi_0))(u)$$

and

$$\|\mathcal{L}_{\mathrm{Re}(z),0}^{n_0}(\beta_0\Phi_0)\|_{L^2(\nu^u)} \leq r_0\|\Phi_0\|_{L^2(\nu^u)}$$

for a uniform choice of $n_0 > 0$. Moreover, r_0 and $\|\beta_0\|_{C^1}$ can be made uniform in ρ, φ and z by choosing δ so that $C(1 + |\mathrm{Im}(z)|)\|\rho\| \cdot \delta$ is constant, as in Lemma 5.1.

We want to iterate this, for which it will be crucial that we can find a uniform C_0 so that we have $\beta_0\Phi_0 \in \mathcal{K}_{C_0(1+|\mathrm{Im}(z)|)\|\rho\|}$ for all $\Phi_0 \in \mathcal{K}_{C_0(1+|\mathrm{Im}(z)|)\|\rho\|}$. For any given $\Phi_0 \in \mathcal{K}_{C(1+|\mathrm{Im}(z)|)\|\rho\|}$, we have

$$\begin{aligned}\sup_{w\in T_u^1U} |(d(\Phi_0\beta_0)_u(w)| &\leq \beta_0(u)(\sup_{w\in T_u^1U} |(d\Phi_0)_u(w)|) + \Phi_0(u)(\sup_{w\in T_u^1U} |(d\beta_0)_u(w)|) \\ &\leq \sup_{w\in T_u^1U} \Phi_0(u)(C(1 + |\mathrm{Im}(z)|)\|\rho\|\beta_0(u) + (d\beta_0)_u(w))\end{aligned}$$

for all $u \in U$. From the construction of β_0, and by our earlier remarks on our choice of δ, it is clear that we can choose a uniform C_0 large enough so that

$$\sup_{w\in T_u^1U} C(1 + |\mathrm{Im}(z)|)\|\rho\|\beta_0(u) + (d\beta_0)_u(w) \leq C_0\beta_0(u)$$

for any choice of β_0 as in Lemma 5.5. We then clearly have $\beta_0\Phi_0 \in \mathcal{K}_{C_0(1+|\mathrm{Im}(z)|)\|\rho\|}$, and, hence, by Lemma 5.6 we obtain

$$\sup_{w\in T_u^1U} \|(d(\mathcal{L}_{z,\rho}^{n_0}\varphi))_u(w)\|_{L^2(G)} \leq C_0(1 + |\mathrm{Im}(z)|)\|\rho\|(\mathcal{L}_{\mathrm{Re}(z),0}^{n}\Phi_0)(u)$$

for all $u \in U$.

Now, we can repeat what we have done so far and inductively choose β_{i-1} as in Lemma 5.5, setting

$$\begin{aligned}\varphi_i(u) &:= (\mathcal{L}_{z,\rho}^{n_0}\varphi_{i-1})(u), \\ \Phi_i(u) &:= (\mathcal{L}_{\mathrm{Re}(z),0}^{n_0}(\beta_{i-1}\Phi_{i-1}))(u)\end{aligned}$$

for $i \geq 1$. Note that we have just shown that $\Phi_1 \in \mathcal{K}_{C_0(1+|\mathrm{Im}(z)|)\|\rho\|}$ and that we have

$$\begin{aligned}\|\varphi_1(u)\|_{L^2(G)} &\leq \Phi_1(u), \\ \sup_{w\in T_u^1U} \|(d\varphi_1)_u(w)\|_{L^2(G)} &\leq C_0(1 + |\mathrm{Im}(z)|)\|\rho\|\Phi_1(u)\end{aligned}$$

for all $u \in U$. As before, we obtain

$$\begin{aligned}\|\varphi_i(u)\|_{L^2(G)} &\leq \Phi_i(u), \\ \sup_{w\in T_u^1U} \|(d\varphi_i)_u(w)\|_{L^2(G)} &\leq C_0(1 + |\mathrm{Im}(z)|)\|\rho\|\Phi_i(u)\end{aligned}$$

https://doi.org/10.1017/etds.2021.162 Published online by Cambridge University Press

inductively for all $i \geq 1$ and, moreover,

$$\|\Phi_i\|_{L^2(\nu^u)} \leq r_0 \|\Phi_{i-1}\|_{L^2(\nu^u)}$$

by construction. Chaining these inequalities together, we have

$$\|\mathcal{L}_{z,\rho}^{i\cdot n_0}\varphi\|_{L^2(\nu^u)} \leq r_0^i \|\varphi\|_{C^1},$$

which is almost what we need. To conclude, observe that we have

$$\|(\mathcal{L}_{z,\rho}^{i\cdot n_0+k}\varphi)(u)\|_{L^2(\nu^u)} \leq r_0^i \|\varphi\|_{C^1} \|\mathcal{L}_{z,\rho}^k\|_{L^2(G)},$$

where $\|\mathcal{L}_{z,\rho}^k\|_{L^2(G)}$ denotes the operator norm of $\mathcal{L}_{z,\rho}^k$. If $k < n_0$ and $|\mathrm{Re}(z) - P(\varsigma)| < 1$, then we can find a uniform bound $D > 0$ so that $\|\mathcal{L}_{z,\rho}^k\|_{L^2(G)} \leq D$, as desired. □

The differentiability of the potential ς was essential to much of what we have done so far in this section; to extend our results to the case when ς is only Hölder, however, is a relatively straightforward approximation argument, identical to that given in [11, §4(II)]. We sketch this in the following.

COROLLARY 5.8. *With notation as in Theorem 5.7, we have*

$$\|\mathcal{L}_{\mathrm{Re}(z),\rho}^n\varphi\|_{L^2(\nu^u)} \leq D r_0^n \|\varphi\|_{C^1}$$

when the potential ς is only Hölder continuous.

Proof. Suppose that ς is Hölder continuous. As in [11], we can find a sequence of smooth potentials $\varsigma^{(\mathrm{Im}(z))} \in C^1(U, \mathbb{R})$ indexed by $\mathrm{Im}(z)$ for which we have

$$\sup_{u\in U} |\varsigma_0(u) - \varsigma^{(\mathrm{Im}(z))}(u)| \leq \|\varsigma_0\|_{C^\alpha} \left(\frac{1}{|\mathrm{Im}(z)|}\right)^{\alpha/2} \tag{5.31}$$

and

$$\|\varsigma^{(\mathrm{Im}(z))}\|_{C^1} < C\sqrt{|\mathrm{Im}(z)|} \tag{5.32}$$

for some uniform constant $C > 0$. We consider the alternatively defined transfer operator

$$(\hat{\mathcal{L}}_{z,\rho}\varphi)(u) := \sum_{\sigma(u')=u} e^{\hat{\alpha}_z(u')}(\rho(\mathrm{Hol}(u')) \cdot \varphi(u')),$$

where we set

$$\hat{\alpha}_z(u) := \varsigma^{(\mathrm{Im}(z))}(u) - \mathrm{Re}(z)\cdot\tau(u,s) - \log(\varphi_{\varsigma^{(x)}}(u)) + \log(\varphi_{\varsigma^{(x)}}(\sigma(u))) - \log P(\varsigma^{(x)})$$

for $u \in U$. By (5.31), $\hat{\alpha}_z$ converges uniformly to $\alpha_{\mathrm{Re}(z)}$ as $\mathrm{Im}(z) \to \infty$ and, hence, $\hat{\mathcal{L}}_{z,\rho}\varphi$ must converge to $\mathcal{L}_{\mathrm{Re}(z),\rho}\varphi$ in the $L^2(\nu^u)$ norm.

Now, we simply observe that the spectral bound in Theorem 5.7 holds for the operator $\hat{\mathcal{L}}_{z,\rho}$, because the main properties required of $\hat{\alpha}_z$, namely that $\|\hat{\alpha}_z\|_{C^1} \leq C(1 + |\mathrm{Im}(z)|)$ for large $|\mathrm{Im}(z)|$ when $|\mathrm{Re}(z) - P(\varsigma)| \leq 1$, are guaranteed by (5.32) and [11, Lemma 1]. Moreover, note that all the constants, and particularly those originating from Proposition 5.6, can be chosen uniformly in $\mathrm{Im}(z)$. As the inequality in Theorem 5.7 holds with the same constants for $\hat{\mathcal{L}}_{z,\rho}$ for each $\mathrm{Im}(z)$, it must hold in the limit, as desired. □

Acknowledgements. I am indebted to my advisor, Ralf Spatzier, for his tireless and persistent support, and to Amie Wilkinson for several brief conversations that proved prescient. I am grateful to Mitul Islam for many productive exchanges, and to Clark Butler, Jon DeWitt and Mark Pollicott for pointing out errors in an earlier version of this paper. This work was supported in part by the National Science Foundation, under grant DMS 1607260.

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