Canad. J. Math. Vol. 71 (2), 2019 pp. 417–435
http://dx.doi.org/10.4153/CJM-2018-028-6

The Steklov Problem on Differential Forms

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Abstract. In this paper we study spectral properties of the Dirichlet-to-Neumann map on differential forms obtained by a slight modification of the definition due to Belishev and Sharafutdinov. The resulting operator Λ is shown to be self-adjoint on the subspace of coclosed forms and to have purely discrete spectrum there. We investigate properties of eigenvalues of Λ and prove a Hersch–Payne–Schiffer type inequality relating products of those eigenvalues to eigenvalues of the Hodge Laplacian on the boundary. Moreover, non-trivial eigenvalues of Λ are always at least as large as eigenvalues of the Dirichlet-to-Neumann map defined by Raulot and Savo. Finally, we remark that a particular case of p-forms on the boundary of a $2p+2$-dimensional manifold shares many important properties with the classical Steklov eigenvalue problem on surfaces.

1 Introduction

Let M be a compact Riemannian manifold of dimension n with smooth boundary ∂M. Recently, there has been much research dedicated to the Steklov eigenvalue problem that is defined in the following way. Number σ is called a *Steklov eigenvalue* of M provided there exists a non-zero solution $u \in C^\infty(M)$ to the following problem:

$$\begin{cases} \Delta u = 0 \text{ on } M, \\ \partial_n u = \sigma u \text{ on } \partial M, \end{cases}$$

where ∂_n stands for the derivative with respect to the unit outer normal vector.

Steklov eigenvalues coincide with eigenvalues of the Dirichlet-to-Neumann operator $\mathcal{D}\colon C^\infty(\partial M) \to C^\infty(\partial M)$. The operator $\mathcal{D}$ sends a function v to the normal derivative of its harmonic extension. Then $\mathcal{D}$ is a self-adjoint elliptic pseudo-differential operator of order 1, *i.e.*, Steklov eigenvalues form a sequence tending to $+\infty$. For details, we refer the reader to [6] and the references therein.

In the present paper we study Steklov eigenvalues on the space of differential forms on M. Several definitions of the Dirichlet-to-Neumann operator are present in the literature [1, 9, 13]. The definition commonly used in spectral theory literature is due to Raulot and Savo [13] and has the advantage of being a positive elliptic self-adjoint pseudo-differential operator of order 1. However, in the literature on inverse problems, different definitions of the Dirichlet-to-Neumann map are used; see the full Dirichlet-to-Neumann map in [9, 16] and the definition due to Belishev and Sharafutdinov [1] that motivated by Maxwell's equations. In the present paper we modify the latter to obtain a self-adjoint operator with a purely discrete spectrum and study its eigenvalues. We plan to tackle spectral theory of the full Dirichlet-to-Neumann map in a subsequent article.

Received by the editors July 24, 2017; revised June 3, 2018.
Published electronically August 31, 2018.
AMS subject classification: **58J50**, 58J32, 35P15.
Keywords: Dirichlet-to-Neumann map, differential form, Steklov eigenvalue, shape optimization.

2 Main Results

2.1 Notations

In the following, (M,g) is always assumed to be a smooth compact orientable manifold of dimension n with smooth nonempty boundary ∂M. It seems that orientability is a purely technical condition that could be eliminated with further investigation. However, Theorem 2.7 requires orientability in an essential way.

Let (X,h) be a compact Riemannian manifold, possibly with boundary. The space of smooth differential p-forms on X will be denoted by $\Omega^p(X)$. We denote the spaces of smooth exact and closed p-forms by

$$\mathcal{E}^p(X) \subset \mathcal{C}^p(X) \subset \Omega^p(X),$$

respectively. A letter c in front of either of them denotes the prefix "co-", concatenation of the letters stands for intersection, *e.g.*, $\mathcal{C}c\mathcal{C}^p(X)$ is the space of closed and coclosed p-forms that will be denoted by $\mathcal{H}^p(X)$ in the following. If $\partial X = \varnothing$, then $\mathcal{H}^p(X)$ coincides with the space of harmonic forms, *i.e.*, the kernel of the Hodge–Laplace operator.

However, if $\partial X \neq \varnothing$, those spaces are different, and we refer to elements of $\mathcal{H}^p(X)$ as *harmonic fields* and reserve the term *harmonic form* for elements of $\ker\Delta$. Let $i\colon \partial X \to X$ be an embedding of the boundary and let i_n denote the contraction of a differential form with the outer unit normal vector field. The form $\omega \in \Omega^p(X)$ satisfies the Dirichlet (resp. Neumann) boundary condition if $i^*\omega = 0$ (resp. $i_n\omega = 0$). We use subscripts D and N to indicate spaces of forms satisfying Dirichlet or Neumann boundary conditions. Finally, for $\omega \in \Omega^p(X)$, we denote by $\mathbf{t}\omega, \mathbf{n}\omega \in \Gamma(\Omega^p(X)|_{\partial X})$ the tangent and normal parts of ω on the boundary, *i.e.*, $\mathbf{t}\omega$ is $i^*\omega$ considered as a section of $\Omega^p(X)|_{\partial X}$ and $\mathbf{n}\omega = dn \wedge i_n\omega$, where dn is a 1-form, dual to the outer unit normal vector field. In practice, the only difference between $\mathbf{n}\omega$ and $i_n\omega$ is the way the Hodge $*$-operator acts on them; see Proposition 3.1. In the following we use i^* exclusively to denote the pullback of differential forms via map i as opposed to the pullback of sections. For that reason, we write $\Omega^p(X)|_{\partial X}$ for the restriction of the bundle.

For a subspace $V \subset \Omega^p(X)$ we denote by $H^sV \subset H^s\Omega^p(X)$ the completion of V with respect to the Sobolev H^s-norm. We write L^2 instead of H^0. For details on Sobolev norms, see [15, §1.3]. We use angle brackets $\langle\cdot,\cdot\rangle$ to denote the pointwise L^2-inner product, double angle brackets $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ to denote the integrated L^2-inner product, round brackets $(\cdot,\cdot)$ to denote the $H^{-s}\times H^s \to \mathbb{R}$ duality pairing and $\|\cdot\|_{H^s}$ to denote H^s-norm. Usually, it is clear from the context whether we are working on the boundary or on the manifold itself. In cases where it needs clarification, we add subscripts indicating the ambient space, *e.g.*, $\|\cdot\|_{L^2(X)}$ or $\|\cdot\|_{H^{1/2}(\partial X)}$.

Finally, we note that for manifolds with boundary, Green's formula states that for $\alpha,\beta \in H^1\Omega^p(M)$

$$\begin{aligned}
\int_M \langle d\alpha,\beta\rangle\, dV &= \int_M \langle\alpha,\delta\beta\rangle\, dV + \int_{\partial M}\langle i^*\alpha, i_n\beta\rangle\, dA \\
&= \int_M \langle\alpha,\delta\beta\rangle + \int_{\partial M} i^*\alpha\wedge *\mathbf{n}\beta.
\end{aligned} \tag{2.1}$$

2.2 Maxwell's Equations

In the modern form, Maxwell's equations are usually written in the language of differential forms on an orientable 3-dimensional Riemannian manifold [20]. In the exposition below, we follow [11]. Maxwell's equations have the following form:

$$\begin{aligned} d\mathcal{E} &= -\partial_t \mathcal{B}, & d\mathcal{H} &= \partial_t \mathcal{D}, \\ \mathcal{D}(x,t) &= *_\epsilon \mathcal{E}(x,t), & \mathcal{B}(x,t) &= *_\mu \mathcal{H}(x,t), \\ d\mathcal{B} &= 0, & d\mathcal{D} &= 0, \end{aligned}$$

where $\mathcal{E}$ and $\mathcal{H}$ are 1-forms corresponding to electric and magnetic fields, $\mathcal{B}$ and $\mathcal{D}$ are 2-forms corresponding to magnetic flux density and electric displacements, $*_\epsilon$ and $*_\mu$ are Hodge operators for some metrics corresponding to electric permittivity and magnetic permeability. When the 3-manifold has a boundary, there is a natural response operator R that sends the component of the electric field tangent to the boundary to the component of the magnetic field tangent to the boundary. In paper [11] the authors studied the inverse problem of recovering the manifold M given the response operator.

Consider the simplest case $*_\epsilon = *_\mu = *$ and the time-harmonic solution to Maxwell's equations, *i.e.*, the t variable is separated and solutions depend on t only via factor e^{ikt} for a fixed angular frequency $k \in \mathbb{R}$. Then Maxwell's equations for $\mathcal{E}$ and $\mathcal{B}$ becomes

$$-ik\mathcal{B} = d\mathcal{E}, \quad d * \mathcal{B} = ik * \mathcal{E}, \quad d\mathcal{B} = 0.$$

In terms of $\mathcal{E}$ this system has the form

$$\Delta\mathcal{E} = k^2\mathcal{E}, \quad \delta\mathcal{E} = 0, \tag{2.2}$$

and the response operator sends $\mathbf{t}\mathcal{E} \mapsto \mathbf{t} * \mathcal{B} = \frac{i}{k} * \mathbf{n} d\mathcal{E}$, *i.e.*, it connects $\mathbf{t}\mathcal{E}$ with $\mathbf{n} d\mathcal{E}$. In the next section we use this calculation to motivate the definition of the Dirichlet-to-Neumann map on differential forms for Riemannian manifolds of arbitrary dimension.

2.3 Definition and Basic Properties

Let M be a compact orientable manifold with smooth non-empty boundary ∂M. Motivated by the particular case $k = 0$ of (2.2), we define the Dirichlet-to-Neumann operator Λ acting on the space of differential forms $\Omega^p(\partial M)$ in the following way. First, the Hodge Laplacian on $\Omega^p(M)$ is defined by the formula $\Delta = d\delta + \delta d$, where δ is the formal adjoint of d with respect to the metric on $\Omega^p(M)$ induced by g. Then for any $\phi \in \Omega^p(\partial M)$, consider the equations

$$\Delta\omega = 0, \quad \delta\omega = 0, \quad i^*\omega = \phi. \tag{2.3}$$

Let us denote the space of solutions ω by $\mathcal{L}(\phi)$. In Proposition 3.11 we prove that $\mathcal{L}(\phi)$ is an affine vector space with an associated vector space $\mathcal{H}^p_D(M)$. We set $\Lambda\phi := i_n d\omega$ for any $\omega \in \mathcal{L}(\phi)$. Since $d\mathcal{H}^p_D(M) = 0$, the definition does not depend on the choice of ω. Let us denote by $\lambda(\phi) \in \mathcal{L}(\phi)$ the unique solution of (2.3) satisfying $\lambda(\phi) \perp \mathcal{H}^p_D(M)$.

Remark 2.1 In [1] the Dirichlet-to-Neumann map is defined up to a sign as $*\Lambda$.

Remark 2.2 Having in mind equation (2.2), it is more natural to consider the operator $\Lambda(\lambda)$ for $\lambda \in \mathbb{R}$ defined in the same way as Λ, but instead of (2.3) one requires ω to be the solution of

$$\Delta\omega = \lambda\omega, \quad \delta\omega = 0, \quad i^*\omega = \phi.$$

However, the study of $\Lambda(\lambda)$ for $\lambda \neq 0$ exceeds the scope of the present article.

Our starting point is the following theorem.

Theorem 2.3 *Operator Λ is identically zero on the space $\mathcal{E}^p(\partial M)$. Restricted to the space $c\mathcal{C}^p(\partial M)$, it is a non-negative self-adjoint operator with compact resolvent. In particular, its spectrum is discrete and is denoted by $0 \leqslant \sigma_1^{(p)} \leqslant \sigma_2^{(p)} \leqslant \cdots \nearrow \infty$, where the eigenvalues are written with multiplicity and all multiplicities are finite. The kernel satisfies* $\ker\Lambda \cap c\mathcal{C}^p(\partial M) = i^*\mathcal{H}_N^p(M) \cap c\mathcal{C}^p(\partial M)$ *and has dimension* $I_p = \dim\operatorname{im}\{i^*: H^p(M) \to H^p(\partial M)\}$.

Moreover, the eigenvalues can be characterised by the following min-max formula,

$$\sigma_k^{(p)} = \max_E \min_{\phi \perp E;\, i^*\widehat{\phi} = \phi} \frac{\|d\widehat{\phi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}},$$

where E runs over all $(k-1)$-dimensional subspaces of $c\mathcal{C}^p(\partial M)$. The maximum is achieved for $E = V_{k-1}$, where V_{k-1} is spanned by the first $(k-1)$ eigenforms, ϕ being the k-th eigenform and $\widehat{\phi} \in \mathcal{L}(\phi)$.

Remark 2.4 An alternative way to prove the first part of Theorem 2.3 is to show that $\Lambda|_{c\mathcal{C}^p}$ is an elliptic pseudo-differential operator. We intend to explore this route in a subsequent paper.

2.4 Main Results

Our main results are concerned with properties of eigenvalues of $\sigma_k^{(p)}$. First, we prove a comparison theorem between eigenvalues of Λ and eigenvalues of the Dirichlet-to-Neumann map L defined by Raulot and Savo [13]. For any $\phi \in \Omega^p(\partial M)$, there exists a unique solution ω to the following problem [15, Theorem 3.4.10]:

$$\Delta\omega = 0, \quad i_n\omega = 0, \quad i^*\omega = \phi.$$

Then $L(\phi)$ is defined to be equal to $i_n d\omega$. Moreover, L is an elliptic pseudo-differential operator of order 1, so its spectrum is discrete and is denoted by

$$0 \leqslant \mu_1^{(p)} \leqslant \mu_2^{(p)} \leqslant \cdots \nearrow \infty.$$

We also use notations $\tilde{\mu}_i^{(p)}$ and $\tilde{\sigma}_i^{(p)}$ to denote the i-th *non-zero* eigenvalue of the corresponding operator.

Theorem 2.5 *Let M be a compact orientable Riemannian manifold of dimension n with boundary. Then for each $0 \leqslant p \leqslant n-2$ and all $k \in \mathbb{N}$, one has $\tilde{\mu}_k^{(p)} \leqslant \tilde{\sigma}_k^{(p)}$.*

Remark 2.6 Let us note that $c\mathcal{C}^{n-1}(\partial M) = \mathcal{H}^{n-1}(\partial M)$ is one-dimensional and from the long exact cohomology sequence of the pair $(M, \partial M)$

$$\cdots \longrightarrow H^{n-1}(M) \longrightarrow H^{n-1}(\partial M) \longrightarrow H^n(M, \partial M) \longrightarrow H^n(M) \longrightarrow 0,$$

one sees that $I_{n-1} = 1$, *i.e.*, $\Lambda \equiv 0$ on $\Omega^{n-1}(\partial M)$.

Recently, there have been several papers [10, 12–14, 17, 18, 21, 22] concerned with estimates for eigenvalues $\tilde{\mu}_k^{(p)}$. Most proofs of upper bounds in these papers can be modified to yield upper bounds for $\sigma_k^{(p)}$. In a sense, proofs of those bounds implicitly make use of Theorem 2.5. In our last theorem, we illustrate that by proving a generalisation of results of Yang and Yu [21].

Theorem 2.7 *Let M be a compact oriented n-dimensional Riemannian manifold with nonempty boundary. Then for any two positive integers m and r and for any $p = 0, \dots, n-2$, one has*

$$\sigma_{m+I_p}^{(p)} \sigma_{r+I_{n-2-p}}^{(n-2-p)} \leqslant \lambda'^{(p)}_{I_p+m+r+b_{n-p-1}-1}, \tag{2.4}$$

where $\lambda'^{(p)}_k$ is the k-th eigenvalue of the Hodge-Laplace operator on the space $c\mathcal{C}^p(\partial M)$.

Remark 2.8 The theorem of Yang and Yu can be obtained from Theorem 2.7 by setting $p = 0$ and applying Theorem 2.5 to the left-hand side. For details, see Section 7.

Remark 2.9 It will be shown in Section 8 that inequality (2.4) is sharp on the Euclidean ball at least for $m, r = 1$. In fact, it is sharp for a wider range of values of m, r; see Section 8 for details.

2.5 Discussion

In this section we discuss a particular case of n even and $p = \frac{n}{2} - 1$.

Proposition 2.10 *Let $n = 2p+2$ and consider operator Λ on the space $\Omega^p(\partial M)$. Then the eigenvalues $\sigma_k^{(p)}$ are invariant under conformal changes of metric with conformal factor identically equal to* 1 *on the boundary.*

Proof The Rayleigh quotient $\frac{\|d\widehat{\phi}\|_{L^2(M)}}{\|\phi\|_{L^2(\partial M)}}$ is invariant under conformal changes of the metric described in the statement. ∎

The case $n = 2$, $p = 0$ corresponds to Steklov eigenvalues on surfaces where conformal invariance is well known. Moreover, under the same relation between n and p, the left-hand side of the bound in Theorem 2.7 only contains the eigenvalues $\sigma^{(p)}$. In particular, setting $m = r$ yields the following theorem.

Theorem 2.11 *Let M be a compact oriented $(2p+2)$-dimensional Riemannian manifold with nonempty boundary. Then for any $m > 0$, one has the inequality*

$$\left(\sigma_{m+I_p}^{(p)}\right)^2 \leqslant \lambda'^{(p)}_{I_p+b_{p+1}+2m-1}. \tag{2.5}$$

The case $n = 2$, $p = 0$ corresponds to a particular case of the Hersch–Payne–Schiffer inequality that is sharp on the disk for all m [4].

From explicit computations of Λ on the unit ball given in Section 8, one can see that inequality (2.5) is sharp on the ball for $m \leqslant \frac{1}{2}\binom{2p+2}{p+1}$. It will be interesting to see if the unit ball is the unique manifold with this property.

Conjecture 2.12 *Suppose that for manifold M inequality (2.5) becomes an equality for $m \leqslant \frac{1}{2}\binom{2p+2}{p+1}$. Then M is a Euclidean ball.*

Moreover, it seems that by using methods similar to the ones developed in [4], it is possible to show that the inequality in Theorem 2.11 is sharp on the ball for all values of m. We formulate it as a conjecture.

Conjecture 2.13 *Inequality (2.5) is sharp for all values of m. To be more precise, for any m and p there exists a sequence M_k of orientable Riemannian manifolds with boundary such that the left-hand side of inequality (2.5) tends to the right-hand side as $k \to \infty$. Moreover, manifolds M_k can be chosen to be a collection of $N = N(m,p)$ Euclidean balls of equal radii glued together in the right way.*

Previous remarks indicate that eigenvalues $\sigma^{(p)}$ for $(2p+2)$-dimensional manifold M have many features similar to Steklov eigenvalues for surfaces. There is a vast literature devoted to the geometric optimisation problem for Steklov eigenvalues [2–4, 6, 10]. Here we propose a similar problem for eigenvalues $\sigma^{(p)}$. Fix an oriented closed Riemannian manifold (Σ, h) of dimension $2p+1$. Assume that the orientable bordism class of Σ is trivial, *i.e.*, there exists an orientable manifold W such that $\partial W = \Sigma$. Denote by $[\Sigma, h]_m$ the set of all orientable Riemannian manifolds (W, g) such that $\partial W = \Sigma$, $g|_{\partial W} = h$, and $b_{p+1} = m$. According to Theorem 2.11, for any element of $[\Sigma, h]_m$, the eigenvalue $\sigma_k^{(p)}$ is bounded from above by a quantity depending only on (Σ, h) and m. For fixed k, m it would be interesting to understand the quantity $\sup_{[\Sigma,h]_m} \sigma_k^{(p)}$. As we pointed out above, for $(\Sigma, h) = (\mathbb{S}^{2p+1}, g_{\mathrm{can}})$ and $m = 0$, Theorem 2.11 yields a sharp bound for the first several values of k and the supremum is attained for $(W, g) = (\mathbb{B}^{2p+2}, g_{\mathrm{can}})$.

2.6 Organisation of the Paper

The paper is organised in the following way. In Section 3 we show preliminary properties of Λ that were essentially demonstrated in [1]. In Section 4 we prove that Λ is an operator with compact resolvent and Section 5 contains the corresponding variational formulae. Sections 6 and 7 are devoted to proofs of Theorem 2.5 and Theorem 2.7, respectively. Finally, in Section 8 we compute the eigenbasis of Λ in the case of the unit ball in $\mathbb{R}^{n+1}$.

3 Preliminaries

3.1 The Hodge–Morrey–Friedrichs decomposition

The cornerstone of our considerations is the Hodge decomposition for manifolds with boundary. First, let us record an elementary result that can be proved by computation in local coordinates.

Proposition 3.1 *One has the following equalities:* $\mathbf{n}\delta = \delta\mathbf{n}$, $\mathbf{t}d = d\mathbf{t}$, *and* $*\mathbf{n} = \mathbf{t}*$. *Equivalently,* $i_n\delta = \pm\delta i_n$, $i^*d = di^*$, *and* $*i_n = \pm i^* *$.

Remark 3.2 It is possible to calculate the exact signs in the above expressions that will depend on the degree of the form and dimension of the manifold. However, the signs are not needed in the following and would make the exposition more cumbersome.

This proposition, together with Green's formula (2.1), clarifies the following theorem.

Theorem 3.3 (Hodge–Morrey–Friedrichs decomposition [15]) *Let M be a compact orientable manifold with non-empty boundary. Then the space of differential p-forms on M admits the following decomposition into a direct sum*

$$\Omega^p(M) = d\Omega_D^{p-1}(M) \oplus \delta\Omega_N^{p+1}(M) \oplus \mathcal{H}^p(M).$$

Note that boundary conditions are taken before applying the operator so that

$$d\Omega_D^{p-1}(M) = \{\omega \in \Omega^p(M) \mid \omega = d\alpha, i^*\alpha = 0\}.$$

The space of harmonic fields $\mathcal{H}^p(M)$ *can be further decomposed in one of two different ways:*

$$\mathcal{H}^p(M) = \mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}_N^p(M)$$

or

$$\mathcal{H}^p(M) = c\mathcal{E}\mathcal{H}^p(M) \oplus \mathcal{H}_D^p(M).$$

Moreover, $\mathcal{H}_N^p(M)$ *is finite-dimensional and constitutes the concrete realisation of the absolute de Rham cohomology group* $H^p(M,\mathbb{R})$, i.e., $\mathcal{H}_N^p(M) \simeq H^p(M,\mathbb{R})$. *Similarly,* $\mathcal{H}_D^p(M)$ *is the concrete realisation of the relative cohomology group* $H^p(M,\partial M,\mathbb{R})$.

In fact, one can say more regarding the connection between spaces $\mathcal{H}_D^p(M)$ and $\mathcal{H}_N^p(M)$.

Theorem 3.4 (DeTurck, Gluck [19]) *Let M be a compact orientable Riemannian manifold with nonempty boundary* ∂M. *Then within the space* $\Omega^p(M)$,

(i) $\mathcal{H}_N^p(M)$ *and* $\mathcal{H}_D^p(M)$ *meet only at the origin;*

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(ii) *each of those spaces has decomposition into boundary and interior subspaces,*

$$\mathcal{H}^p_N(M) = c\mathcal{E}\mathcal{H}^p_N(M) \oplus \mathcal{E}_\partial\mathcal{H}^p_N(M),$$
$$\mathcal{H}^p_D(M) = \mathcal{E}\mathcal{H}^p_D(M) \oplus c\mathcal{E}_\partial\mathcal{H}^p_D(M),$$

where $\mathcal{E}_\partial(c\mathcal{E}_\partial)$ denotes the spaces of forms ω such that $i^*\omega(i_n\omega)$ is a closed (co-closed) form on ∂M;

(iii) $c\mathcal{E}\mathcal{H}^p_N(M) \perp \mathcal{H}^p_D(M)$ *and* $\mathcal{E}\mathcal{H}^p_D(M) \perp \mathcal{H}^p_N(M)$*;*

(iv) *no larger subspace of $\mathcal{H}^p_N(M)$ is orthogonal to all of $\mathcal{H}^p_D(M)$ and no larger subspace of $\mathcal{H}^p_D(M)$ is orthogonal to all of $\mathcal{H}^p_N(M)$;*

(v) $\dim \mathcal{E}_\partial\mathcal{H}^p_N(M) = \dim c\mathcal{E}_\partial\mathcal{H}^p_D(M)$.

The Hodge–Morrey–Friedrichs decomposition (simply the Hodge decomposition in the following) can be used to solve boundary problems for differential forms. It is the subject of Schwarz's book [15]. Here we collect several results from that book.

Theorem 3.5 ([15, Theorem 3.1.1, Lemma 3.1.2]) *The system*

$$d\omega = \chi, \quad \delta\omega = 0, \quad i^*\omega = \phi,$$

has a solution if and only if $d\chi = 0$, $\mathbf{t}\chi = \mathbf{t}d\phi$, and for any $\lambda \in \mathcal{H}^{p+1}_D(M)$, $\langle\!\langle \chi, \lambda \rangle\!\rangle = \int_{\partial M} \phi \wedge *\mathbf{n}\lambda$. The solution is unique up to an element of $\mathcal{H}^p_D$.

As an immediate corollary we obtain the following.

Corollary 3.6 *One has the following description.*

$$i^*\mathcal{H}^p(M) = \{\psi \in \mathcal{C}^p(\partial M) \mid \psi \perp i_n\mathcal{H}^{p+1}_D(M)\}.$$

Moreover, $\mathcal{E}^p(\partial M) \subset i^*\mathcal{H}^p(M)$.

Proof The equality is a direct consequence of Theorem 3.5. The inclusion follows from the following calculation. For any $d\alpha \in \mathcal{E}^p(\partial M)$ and any $\lambda \in \mathcal{H}^{p+1}_D(M)$, one has

$$\langle\!\langle d\alpha, i_n\lambda \rangle\!\rangle = \int_{\partial M} d\alpha \wedge *\mathbf{n}\lambda = \int_{\partial M} d(\alpha \wedge *\mathbf{n}\lambda) \pm \int_{\partial M} \alpha \wedge *\mathbf{n}\delta\lambda = 0,$$

where we used the Stokes theorem and identities $\mathbf{n}\delta = \delta\mathbf{n}$, $\delta\lambda = 0$. ■

By applying the Hodge $*$-operator to the statement of Theorem 3.5, one obtains the next theorem.

Theorem 3.7 ([15, Corollary 3.1.3]) *The system*

$$d\omega = 0, \quad \delta\omega = \chi, \quad i_n\omega = \phi,$$

has a solution if and only if $\delta\chi = 0$, $\mathbf{n}\chi = \mathbf{n}\delta\phi$ and for any $\lambda \in \mathcal{H}^{p-1}_N(M)$,

$$\langle\!\langle \chi, \lambda \rangle\!\rangle = -\int_{\partial M} \mathbf{t}\lambda \wedge *\phi.$$

The solution is unique up to an element of $\mathcal{H}^p_N(M)$.

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Corollary 3.8 *One has the following equalities:*

(3.1) $$i_n\mathcal{H}^p(M) = \left\{ \psi \in c\mathcal{C}^{p-1}(\partial M) \middle| \, \psi \perp i^*\mathcal{H}_N^{p-1}(M) \right\},$$

(3.2) $$i_n\mathcal{H}^p(M) = (i^*\mathcal{H}^{p-1}(M))^\perp.$$

Proof The first equality is a direct consequence of Theorem 3.7.

Let us prove the second. Note that $i^*\mathcal{H}^{p-1}(M) = i^*\mathcal{E}\mathcal{H}^{p-1}(M) + i^*\mathcal{H}_N^{p-1}(M)$, where "+" denotes the sum of the subspaces (not necessarily direct). Moreover,

$$i^*\mathcal{E}\mathcal{H}^{p-1}(M) \subset \mathcal{E}^{p-1}(\partial M),$$

and by Corollary 3.6, $\mathcal{E}^{p-1}(\partial M) \subset i^*\mathcal{H}^{p-1}(M)$. Therefore,

$$i^*\mathcal{H}^{p-1}(M) = \mathcal{E}^{p-1}(\partial M) + i^*\mathcal{H}_N^{p-1}(M).$$

Taking the orthogonal complement of both sides yields

$$(i^*\mathcal{H}^{p-1}(M))^\perp = (\mathcal{E}^{p-1}(\partial M))^\perp \cap (i^*\mathcal{H}_N^{p-1}(M))^\perp = c\mathcal{C}^{p-1}(\partial M) \cap (i^*\mathcal{H}_N^{p-1}(M))^\perp,$$

which is exactly the right-hand side of equality (3.2). ∎

3.2 Properties of the Dirichlet-to-Neumann Map

In this section we study elementary properties of the map Λ.

Proposition 3.9 *Any solution of*

$$\Delta\omega = 0, \quad i^*\delta\omega = 0,$$

satisfies $\delta\omega = 0$. Similarly, any solution of

$$\Delta\omega = 0, \quad i_n d\omega = 0,$$

satisfies $d\omega = 0$.

Proof To prove the first statement, note that the form $\xi = \delta\omega$ satisfies

$$\Delta\xi = 0, \quad \delta\xi = 0, \quad i^*\xi = 0.$$

Therefore, by Green's formula $\|d\xi\|^2 = \langle\!\langle \delta d\xi, \xi \rangle\!\rangle + \int_{\partial M} \xi \wedge * \mathbf{n}\, d\xi = 0$, *i.e.*, $\xi \in \mathcal{H}_D^{p-1}(M)$ and by construction $\xi \in c\mathcal{E}\mathcal{H}^{p-1}(M)$. Since those spaces are orthogonal, $\delta\omega = \xi = 0$.

An application of the first statement to the form $*\omega$ yields the second statement. ∎

In view of this proposition, the requirement $\delta\omega = 0$ for the harmonic extension is equivalent to $i^*\delta\omega = 0$. Thus, equation (2.3) is a particular case of the following theorem.

Theorem 3.10 ([15, Lemma 3.4.7]) *The system*

$$\Delta\omega = \eta, \quad i^*\delta\omega = \psi, \quad i^*\omega = \phi$$

has a solution if and only if, for any $\lambda \in \mathcal{H}_D^p(M)$, $\langle\!\langle \eta, \lambda \rangle\!\rangle = \int_{\partial M} \psi \wedge * \mathbf{n}\lambda$. The solution is unique up to an element of $\mathcal{H}_D^p(M)$.

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Belishev and Sharafutdinov proved the following proposition. Since the notations in [1] slightly differ from ours, the proofs are provided for the sake of completeness.

Proposition 3.11 *The space $\mathcal{L}(\phi)$ of solutions ϕ to equation (2.3) is an affine space with an associated vector space $\mathcal{H}^p_D(M)$. Therefore there exists unique $\lambda(\phi) \in \mathcal{L}(\phi)$ such that $\lambda(\phi) \perp \mathcal{H}^p_D(M)$.*

Proof It suffices to check the solvability condition in Theorem 3.10, which is obvious as $\eta = 0$ and $\psi = 0$. ∎

Proposition 3.12 $\ker \Lambda = i^*\mathcal{H}^p(M)$.

Proof The inclusion $i^*\mathcal{H}^p(M) \subset \ker \Lambda$ is obvious.

For the inverse, suppose $\phi \in \ker \Lambda$ and let $\omega \in \mathcal{L}(\phi)$. Then ω satisfies $\Delta\omega = 0$ and $i_n d\omega = 0$. Therefore, by Proposition 3.9, $d\omega = 0$. Moreover, $\delta\omega = 0$ by definition of $\mathcal{L}(\phi)$. Thus, $\omega \in \mathcal{H}^p(M)$ ∎

Proposition 3.13 *Operator Λ is symmetric with respect to the L^2-inner product on $\Omega^p(M)$.*

Proof Let $\phi, \psi \in \Omega^p(\partial M)$. Then Green's formula (2.1) implies

$$0 = \int_M \langle \delta d\lambda(\phi), \lambda(\psi)\rangle = \langle\!\langle d\lambda(\phi), d\lambda(\psi)\rangle\!\rangle - \int_{\partial M} \langle \phi, \Lambda\psi\rangle,$$

i.e., $\langle\!\langle d\lambda(\phi), d\lambda(\psi)\rangle\!\rangle = \langle\!\langle \phi, \Lambda\psi\rangle\!\rangle$. Switching ϕ and ψ in the computation above completes the proof. ∎

3.3 Image of Λ

In this section, we identify the image of Λ. From the previous section, one has the following sequence of inclusions

$$c\mathcal{E}^p(\partial M) \subset (\ker \Lambda)^\perp = (i^*\mathcal{H}^p(M))^\perp = i_n\mathcal{H}^{p+1}(M) \subset c\mathcal{C}^p(\partial M).$$

There are two natural ways to look at the domain of Λ. One can either set the domain to be $c\mathcal{C}^p(\partial M)$, which reflects intrinsic geometry of ∂M, or set it to be $(i^*\mathcal{H}^p(M))^\perp = i_n\mathcal{H}^{p+1}(M)$, which emphasises the role of M. A nice feature of the latter is that Λ is strictly positive on that domain. However, in most of this article we adapt the former convention and consider Λ as an operator on $c\mathcal{C}^p(\partial M)$

From symmetry, it follows that $\operatorname{im} \Lambda \subset (i^*\mathcal{H}^p(M))^\perp$. In fact, this inclusion is an equality.

Proposition 3.14 *The operator*

$$\Lambda\colon i_n\mathcal{H}^{p+1}(M) \longrightarrow i_n\mathcal{H}^{p+1}(M) \tag{3.3}$$

is a bijection.

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Proof It is sufficient to show surjectivity. Let $\psi \in i_n\mathcal{H}^{p+1}(M)$. Then there exists $\xi \in \Omega^{p+1}(M)$ satisfying

$$d\xi = 0, \quad \delta\xi = 0, \quad i_n\xi = \psi.$$

According to the Hodge decomposition for harmonic fields one can write $\xi = d\beta + \gamma$, where $\beta \in \Omega^p(M)$ and $\gamma \in \mathcal{H}_N^{p+1}$. Moreover, β can be chosen coclosed. Indeed, consider its Hodge decomposition $\beta = d\tilde{\alpha} + \delta\tilde{\beta} + \tilde{\gamma}$, where $d(d\tilde{\alpha} + \tilde{\gamma}) = 0$, *i.e.*, $d\delta\tilde{\beta} = d\beta$. Thus, replacing β with $\delta\tilde{\beta}$ does not change ξ. Therefore, β solves the system

$$\Delta\beta = 0, \quad \delta\beta = 0, \quad i_n d\beta = \psi,$$

i.e., $\Lambda i^*\beta = \psi$. ∎

In view of this proposition, in the next section we use Λ^{-1} to denote the inverse of Λ as an operator in (3.3). Our next goal is to prove compactness of Λ^{-1} as an operator on the Hilbert space $L^2(i_n\mathcal{H}^{p+1}(M))$, that, together with symmetry, yields discreteness of the spectrum.

4 Compactness of Λ^{-1}

In order to prove the compactness of Λ^{-1} we would like to use the following theorem.

Theorem 4.1 ([15, Theorem 3.4.9]) *For any form $\psi \in (i^*\mathcal{H}^p(M))^\perp$, there exists a unique solution ω to*

$$\Delta\omega = 0, \quad i^*\delta\omega = 0, \quad i_n d\omega = \psi, \tag{4.1}$$

orthogonal to the space $\mathcal{H}^p(M)$. Moreover, that solution satisfies the following Sobolev bounds

$$\|\omega\|_{H^{s+2}} \leqslant C\|\psi\|_{H^{s+1/2}}, \tag{4.2}$$

for any $s \in \mathbb{Z}_{\geqslant 0}$.

However, for our purposes we need inequality (4.2) for $s = -1$, which is not guaranteed by the theorem above.

Theorem 4.2 *For the solution of equation (4.1), one has the following bound*

$$\|\omega\|_{H^1} \leqslant C\|\psi\|_{H^{-1/2}}. \tag{4.3}$$

This theorem is proved below. For now assume that inequality (4.3) holds.

Theorem 4.3 *Operator $\Lambda^{-1}: L^2((i^*\mathcal{H}^p(M))^\perp) \to L^2((i^*\mathcal{H}^p(M))^\perp)$ is compact. Moreover, it is a bounded operator from*

$$H^{s+1/2}((i^*\mathcal{H}^p(M))^\perp) \quad \textit{to} \quad H^{s-1/2}((i^*\mathcal{H}^p(M))^\perp)$$

for $s \in \mathbb{Z}_{\geqslant 0}$.

Proof Note that $\Lambda^{-1}(\psi) = P(i^*\omega)$, where ω is a solution to (4.1) and P is an L^2-orthogonal projection from $L^2\Omega^p(\partial M)$ onto $L^2((i^*\mathcal{H}^p(M))^\perp)$. Since

$$H^s(\operatorname{im}\delta) \subset H^s((i^*\mathcal{H}^p(M))^\perp) \subset H^s(\ker\delta)$$

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and $H^s(\operatorname{im}\delta) \subset H^s(\ker\delta)$ is a finite codimension closed subspace in a closed space for any s (Hodge decomposition theorem for closed manifolds), then

$$H^{1/2}\Big(\big(i^*\mathcal{H}^p(M)\big)^\perp\Big)$$

is a split subspace. Thus, using (4.3) and trace formula one has

$$\|\Lambda^{-1}(\psi)\|_{H^{1/2}} \leqslant C\|i^*\omega\|_{H^{1/2}} \leqslant C'\|\omega\|_{H^1} \leqslant C''\|\psi\|_{H^{-1/2}}.$$

Bounds for $H^{s+1/2}$ norms with natural s are proved in a similar fashion using inequality (4.2). Compactness of Λ^{-1} follows from inclusion $L^2 \subset H^{-1/2}$ and compactness of $H^{1/2} \hookrightarrow L^2$. ∎

This completes the proof of the first part of Theorem 2.3. Note that Sobolev bounds for Λ^{-1} imply smoothness of Λ-eigenforms.

4.1 Proof of Theorem 4.2

First, let us provide a weak formulation of equation (4.1). For any

$$\psi \in H^{-1/2}(\Omega^p(\partial M)) := \big(H^{1/2}(\Omega^p(\partial M))\big)^*$$

such that $(\psi,\cdot)$ is identically zero on $i^*\mathcal{H}^p(M)$, find $\omega \in H^1(\mathcal{H}^p(M)^\perp)$ such that for any $\eta \in H^1(\Omega^p(M))$ one has

$$\int_M \big(\langle d\omega, d\eta\rangle + \langle\delta\omega,\delta\eta\rangle\big) = (\psi, i^*\eta), \tag{4.4}$$

where the round brackets denote duality pairing.

First note that both sides of the equation are invariant under transformation

$$\eta \longmapsto \eta + \xi,$$

where $\xi \in \mathcal{H}^p(M)$. Therefore, without loss of generality $\eta \perp_{L^2} \mathcal{H}^p(M)$. By [15, Lemma 2.4.10.(i)] the left-hand side of equation (4.4) defines a scalar product on $H^1(\mathcal{H}^p(M)^\perp)$ equivalent to the usual H^1-scalar product. Moreover, the right-hand side is a bounded linear functional on $H^1(\Omega^p(M))$ as by the trace formula

$$|(\psi, i^*\eta)| \leqslant \|\psi\|_{H^{-1/2}}\|i^*\eta\|_{H^{1/2}} \leqslant C\|\psi\|_{H^{-1/2}}\|\eta\|_{H^1}.$$

Thus, by the Riesz representation theorem, there exists solution ω to (4.4) satisfying bound (4.3).

Easy application of Green's formula shows that if solution ω is in H^2, then it is a strong solution in the sense of Theorem 4.1 and $\psi = i_n d\omega \in H^{1/2}(\Omega^p(\partial M))$.

5 Min-max Principle

The goal of this section is to prove the second half of Theorem 2.3, *i.e.*, to obtain a min-max characterisation of eigenvalues similar to the one for Steklov eigenvalues on functions. By Proposition 3.13, for $\omega_1 \in \mathcal{L}(\phi_1)$, $\omega_2 \in \mathcal{L}(\phi_2)$ one has

$$\int_{\partial M}\langle\Lambda\phi_1,\phi_2\rangle = \int_M\langle d\omega_1,\, d\omega_2\rangle.$$

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This equality suggests that the Rayleigh quotient for operator Λ is a ratio of squares of L^2-norms of $d\omega_i$ and ϕ_i. The following proposition makes it possible to omit the condition $\omega_i \in \mathcal{L}(\phi_i)$.

Proposition 5.1 *Any form ω in the space $\mathcal{L}(\phi)$ minimises the quadratic form $Q(\omega) = \|d\omega\|^2_{L^2}$ in the class of p-forms ρ on M satisfying $i^*\rho = \phi$.*

Proof First note that $Q(\omega)$ is constant on $\mathcal{L}(\phi)$ as $d\mathcal{H}^p_D(M) = 0$. Thus, it is sufficient to prove that for any ρ with $i^*\rho = \phi$, one has $Q(\rho) \geqslant Q(\omega)$ for some $\omega \in \mathcal{L}(\phi)$.

Let ρ and ω be as above. Then $d\rho = d(\rho - \omega) + d\omega$, where $i^*(\rho - \omega) = 0$ and $d\omega \in \mathcal{H}^p(M)$. Therefore, by Green's formula $d(\rho - \omega) \perp d\omega$ and

$$Q(\rho) = Q(\rho - \omega) + Q(\omega) \geqslant Q(\omega) \qquad \blacksquare$$

.

Theorem 5.2 (Min-max principle) *The k-th eigenvalue $\sigma_k^{(p)}$ of*

$$\Lambda : c\mathcal{C}^p(\partial M) \longrightarrow c\mathcal{C}^p(\partial M)$$

can be characterised in the following way:

$$\sigma_k^{(p)} = \max_E \min_{\phi \perp E;\, i^*\widehat{\phi} = \phi} \frac{\|d\widehat{\phi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}},$$

where E runs over all $(k-1)$-dimensional subspaces of $c\mathcal{C}^p(\partial M)$. The maximum is achieved for $E = V_{k-1}$, where V_{k-1} is spanned by the first $(k-1)$ eigenforms, ϕ being the k-th eigenform and $\widehat{\phi} \in \mathcal{L}(\phi)$. In particular,

$$\sigma_k^{(p)} \leqslant \frac{\|d\widehat{\phi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}}$$

for any $\phi \perp V_{k-1}$ and any $\widehat{\phi}$ satisfying $i^*\widehat{\phi} = \phi$.

Proof Application of the min-max theorem for positive self-adjoint operator Λ guarantees that

$$\sigma_k^{(p)} = \max_E \min_{\phi \perp E} \frac{\|d\lambda(\phi)\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}},$$

where E runs over all $(k-1)$-dimensional subspaces of $H^{1/2}(c\mathcal{C}^p(\partial M))$. Elliptic regularity estimates of Theorem 4.3 guarantee that it is sufficient to consider $E \subset c\mathcal{C}^p(\partial M)$. Therefore, the min-max formula of the theorem follows from Proposition 5.1. $\blacksquare$

6 Proof of Theorem 2.5

Raulot and Savo defined the operator L [13]. By [15, Theorem 3.4.10] for any $\phi \in \Omega^p(\partial M)$ there exists a unique $\widehat{\omega} \in \Omega^p(M)$ satisfying

$$\Delta\widehat{\omega} = 0, \quad i_n\widehat{\omega} = 0, \quad i^*\widehat{\omega} = \phi. \tag{6.1}$$

Then $L\phi$ is defined to be $i_n d\widehat{\omega}$. Raulot and Savo demonstrated that L is an elliptic, self-adjoint pseudo-differential operator of first order. Therefore, its spectrum consists of

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eigenvalues that will be denoted by $0 \leqslant \mu_1^{(p)} \leqslant \mu_2^{(p)} \leqslant \cdots$. The kernel of this map is the space $i^*\mathcal{H}_N^p(M)$. Eigenvalues $\mu_k^{(p)}$ have a min-max characterisation that is the subject of the next theorem.

Theorem 6.1 (Min-max principle [13]) *The k-th eigenvalue $\mu_k^{(p)}$ can be computed in the following way:*

$$\mu_k^{(p)} = \max_E \min_{\phi \perp E;\, i^*\widehat{\phi}=\phi,\, i_n\widehat{\phi}=0} \frac{\|d\widehat{\phi}\|^2_{L^2(M)} + \|\delta\widehat{\phi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}},$$

where E runs over $(k-1)$-dimensional subspaces of $\Omega^p(\partial M)$. The maximum is achieved for $E = V_{k-1}$, where V_k is spanned by the first $(k-1)$-eigenforms, ϕ being the k-th eigenform and $\widehat{\phi}$ is a solution to (6.1). In particular,

$$\mu_k^{(p)} \leqslant \frac{\|d\widehat{\phi}\|^2_{L^2(M)} + \|\delta\widehat{\phi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}}$$

for any $\phi \perp V_{k-1}$ and $i^*\widehat{\phi} = \phi$, $i_n\widehat{\phi} = 0$.

We turn to Theorem 2.5. Let us recall the statement.

Theorem 6.2 *Let $\tilde{\sigma}_k^{(p)}$ and $\tilde{\mu}_k^{(p)}$ denote the k-th non-zero eigenvalue of Λ and L, respectively. Then for any $0 \leqslant p \leqslant (n-2)$, $\tilde{\mu}_k^{(p)} \leqslant \tilde{\sigma}_k^{(p)}$.*

For completeness, let us state the same inequality for eigenvalues without the tilde.

Corollary 6.3 *One has the following inequality $\mu_{k+b_p} \leqslant \sigma_{k+I_p}$, where*

$$b_p = \dim H^p(M) \quad \text{and} \quad I_p = \dim \operatorname{im}\{i_p : H^p(M) \to H^p(\partial M)\}.$$

We start the proof with some preliminary results.

Proposition 6.4 *For any $\phi \in \mathcal{E}^p(\partial M)$, there exists $\xi \in \mathcal{EH}^p(M)$ satisfying $i^*\xi = \phi$.*

Proof Let $\phi = d\alpha$. Then $\xi = d\lambda(\alpha)$ is the form in question. Indeed, $i^*\xi = di^*\lambda(\alpha) = d\alpha = \phi$ and $\delta\xi = \delta d\lambda(\alpha) = \Delta\lambda(\alpha) = 0$. ■

Proposition 6.5 *For any $\phi \in \Omega^p(\partial M)$, there exists (not necessarily unique) $\psi \in \Omega^p(\partial M)$ such that $\psi - \phi \in i^*\mathcal{H}^p(M)$, $\psi \perp i^*\mathcal{H}_N^p(M)$, and there exists a solution ω to*

$$\Delta\omega = 0, \quad \delta\omega = 0, \quad i_n\omega = 0, \quad i^*\omega = \psi. \tag{6.2}$$

Proof By Proposition 6.4 there exists $\chi \in \mathcal{EH}^{p+1}(M)$ such that $i^*\chi = d\phi$ and χ is unique up to $\mathcal{EH}_D^{p+1}(M)$. Let ω' be a primitive of χ, *i.e.*, $d\omega' = \chi$. Consider Hodge decomposition $\omega' = d\alpha + \delta\beta + \gamma$. Then $\omega = \delta\beta + \gamma_N$ solves

$$\Delta\omega = 0, \quad \delta\omega = 0, \quad i_n\omega = 0,$$

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for any $\gamma_N \in \mathcal{H}^p_N(M)$. Set ω_χ to be a unique choice of γ_N such that $i^*\omega_\chi \perp i^*\mathcal{H}^p_N(M)$. Consider the space $W = \{i^*\omega_\chi - \phi \mid \chi \in \mathcal{EH}^{p+1}(M), i^*\chi = d\phi\}$. Then one has the following properties.

- The space W is an affine space of dimension $\dim \mathcal{EH}^{p+1}_D(M)$. Indeed, if $i^*\omega_{\chi_1} = i^*\omega_{\chi_2}$, then $\omega_{\chi_1} - \omega_{\chi_2}$ is a harmonic form with zero tangent and normal parts on the boundary. By Green's formula, $\omega_{\chi_1} - \omega_{\chi_2} \in \mathcal{H}^p_N(M) \cap \mathcal{H}^p_D(M)$. Therefore, it is zero by Theorem 3.4 (i).
- There exists $\phi_0 \in W$ such that $\phi_0 \perp i_n \mathcal{EH}^{p+1}_D(M)$.
- Since $W \subset \mathcal{C}^p(\partial M)$, Corollary 3.6 and Theorem 3.4 (ii) imply that

$$\phi_0 \in i^*\mathcal{H}^p(M).$$

- By definition, $\phi + W \perp i^*\mathcal{H}^p_N(M)$. Thus $\psi = \phi + \phi_0$ satisfies all the requirements of the theorem. ∎

Proof of Theorem 2.5 The idea is that if for ψ there exists a solution to equation (6.2), then $\Lambda(\psi) = L(\psi)$, which allows us to connect operators Λ and L.

Let V_k be the space spanned by the eigenforms of Λ corresponding to the first k non-zero eigenvalues, *i.e.*, V_k is spanned by $\phi_1, \dots, \phi_k$, where $\Lambda\phi_k = \tilde{\sigma}^{(p)}_k$. In particular, $V_k \perp i^*\mathcal{H}^p(M)$. Let ψ_i be forms constructed from ϕ by applying Proposition 6.5 and let $\tilde{V}_k$ be the vector space spanned by $\psi_1, \dots, \psi_k$. Then Proposition 6.5 implies the following properties of $\tilde{V}_k$.

(i) For any $\psi \in \tilde{V}_k$ there exists a solution to (6.2).

(ii) $\tilde{V}_k \perp i^*\mathcal{H}^p_N(M)$.

(iii) If $\psi = \sum_{i=1}^k a_i\psi_i \in \tilde{V}_k$, then $\phi = \sum_{i=1}^k a_i\phi_i \in V_k$ satisfies $\phi - \psi \in i^*\mathcal{H}^p(M)$. If there exist non-trivial a_is such that $\psi = 0$, then $\phi \in i^*\mathcal{H}^p(M)$. But $V_k \perp i^*\mathcal{H}^p(M)$, therefore, the map $\sum_{i=1}^k a_i\psi_i \mapsto \sum_{i=1}^k a_i\phi_i$ is an isomorphism.

(iv) $\dim \widetilde{V}_k = k$.

By property (iv), there exists $\psi \in \widetilde{V}_k$ orthogonal to the first $k-1$ eigenforms of L corresponding to non-zero eigenvalues. By property (ii) $\psi \perp \ker L$, and by property (iii), there exists $\phi \in V_k$ such that $\psi - \phi \in \ker\Lambda$. Let $\widehat{\psi} \in \mathcal{L}(\psi)$ be the solution to (6.2) and let $\widehat{\phi}$ belong to $\mathcal{L}(\phi)$. Then $i^*(d\widehat{\psi} - d\widehat{\phi}) = 0$ and $i_n d(\widehat{\psi} - \widehat{\phi}) = \Lambda(\phi - \psi) = 0$. Therefore, $d\widehat{\psi} = d\widehat{\phi}$. The min-max theorem yields the following estimates:

$$\begin{aligned}
\tilde{\mu}^{(p)}_k &\leqslant \frac{\|d\widehat{\psi}\|^2_{L^2(M)} + \|\delta\widehat{\psi}\|^2_{L^2(M)}}{\|\psi\|^2_{L^2(\partial M)}} = \frac{\|d\widehat{\psi}\|^2_{L^2(M)}}{\|\psi\|^2_{L^2(\partial M)}} = \frac{\|d\widehat{\psi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)} + \|\psi - \phi\|^2_{L^2(\partial M)}} \\
&\leqslant \frac{\|d\widehat{\psi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}} = \frac{\|d\widehat{\phi}\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}} \leqslant \sup_{\phi \in V_k} \frac{\|d\lambda(\phi)\|^2_{L^2(M)}}{\|\phi\|^2_{L^2(\partial M)}} = \tilde{\sigma}^{(p)}_k.
\end{aligned}$$ ∎

7 Proof of Theorem 2.7

Yang and Yu [21] used the concept of conjugate harmonic forms to generalise the famous result of Hersch, Payne, and Schiffer [7]. They proved the following theorem.

https://doi.org/10.4153/CJM-2018-028-6 Published online by Cambridge University Press

Theorem 7.1 *Let M be a compact oriented n-dimensional Riemannian manifold with nonempty boundary. Let λ_m be the m-th eigenvalue for the Laplacian operator on ∂M. Then for any two positive integers m and r, one has $\mu^{(0)}_{m+1}\mu^{(n-2)}_{b_{n-2}+r} \leqslant \lambda_{m+r+b_{n-1}}$.*

Let $\lambda'^{(p)}_k$ denote the k-th eigenvalue of the Hodge Laplacian Δ_∂ on ∂M restricted to the space $c\mathcal{C}^p(\partial M)$. We will prove the following.

Theorem 7.2 *Let M be a compact oriented n-dimensional Riemannian manifold with nonempty boundary. Then for any two positive integers, m and r, and for any $p = 0,\dots,n-2$, one has $\sigma^{(p)}_{m+I_p}\sigma^{(n-2-p)}_{r+I_{n-2-p}} \leqslant \lambda'^{(p)}_{I_p+m+r+b_{n-p-1}-1}$.*

The proof is based on the notion of conjugate harmonic fields. Two harmonic fields $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^{n-p-2}(M)$ are called *harmonic conjugates* if $*d\omega_1 = d\omega_2$. This is a higher-dimensional generalisation of the notion of harmonic conjugate functions on the plane.

We define a duality relation between $(i^*\mathcal{H}^p(M))^\perp$ and $(i^*\mathcal{H}^{n-p-2}(M))^\perp$. We say that ϕ is *dual* to ψ if $\lambda(\phi)$ and $\lambda(\psi)$ are harmonic conjugates.

Proposition 7.3 *Let $\phi \perp i^*\mathcal{H}^p(M)$. There exists ψ dual to ϕ if and only if $*d\lambda(\phi) \perp \mathcal{H}^{n-p-1}_N(M)$. If it exists, then ψ is unique, depends linearly on ϕ, and is nonzero unless ϕ is zero.*

Proof Let $\xi = *d\lambda(\phi) \in \mathcal{H}^{n-p-1}(M)$. If ϕ has a dual, then ξ is exact. At the same time, by the Hodge decomposition theorem ξ is exact if and only if $\xi \perp \mathcal{H}^{n-p-1}_N(M)$. This proves implication $\Rightarrow$.

Assume $\xi \perp \mathcal{H}^{n-p-1}_N(M)$. Then ξ is exact. Let ρ_0 be a primitive of ξ and let its Hodge decomposition be $\rho_0 = d\alpha + \delta\beta + \gamma$, where $\gamma \in \mathcal{H}^{n-2-p}(M)$ and $\beta \in \Omega^{n-p-1}_N(M)$. There exists $\gamma_0 \perp \mathcal{H}^{n-p-2}(M)$ such that $i^*(\delta\beta+\gamma_0) \perp i^*\mathcal{H}^{n-p-2}(M)$. We set $\psi = i^*(\delta\beta+\gamma_0)$. Let $\rho = \delta\beta+\gamma_0$. Then $\delta\rho = 0$ and $\Delta\rho = \delta d\rho = \delta\xi = 0$, *i.e.*, $\rho \in \mathcal{L}(\psi)$. In particular, $d\rho = d\lambda(\psi) = *d\lambda(\phi)$. This proves implication $\Leftarrow$.

Suppose that ψ_1 and ψ_2 are both dual to ϕ. Then $d(\lambda(\psi_1)-\lambda(\psi_2)) = 0$, *i.e.*, $\psi_1-\psi_2 \in \ker\Lambda$. At the same time, $(\psi_1-\psi_2) \perp \ker\Lambda$, therefore $\psi_1 = \psi_2$. Linearity is obvious.

If $\psi = 0$, then $d\phi = 0$ and similar arguments as above assert that $\phi = 0$. ■

Proof of Theorem 7.2 Suppose that ψ is dual to ϕ. Then

$$\begin{aligned}(7.1)\qquad \|d\lambda(\psi)\|^4_{L^2(M)} &= \Big(\int_{\partial M}\langle\psi, i_n\, d\lambda(\psi)\rangle\Big)^2 \leqslant \int_{\partial M}|\psi|^2\int_{\partial M}|i_n d\lambda(\psi)|^2\\ &= \int_{\partial M}|\psi|^2\int_{\partial M}|d\phi|^2,\end{aligned}$$

where we used Green's formula, the Cauchy–Schwarz inequality, and

$$i_n d\lambda(\psi) = i_n * d\lambda(\phi) = \pm * i^* d\lambda(\phi) = \pm d\phi.$$

Let ϕ_i be the eigenforms of Δ_∂. Since the kernel of the Hodge Laplacian is the space of harmonic p-forms on ∂M, one can choose ϕ_i to satisfy $\phi_1,\dots,\phi_{I_p} \in \ker\Lambda$, $\phi_j \perp \ker\Lambda$ for $j > I_p$. Let $\psi^{(q)}_i$ be eigenforms of Λ on $c\mathcal{C}^q(\partial M)$. Let ϕ belong to the space $\operatorname{span}\{\phi_{I_p+1},\dots,\phi_{I_p+m+r-1+b_{n-p-1}}\}$ such that $\phi \perp \operatorname{span}\{\psi^{(p)}_{I_p+1},\dots,\psi^{(p)}_{I_p+m-1}\}$ and $*d\lambda(\phi) \perp \mathcal{H}^{n-p-1}_N(M)$. The latter guarantees the existence of the form ψ dual to ϕ. Moreover, ϕ can be chosen so that $\psi \perp \operatorname{span}\{\psi^{(n-p-2)}_{I_{n-p-2}+1},\dots,\psi^{(n-p-2)}_{I_{n-p-2}+r-1}\}$. By dimension count, it is easy to see that such a nonzero ϕ exists. Then ψ is also non-zero and by min-max principles for Λ and Δ_∂ and inequality (7.1) one has

$$\begin{aligned}\sigma^{(p)}_{m+I_p}\sigma^{(n-2-p)}_{r+I_{n-2-p}} &\leqslant \frac{\|d\lambda(\phi)\|^2_{L^2(M)}\|d\lambda(\psi)\|^2_{L^2(M)}}{\|\psi\|^2_{L^2(\partial M)}\|\phi\|^2_{L^2(\partial M)}} = \frac{\|d\lambda(\psi)\|^4_{L^2(M)}}{\|\psi\|^2_{L^2(\partial M)}\|\phi\|^2_{L^2(\partial M)}}\\ &\leqslant \frac{\|d\phi\|^2_{L^2(\partial M)}}{\|\phi\|^2_{L^2(\partial M)}} \leqslant \lambda'^{(p)}_{I_p+m+r-1+b_{n-p-1}},\end{aligned}$$

where in the first equality we used the isometry property of Hodge star and equality $*d(\lambda(\phi)) = d\lambda(\psi)$. ∎

The combination of Theorem 2.5 and Theorem 2.7 yields the following generalisation of Theorem 7.1.

Corollary 7.4 *Let M be a compact oriented n-dimensional Riemannian manifold with nonempty boundary. Then for any two positive integers m and r and for any $p = 0,\dots,n-2$, one has $\mu^{(p)}_{m+b_p}\mu^{(n-2-p)}_{r+b_{n-2-p}} \leqslant \lambda'^{(p)}_{I_p+m+r+b_{n-p-1}-1}$.*

Note that $I_0 = 1$; so for $p = 0$, this corollary yields the statement of Theorem 7.1.

8 Eigenvalues of the Unit Euclidean Ball $\mathbb{B}^{n+1}$

In this section, we compute an eigenbasis and eigenvalues for Λ on $\mathbb{S}^n = \partial\mathbb{B}^{n+1}$. We follow [14] where Raulot and Savo computed eigenspaces and eigenvalues for operator L on $\mathbb{S}^n = \partial\mathbb{B}^{n+1}$. Note that in order to preserve notations from [14], we deviate from the convention that the ambient manifold has dimension n and instead in this section the ambient manifold has dimension $n+1$. In the case of the ball $\mathbb{B}^{n+1}$, operators L, Λ, and Δ have a common basis of eigenforms that we describe below.

Let $P_{k,p}$ denote the space of homogeneous polynomial p-forms of degree k in $\mathbb{R}^{n+1}$. We introduce the following subspaces of $P_{k,p}$:

- $H_{k,p} = \{\omega \in P_{k,p} \mid \Delta_{\mathbb{R}^{n+1}}\omega = 0, \delta_{\mathbb{R}^{n+1}}\omega = 0\}$,
- $H'_{k,p} = \{\omega \in H_{k,p} \mid d_{\mathbb{R}^{n+1}}\omega = 0\}$,
- $H''_{k,p} = \{\omega \in H_{k,p} \mid i_n\omega = 0\}$.

Assume $1 \leqslant p \leqslant (n-1)$. Then $\mathcal{H}^p(\mathbb{S}^n) = 0$ and $\Omega^p(\mathbb{S}^n) = \mathcal{E}^p(\mathbb{S}^n) \oplus c\mathcal{E}^p(\mathbb{S}^n)$. It was shown in [8] that $\mathcal{E}^p(\mathbb{S}^n) = \bigoplus_k(i^*H'_{k,p})$, $c\mathcal{E}^p(\mathbb{S}^n) = \bigoplus_k(i^*H''_{k,p})$ and

$$\delta\colon i^*H'_{k,p} \longrightarrow i^*H''_{k+1,p-1}$$

is an isomorphism. Thus, $\dim i^*H''_{1,p} = \dim i^*H'_{0,p+1} = \binom{n+1}{p+1}$ as all forms with constant coefficients lie in $H'_{0,p+1}$.

https://doi.org/10.4153/CJM-2018-028-6 Published online by Cambridge University Press

We see that $H'_{k,p} \subset \mathcal{H}^p(\mathbb{B}^{n+1})$; therefore Λ is identically zero on each $i^*(H'_{k,p})$. Moreover, for $\phi \in i^*(H''_{k,p})$, the form $\lambda(\phi)$ satisfies $i_n\lambda(\phi) = 0$. Therefore, $L(\phi) = \Lambda(\phi)$.

We summarise the above observations and results of [8,14] in the following theorem.

Theorem 8.1 *Spaces $i^*H'_{k-1,p}$ and $i^*H''_{k,p}$ for $k \geqslant 1$ form common eigenbases of Λ, L and Δ. The corresponding eigenvalues are given below.*

- *If $\phi \in i^*H'_{k-1,p}$, then $\Lambda\phi = 0$, $L\phi = (k+p-1)\frac{n+2k+1}{n+2k-1}\phi$, and $\Delta\phi = (k+p-1)(n+k-p)\phi$.*
- *If $\phi \in i^*H''_{k,p}$, then $\Lambda\phi = L\phi = (k+p)\phi$ and $\Delta\phi = (k+p)(n+k-p-1)\phi$.*

This theorem implies the sharpness properties of inequality (2.4) stated in Section 2.5 and Remark 2.9. Indeed, according to Theorem 8.1, inequality (2.4) is sharp for $m = r = 1$. Moreover, it is sharp as long as the eigenvalues involved coincide with the first eigenvalue. The statement after Theorem 2.11 follows from the fact that the multiplicities of $\sigma_1^{(p)}$ and $\lambda_1'^{(p)}$ are equal to $\dim i^*H''_{1,p} = \dim i^*H'_{0,p+1} = \dim H'_{0,p+1} = \binom{n+1}{p+1}$.

Acknowledgements The author is grateful to D. Jakobson, N. Nigam, I. Polterovich, A. Savo and A. Strohmaier for fruitful discussions. The author also thanks the referee for remarks that helped to improve the presentation. Part of this project was completed when the author was visiting N. Nigam at Simon Fraser University. Its hospitality is greatly acknowledged.

This research was partially supported by the Tomlinson Fellowship. This work is a part of the author's PhD thesis at McGill University under the supervision of Dmitry Jakobson and Iosif Polterovich.

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