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THE IDEAL STRUCTURE OF IDEMPOTENT-GENERATED TRANSFORMATION SEMIGROUPS

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1. Introduction

Let X be a set and $\mathscr{T}_X$ the semigroup (under composition) of all total transformations from X into itself. In ([6], Theorem 3) Howie characterised those elements of $\mathscr{T}_X$ that can be written as a product of idempotents in $\mathscr{T}_X$ different from the identity. We gather from review articles that his work was later extended by Evseev and Podran [3, 4] (and independently for finite X by Sullivan [15]) to the semigroup $\mathscr{P}_X$ of all partial transformations of X into itself. Howie's result was generalized in a different direction by Kim [8], and it has also been considered in both a topological and a totally ordered setting (see [11] and [14] for brief summaries of this latter work). In addition, Magill [10] investigated the corresponding idea for endomorphisms of a Boolean ring, while J. A. Erdos [2] resolved the analogous problem for linear transformations of a finite-dimensional vector space.

In this paper we return to Howie's original article and first determine the ideals of the semigroup $\mathscr{E}_X$ generated by the idempotents in $\mathscr{T}_X$ different from the identity. Next we characterise Green's relations on $\mathscr{E}_X$ and use our result to produce a new class of 0-bisimple regular semigroups. Finally we consider the extension of our work to the partial case.

2. Ideal structure

Throughout this paper we shall in the main use the notation of [1] but occasionally abbreviate it for the purpose of convenience. In particular, if $\alpha\in\mathscr{T}_X$ we write $r(\alpha)=|X\alpha|$ and can call this the *rank* of α.

Howie showed in ([6], Theorem 1) that if X is finite then $\mathscr{E}_X=\{\alpha\in\mathscr{T}_X: r(\alpha)<|X|\}$. Since the ideals of this semigroup are well-known (cf. [1], Vol. 2, Theorem 10.59) we assume for the remainder of this section that $|X|=\kappa\geqq\aleph_0$. In [6] Howie described the elements of $\mathscr{E}_X$ in this case via three concepts: if $\alpha\in\mathscr{T}_X$ we put

$$D(\alpha)=X\backslash X\alpha \qquad \text{and } d(\alpha)=|D(\alpha)|$$

$$S(\alpha)=\{x\in X: x\alpha\neq x\} \qquad \text{and } s(\alpha)=|S(\alpha)|$$

$$C(\alpha)=\cup\{t\alpha^{-1}: |t\alpha^{-1}|\geqq 2\} \quad \text{and } c(\alpha)=|C(\alpha)|$$

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and we refer to the cardinals $d(\alpha)$, $s(\alpha)$ and $c(\alpha)$ as the *defect*, *shift* and *collapse* of α, respectively. Howie proved in ([6], Theorem 3) that $\mathscr{E}_X$ is the disjoint union of two semigroups:

$$V = \{\alpha \in \mathscr{T}_X : s(\alpha) < \aleph_0 \text{ and } d(\alpha) \neq 0\}$$

$$H = \{\alpha \in \mathscr{T}_X : s(\alpha) = d(\alpha) = c(\alpha) \geqq \aleph_0\}.$$

That V is in fact a semigroup follows from ([6], Lemmas 2 and 5), and that H is a semigroup follows from ([6], Lemmas 6 and 7). Since we will need to refer to the last of these Lemmas quite often, we re-state it here for convenience (and note in passing that the original proof contained a significant error that was neatly corrected in [7]).

Lemma 1. *If $\alpha \in H$, $\beta \in \mathscr{T}_X$ and $s(\beta) < s(\alpha)$ then both $\alpha\beta$ and $\beta\alpha$ have shift, defect and collapse equal to that of α.*

In this section we aim to describe the ideals I of $\mathscr{E}_X$: note that $I = (I \cap V) \cup (I \cap H)$ and if $I \cap V$ and $I \cap H$ are non-empty they are ideals of V and H respectively; hence our first task will be to determine the ideals of V and of H. It seems that Vorobev [18] has described the ideals of a semigroup closely allied to V: namely, the set of all $\alpha \in \mathscr{T}_X$ with $s(\alpha) < \aleph_0$ ([17] may also be relevant: it is listed in [9] but has not been reviewed and was unavailable to us). For completeness we provide a proof of the following.

Theorem 1. *Let $n \in \mathbb{Z}^+$ and $V_n = \{\alpha \in V : d(\alpha) \geqq n\}$. Then V_n is an ideal of V and every ideal of V equals some V_n. Moreover, each V_n is principal and generated by an element with defect n.*

Proof. Let $\alpha \in V_n$ and $\beta \in V$. Since $D(\alpha) \subseteq D(\beta\alpha)$, we have $\beta\alpha \in V_n$. Although after some reflection it is intuitively clear that also $\alpha\beta \in V_n$, a convincing argument is somewhat longer. Firstly we assert that

$$[S(\beta) \cup D(\alpha)] \cap X\alpha\beta \subseteq [S(\beta) \backslash D(\alpha)]\beta.$$

For, if $x \in S(\beta) \cup D(\alpha)$ and $x = y\alpha\beta$ for some $y \in X$ then $y\alpha \in S(\beta)$: otherwise, $x = (y\alpha)\beta = y\alpha \notin S(\beta)$ implies $x = y\alpha \in D(\alpha)$, a contradiction. Hence $y\alpha \in S(\beta) \backslash D(\alpha)$, and our assertion follows. Now we put $Y = S(\beta) \cup D(\alpha)$ and note that

$$|Y \cap X\alpha\beta| \leqq |S(\beta) \backslash D(\alpha)|$$

where

$$|Y| = |Y \cap X\alpha\beta| + |Y \cap D(\alpha\beta)|$$

Hence we have

$$d(\alpha\beta) \geqq |Y \cap D(\alpha\beta)| = |Y| - |Y \cap X\alpha\beta| \geqq |Y| - |S(\beta) \backslash D(\alpha)| = d(\alpha) \geqq n.$$

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For the converse we assume I is an ideal of V, choose $\alpha \in I$ with minimal defect, and put $d(\alpha)=n$. Then $I \subseteq V_n$. Let $\beta \in V_n$ and put $Z=E(\alpha) \cup E(\beta)$, where $E(\gamma)$ denotes $S(\gamma) \cup S(\gamma)\gamma$ for each $\gamma \in V$. Note that both α and β map Z into itself, and fix $X \backslash Z$ pointwise. Hence $D(\alpha) \cup D(\beta) \subseteq Z$. Put $\alpha_1 = \alpha | Z$ and $\beta_1 = \beta | Z$, and note that since α_1 fixes $E(\beta) \backslash E(\alpha)$, we have $d(\alpha_1)=d(\alpha)$. Likewise $d(\beta_1)=d(\beta)$ and, since

$$r(\alpha_1)+d(\alpha_1)=r(\beta_1)+d(\beta_1)=|Z|<\aleph_0$$

where $d(\beta_1) \geqq d(\alpha_1) \neq 0$, we conclude that $r(\beta_1) \leqq r(\alpha_1) < |Z|$. At this point we can invoke the well-known characterisation of Green's $\mathscr{J}$-relation on $\mathscr{T}_Z$ (cf. [1], Vol. 1, pp. 52–53) to assert that $\beta_1 = \lambda_1 \alpha_1 \mu_1$ for some $\lambda_1, \mu_1 \in \mathscr{T}_Z$. In fact, since $r(\alpha_1) < |Z|$, we can ensure that $d(\lambda_1)$ and $d(\mu_1)$ are both non-zero. So, by extending λ_1 and μ_1 to the whole of X in an obvious way, we obtain $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in V$; that is, $\beta \in I$ and we have shown $I = V_n$. That V_n is a principal ideal is clear from the foregoing discussion.

According to ([1], Vol. 2, Theorem 10.59) the ideals of $\mathscr{T}_X$ take the form I_ξ where for $1 < \xi \leqq \kappa'$,

$$I_\xi = \{\alpha \in \mathscr{T}_X : r(\alpha) < \xi\}.$$

If ξ is finite then the Rees quotient semigroup $I_{\xi+1}/I_\xi$ is completely 0-simple ([1], Vol. 2, Lemma 10.54). We assert that this is also true for the semigroups V_n/V_{n+1} where $1 \leqq n < \aleph_0 \leqq \kappa$. To show this we again use the set $E(\alpha)=S(\alpha) \cup S(\alpha)\alpha$ (called by Symons [16] the *essential domain* of $\alpha \in \mathscr{T}_X$); namely, if $\alpha, \beta \in V$ with $d(\alpha)=d(\beta)=n$, we put $Y=E(\alpha) \cup E(\beta)$ and observe that $\alpha_1 = \alpha | Y$ and $\beta_1 = \beta | Y$ are elements of $\mathscr{T}_Y$ with $r(\alpha_1)=r(\beta_1)<|Y|$. We can now follow the proof of ([1], Vol. 2, Lemma 10.54) to eventually conclude that V_n/V_{n+1} is 0-simple. Clearly, $V_n \backslash V_{n+1}$ contains idempotents. To show each of these is primitive, we again put $Y=E(\alpha) \cup E(\beta)$ where α, β are idempotents in V with $\alpha\beta=\beta\alpha=\alpha$ and $d(\alpha)=d(\beta)$. Then $\alpha_1\beta_1=\beta_1\alpha_1=\alpha_1$ for idempotents $\alpha_1, \beta_1 \in \mathscr{T}_Y$ with $r(\alpha_1)=r(\beta_1)<\aleph_0$. An argument similar to that in the reference already cited eventually leads us to $\alpha=\beta$. We have therefore shown

Theorem 2. If $1 \leqq n < \aleph_0$ *then* V_n/V_{n+1} *is a completely* 0-*simple semigroup.*

We assert that V_n/V_{n+1} is not isomorphic to any I_{m+1}/I_m with m finite ... simply because the cardinal of the first is κ while that of the second is 2^κ. To see this, recall that the set $\mathscr{F}$ of all finite subsets of X has cardinal κ ([12], Theorem 22.17). If $\mathscr{F}$ is any finite subset of X such that $|F|=n+1 \geqq 2$ then V_n/V_{n+1} contains an idempotent that is constant on F and fixes $X \backslash F$; hence, if $|V_n/V_{n+1}|=\varepsilon$ then $\varepsilon \geqq \kappa$. Now, to each $\alpha \in V_n \backslash V_{n+1}$ we can associate in a one-to-one fashion the element $\alpha | E(\alpha)$ of $\mathscr{T}_{E(\alpha)}$. Hence if m_F denotes the (finite) cardinal of $\mathscr{T}_F$ for each $F \in \mathscr{F}$ then $\varepsilon \leqq \Sigma m_F = \kappa$. On the other hand, if $F \in \mathscr{F}$ and $|F|=m$ then I_{m+1}/I_m contains all maps from X onto F and there are 2^κ such maps (since there are $(2^\kappa)^m = 2^\kappa$ ways of partitioning X into a family of m subsets of X: cf. [12], Exercise 22.20). However the cardinal of $\mathscr{T}_X$ is 2^κ and so we have $|I_{m+1}/I_m| = 2^\kappa$.

It will become apparent after we have determined Green's $\mathscr{H}$-relation on V in Section 3 that V_n/V_{n+1} and I_{m+1}/I_m are non-isomorphic for a less trivial reason: namely, the non-zero group $\mathscr{H}$-classes of I_{m+1}/I_m are all isomorphic to the symmetric group $\mathscr{G}_m$ on m letters (as observed in [**1**], Vol. 2, p. 226) whereas those of V_n/V_{n+1} are all isomorphic to the group $\mathscr{G}(\kappa, \aleph_0)$ of all permutations of κ letters with finite shift (as can be readily checked by mimicking the proof of Theorem 2.10(ii) in [**1**], Vol. 1).

We now turn to the problem of describing the ideal of H: it happens that, just as in V, they form a chain, even though two cardinals are required for their description. To show this, we let

$$H(\delta,\xi)=\{\alpha\in H: d(\alpha)\geqq\delta \quad \text{and} \quad r(\alpha)<\xi\}$$

where $\aleph_0\leqq\delta\leqq\kappa$ and $2\leqq\xi\leqq\kappa'$.

Lemma 2. *Each $H(\delta,\xi)$ is an ideal of H and the set of all such distinct ideals forms a chain:*

$$H(\kappa,2)\subseteq\cdots\subseteq H(\kappa,\xi)\subseteq\cdots\subseteq H(\kappa,\kappa')\subseteq\cdots\subseteq H(\aleph_1,\kappa')\subseteq H(\aleph_0,\kappa') \qquad (*)$$

Proof. Let $\alpha\in H(\delta,\xi)$ and $\beta\in H$, and suppose $s(\alpha)=d(\alpha)=c(\alpha)=a$ and $s(\beta)=d(\beta)=c(\beta)=b$. If $b\leqq a$ then ([**6**], Lemmas 6 and 7) imply that both $\alpha\beta$ and $\beta\alpha$ have defect equal to a ($\geqq\delta$). If $a>b$ then Lemma 1 above implies that both $\alpha\beta$ and $\beta\alpha$ have defect equal to $b>a\geqq\delta$. Since $r(\alpha\beta)\leqq\min\{r(\alpha),r(\beta)\}$ it therefore follows that $\alpha\beta,\beta\alpha\in H(\delta,\xi)$ and $H(\delta,\xi)$ is an ideal of H.

Now consider an arbitrary $H(\delta,\xi)$. If $\delta=\kappa$ (and $2\leqq\xi\leqq\kappa'$) we have an ideal in the first portion of the above chain, and if $\xi=\kappa'$ (and $\aleph_0\leqq\delta\leqq\kappa$) we are in the second portion of the chain. On the other hand, since $|X|=\kappa\geqq\aleph_0$ and $X=X\alpha\cup(X\backslash X\alpha)$ for each $\alpha\in H$, we must have $d(\alpha)=\kappa$ if $r(\alpha)<\xi\leqq\kappa$; that is if $\delta<\kappa$, $\xi\leqq\kappa$ and $\alpha\in H(\delta,\xi)$ then $\alpha\in H(\kappa,\xi)$. Since $H(\kappa,\xi)\subseteq H(\delta,\xi)$, we deduce that $H(\delta,\xi)=H(\kappa,\xi)$ when $\delta<\kappa$ and $\xi\leqq\kappa$.

Following ([**1**], Vol. 2, p. 241), for each $\alpha\in\mathscr{T}_X$, we write

$$\alpha=\begin{pmatrix} C_m \\ x_m \end{pmatrix}$$

where $X\alpha=\{x_m: m\in M\}$ for some index set M and $C_m=x_m\alpha^{-1}$ for each $m\in M$. To abbreviate notation, we adopt the convention (as in the reference just cited) of writing $\{x_m\}$ for $\{x_m: m\in M\}$, taking the subscript m to signify the index set M within a specific context.

Theorem 3. *Every ideal of H has the form $H(\delta,\xi)$ for some δ,ξ. In particular, the principal ideals of H are $H(\kappa,\eta')$ and $H(\varepsilon,\kappa')$ for some η,ε satisfying $1\leqq\eta\leqq\kappa$ and $\aleph_0\leqq\varepsilon\leqq\kappa$.*

Proof. Suppose I is an ideal of H. Let δ be the defect of an element of I with minimal defect and let ξ be the least cardinal greater than the ranks of all the elements of I. We assert that $I=H(\delta,\xi)$. Since $I\subseteq H(\delta,\xi)$, we therefore proceed to show that if $\beta\in H(\delta,\xi)$ then there exist $\alpha\in I$ and $\lambda,\mu\in H$ such that $\beta=\lambda\alpha\mu$. So, let $\beta\in H(\delta,\xi)$ and note that $r(\alpha)<r(\beta)<\xi$ *for all* $\alpha\in I$ contradicts the choice of ξ. Hence there exists $\alpha\in I$ with $r(\beta)\leqq r(\alpha)$. Put

$$\beta=\begin{pmatrix}B_m\\ x_m\end{pmatrix}\quad\text{and}\quad\alpha=\begin{pmatrix}A_n\\ y_n\end{pmatrix}$$

and choose a cross-section $\{a_n\}$ of $\{A_n\}$. Write $\{a_n\}=\{a_m\}\mathbin{\dot\cup}\{a_s\}$, which is possible since $r(\alpha)\geqq r(\beta)$, and put

$$\lambda=\begin{pmatrix}B_m\\ a_m\end{pmatrix}.$$

Now suppose $d(\alpha)=d(\beta)=\kappa$. Since $\beta\in H$, we have $c(\beta)=\kappa$ and this means $c(\lambda)=s(\lambda)=\kappa$ (using [**6**], Lemma 3). If $r(\beta)<\kappa$ then $|X\backslash\{a_m\}|=\kappa$ and so $d(\lambda)=\kappa$; that is, $\lambda\in H$. If on the other hand $r(\beta)=\kappa$ then $r(\alpha)=\kappa$ and we can ensure that $|S|=\kappa$; that is, $\{a_s\}\subseteq D(\lambda)$ and again $\lambda\in H$. To define μ, put $C=X\backslash\{y_m\}$ and note that $D(\alpha)\subseteq C$. Hence if we choose $z\in C$ and define

$$\mu=\begin{pmatrix}y_m & C\\ x_m & z\end{pmatrix}$$

then $c(\mu)=s(\mu)=\kappa$. In addition, $D(\beta)\backslash z\subseteq D(\mu)$ and so $d(\mu)=\kappa$. That is, $\mu\in H$ and we have $\beta=\lambda\alpha\mu$ as required.

Before considering the next case, note that if $r(\beta)\leqq r(\alpha)<\kappa$ then $d(\alpha)=d(\beta)=\kappa$ as above. Hence we may suppose $r(\alpha)=\kappa$. Suppose further that $d(\beta)=\kappa$. In this case, with the same notation as before, we immediately have $c(\lambda)=s(\lambda)=\kappa$. Moreover, since $r(\alpha)=\kappa$ we can ensure that $|S|=\kappa$. Then $d(\lambda)=\kappa$ and, since $\{y_s\}\subseteq C$, we also have $c(\mu)=s(\mu)=\kappa$ together with $d(\mu)=\kappa$ (as before).

Hence we may now assume $r(\alpha)=\kappa$ and $\delta\leqq d(\beta)<\kappa$. This implies $r(\beta)=\kappa$. In addition, by choice of δ, there exists $\gamma\in I$ with $d(\gamma)=\delta<\kappa$ (in this case) and so $r(\gamma)=\kappa$; that is, we can assume without loss of generality that $d(\alpha)=\delta\leqq d(\beta)=\varepsilon$, say. Given all this, we now restrict α,β (as in the proof of Theorem 1) to $Y=E(\alpha)\cup E(\beta)$ and obtain $\alpha_1,\beta_1\in\mathscr{T}_Y$ with the same shift, defect and collapse as α,β respectively. However, $|Y|=\varepsilon$ and so, from our very first case, $\beta_1=\lambda_1\alpha_1\mu_1$ for some $\lambda_1,\mu_1\in\mathscr{T}_Y$ where both λ_1 and μ_1 have equal infinite shift, defect and collapse. By extending this equation to the whole of X in an obvious way, we have $\beta=\lambda\alpha\mu$ for some $\lambda,\mu\in H$ and so $\beta\in I$.

Finally, observe that we have indirectly proved

$$H(\delta_1,\xi_1)\subseteq H(\delta_2,\xi_2)\text{ if and only if }\delta_1\geqq\delta_2\text{ and }\xi_1\leqq\xi_2,$$

and $H(d(\alpha),r(\alpha)')=H^1\alpha H^1$ for each $\alpha\in H$.

As noted in ([**1**], Vol. 2, p. 227, Exercise 3), each $I_{\xi'}/I_\xi$ is a 0-bisimple semigroup for $\aleph_0\leqq\xi\leqq\kappa$. Hence, since $I_{\xi'}/I_\xi$ contains non-zero idempotents, it is also regular (by [**1**], Vol. 1, Theorem 2.11). We shall consider the Rees factor semigroups corresponding to the ideals in (*) after we have determined Green's $\mathscr{D}$ and $\mathscr{J}$ relations on H in Section 3.

At this point we simply remark that $H(\kappa,\xi)=I_\xi$ for each ξ satisfying $1<\xi\leqq\kappa$. For. if $\alpha\in I_\xi$ and $1<\xi\leqq\kappa$ then $d(\alpha)=\kappa$ and, since $D(\alpha)\subseteq S(\alpha)$, we also have $s(\alpha)=\kappa$. But α can be written as

$$\alpha=\begin{pmatrix} A_m & a_n & a_p \\ x_m & x_n & a_p \end{pmatrix}$$

where $C(\alpha)=\cup\{A_m: m\in M\}$, $a_n\neq x_n$ for all $n\in N$ and $\{a_p\}=X\backslash[C(\alpha)\cup S(\alpha)]$. Since $|M\cup N|<\kappa$ and $s(\alpha)=\kappa$, we must have $|C(\alpha)\backslash\{x_n\}|=\kappa$ and so $\alpha\in H(\kappa,\xi)$.

On the other hand, $H(\kappa,\kappa')$ is a proper subset of $I_{\kappa'}$. For, if $\xi<\kappa$ we can partition X into sets A and $B_1, B_2, B_3,\ldots$ where $|A|=\xi$ and $|B_i|=\kappa$ for each $i\geqq 1$, choose bijections $\theta_i: B_i\to B_{i+1}$ and $a\in A$, and then define $\alpha\in\mathscr{T}_X$ by

$$\begin{aligned} x\alpha &= a && \text{if } x\in A,\\ &= x\theta_i && \text{if } x\in B_i. \end{aligned}$$

Then $D(\alpha)=B_1$, $S(\alpha)=\cup\{B_i: i\geqq 1\}$ and $C(\alpha)=A$; that is $\alpha\notin H(\kappa,\kappa')$.

Theorem 4. *The ideals of $\mathscr{E}_X$ are precisely the ideals of H together with the sets $V_n\cup H$ for $n\geqq 1$.*

Proof. By Lemma 1 and Theorem 3 the ideals of H, as well as the sets $V_n\cup H$, are all ideals of $\mathscr{E}_X$. Conversely, suppose I is an ideal of $\mathscr{E}_X$; the desired result follows immediately from Theorem 1 since if $I\cap H$ is a proper subset of H we can use Lemma 1 to obtain a contradiction.

3. Green's Relations

For convenience we start this section by re-stating certain information from ([**1**], Vol. 1, pp. 52–53).

Lemma 3. *If $\alpha,\beta\in\mathscr{T}_X$ then*

(*a*) *$\beta=\lambda\alpha$ for some $\lambda\in\mathscr{T}_X$ if and only if $X\beta\subseteq X\alpha$,*

(*b*) *$\beta=\alpha\mu$ for some $\mu\in\mathscr{T}_X$ if and only if $\alpha\circ\alpha^{-1}\subseteq\beta\circ\beta^{-1}$,*

(*c*) *$\beta=\lambda\alpha\mu$ for some $\lambda,\mu\in\mathscr{T}_X$ if and only if $r(\alpha)\leqq r(\beta)$,*

(*d*) *$\mathscr{D}=\mathscr{J}$.*

Comparable statements can be made for $\mathscr{P}_X$ and the symmetric inverse semigroup $\mathscr{I}_X$ on X (see [**5**] for a brief summary of this idea and its extension to a categorical setting). Our task in this section is to show that statements analogous to (*a*), (*b*) and (*d*) hold for both V and H, but that something different occurs for (*c*).

Theorem 5. *If $\alpha,\beta\in V$ then*

(*a*) *$\beta=\lambda\alpha$ for some $\lambda\in V$ if and only if $X\beta\subseteq X\alpha$,* (*b*) $\beta=\alpha\mu$ *for some* $\mu\in V$ *if and only if* $\alpha\circ\alpha^{-1}\subseteq\beta\circ\beta^{-1}$,

(*c*) $\beta=\lambda\alpha\mu$ *for some* $\lambda,\mu\in V$ *if and only if* $d(\beta)\geqq d(\alpha)$,

(*d*) $\mathscr{D}=\mathscr{J}$.

Proof. Suppose $X\alpha\subseteq X\beta$ where $X\beta$ is a proper subset of X. Put $Y=E(\alpha)\cup E(\beta)$ and restrict α,β to Y to produce $\alpha_1,\beta_1\in\mathscr{T}_Y$ where $Y\beta_1$ is a proper subset of Y. Now Y is finite and $Y\beta_1\subseteq Y\alpha_1$: since if $y\in Y$ and $y\beta=x\alpha$ for $x\in X$ then $x\alpha\neq x$ (and so $x\in Y$) or $x\alpha=x$ (and so $x=y\beta\in Y$). By Lemma 3(a), $\beta_1=\lambda_1\alpha_1$ for some $\lambda_1\in\mathscr{T}_Y$ which moreover can be chosen with $d(\lambda_1)\neq 0$ (since $d(\beta_1)\neq 0$). Then $\beta=\lambda\alpha$ where $\lambda\in V$ and part (a) follows; a similar argument establishes part (b).

Suppose $\beta=\lambda\alpha\mu$ where $d(\alpha)=n$. By Theorem 1, $V^1\alpha V^1=V_n$ and so $d(\beta)\geqq n$. Conversely, suppose $d(\beta)\geqq d(\alpha)$. Then, using our customary notation, $r(\beta_1)\leqq r(\alpha_1)$ and Lemma 3(c) implies $\beta_1=\lambda_1\alpha_1\mu_1$ for some $\lambda_1,\mu_1\in\mathscr{T}_Y$. In fact, since Y is finite and $d(\beta_1)\neq 0$, both λ_1 and μ_1 can be chosen with non-zero defect; hence we have $\beta=\lambda\alpha\mu$ with $\lambda,\mu\in V$, as required.

Finally, if $V^1\alpha V^1=V^1\beta V^1$ then $d(\alpha)=d(\beta)\neq 0$ and so $r(\alpha_1)=r(\beta_1)\neq|Y|$. By Lemma 3(d), this implies $\alpha_1\mathscr{L}\gamma_1\mathscr{R}\beta_1$ for some $\gamma_1\in\mathscr{T}_Y$ which can in fact be chosen with non-zero defect. Hence, $\alpha\mathscr{L}\gamma\mathscr{R}\beta$ for some $\gamma\in V$ and the proof is complete.

The proof of the corresponding result for H is much longer since our technique of restricting $\alpha,\beta\in\mathscr{T}_X$ to $Y=E(\alpha)\cup E(\beta)$ does not seem to help matters.

Theorem 6. *If* $\alpha,\beta\in H$ *then*

(*a*) $\beta=\lambda\alpha$ *for some* $\lambda\in H$ *if and only if* $X\beta\subseteq X\alpha$,

(*b*) $\beta=\alpha\mu$ *for some* $\mu\in H$ *if and only if* $\alpha\circ\alpha^{-1}\subseteq\beta\circ\beta^{-1}$,

(*c*) $\beta=\lambda\alpha\mu$ *for some* $\lambda,\mu\in H$ *if and only if* $r(\beta)\leqq r(\alpha)$ *and* $d(\beta)\geqq d(\alpha)$,

(*d*) $\mathscr{D}=\mathscr{J}$.

Proof. Suppose $X\beta\subseteq X\alpha$ and put $Z=X\backslash[C(\beta)\cup S(\beta)]$. Then $d(\beta)\geqq d(\alpha)$ and $|Z\cap S(\alpha)|\leqq s(\alpha)\leqq s(\beta)$. Now write

$$\beta=\begin{pmatrix}B_p & b_q & b_m & b_n\\ x_p & x_q & b_m & b_n\end{pmatrix}$$

where $C(\beta)=\cup\{B_p: p\in P\}$, $b_q\neq x_q$ for all $q\in Q$, $\{b_m\}=Z\cap S(\alpha)$, and $Z=\{b_m\}\,\dot{\cup}\,\{b_n\}$. Then $b_n\alpha=b_n$ for all $n\in N$ and we can write

$$\alpha=\begin{pmatrix}A_p & A_q & A_m & A_n & A_s\\ x_p & x_q & b_m & b_n & x_s\end{pmatrix}$$

where $b_n\in A_n$ for each $n\in N$ and $\{x_s\}=X\alpha\backslash X\beta$ (if non-empty). We now choose a partial cross-section $\{a_p\}\,\dot{\cup}\,\{a_q\}\,\dot{\cup}\,\{a_m\}\,\dot{\cup}\,\{b_n\}$ of $X/\alpha\circ\alpha^{-1}$ and put

$$\lambda=\begin{pmatrix}B_p & b_q & b_m & b_n\\ a_p & a_q & a_m & b_n\end{pmatrix}.$$

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Note that since $|M| \leqq s(\beta)$, we have $c(\lambda)=s(\lambda)=s(\beta)$. In addition, we have

$$D(\lambda) \subseteq [C(\beta) \cup S(\beta) \cup \{b_m\}] \backslash [\{a_p\} \cup \{a_q\} \cup \{a_m\}]$$

and so $d(\lambda) \leqq s(\beta)$. If $d(\alpha) < d(\beta)$ then $|S| = d(\beta)$ and so, since $\cup\{A_s : s \in S\} \subseteq D(\lambda)$, we have $d(\lambda)=s(\beta)$. Hence, we may suppose $d(\alpha)=d(\beta)=\varepsilon$ say. Now write $C(\alpha)=\cup\{A_t : t \in T\}$ and note that, when selecting the partial cross-section of $X/\alpha \circ \alpha^{-1}$ to form λ, we choose at most one element from each A_t. Consider the worst case and suppose we have in fact chosen some $a_t \in A_t$ for each $t \in T$. Since each A_t contains at least 2 elements, we have $|C(\alpha) \backslash \{a_t\}| = \varepsilon$. However, $C(\alpha) \backslash \{a_t\} \subseteq D(\lambda)$ and so $d(\lambda)=\varepsilon$. That is, $\lambda \in H$ and $\beta = \lambda\alpha$.

For part (b), we now suppose $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$ and write

$$\beta = \begin{pmatrix} B_m & B_p & x_i & w_r & w_s \\ b_m & b_p & y_i & w_r & w_s \end{pmatrix}$$

$$\alpha = \begin{pmatrix} B_{mn} & B_p & x_i & w_r & w_s \\ c_{mn} & c_p & z_i & v_r & w_s \end{pmatrix}$$

where the sets B_m and B_p contain at least 2 elements, $B_m = \cup\{B_{mn} : n \in N_m\}$ for some index set N_m, and $x_i \neq y_i$, $w_r \neq v_r$ (note that possibly $x_i = z_i$ for some i, and also some B_{mn} may consist of a single element). The above display is possible since each $\beta \circ \beta^{-1}$-class is the union of one or more $\alpha \circ \alpha^{-1}$-classes; the sets N_m are therefore chosen to satisfy $2 \leqq |N_m| \leqq c(\beta)$. Put $C_m = \{c_{mn} : n \in N_m\}$ and $D = D(\alpha)$, choose $d \in D$ and let

$$\mu = \begin{pmatrix} C_m & c_p & z_i & v_r & w_s & D \\ b_m & b_p & y_i & w_r & w_s & d \end{pmatrix}$$

Then $\beta = \alpha\mu$ and $D(\mu) = D(\beta) \backslash d$: that is, $d(\mu) = d(\beta)$. In addition, $C(\mu) = (\cup C_m) \cup D$ and

$$S(\mu) \subseteq (\cup C_m) \cup \{c_p\} \cup \{z_i\} \cup \{v_r\} \cup D.$$

However, $C(\alpha) \subseteq C(\beta)$ and so $d(\alpha) \leqq d(\beta)$; also, $|M \cup P| \leqq c(\beta)$, $|I| \leqq s(\beta)$, $|R| \leqq s(\alpha)$ and $|C_m| \leqq c(\beta)$ for each m. Hence both $c(\mu)$ and $s(\mu)$ are at most $d(\beta)$. In fact, it is clear from the very definition of μ that $c(\mu) = s(\mu) = d(\beta)$ when $d(\alpha) = d(\beta)$. So, suppose $d(\alpha) < d(\beta)$. This implies $|\cup B_m| = c(\beta)$ since $|\cup B_p| \leqq c(\alpha)$; also $|(\cup B_m) \cap S(\alpha)| < c(\beta)$. Hence, $|(\cup B_m) \cap F(\alpha)| = c(\beta)$ where $F(\alpha) = X \backslash S(\alpha)$. But $\cup B_m = \bigcup_m \bigcup_n B_{mn}$ and so in this case there are $c(\beta)$ elements in $\cup C_m$ that are fixed by α. Consequently, $|\cup C_m| = c(\beta)$ and so $c(\mu) = c(\beta)$. Moreover, each C_m contains at least 2 elements and $|M| \leqq c(\beta)$. So, $|\cup (C_m \backslash b_m)| = c(\beta)$ and therefore $s(\mu) \geqq c(\beta)$. That is, $\mu \in H$ as required.

To prove parts (c) and (d), we first show that $\alpha \mathscr{D} \beta$ in H if and only if $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\beta)$. Suppose $\alpha \mathscr{L} \gamma \mathscr{R} \beta$ for some $\gamma \in H$. By parts (a) and (b), $X\alpha = X\gamma$ and $\gamma \circ \gamma^{-1} = \beta \circ \beta^{-1}$. Hence, $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\gamma) = c(\gamma) = c(\beta) = d(\beta)$. For the converse we assume $r(\alpha) = r(\beta)$ and $d(\alpha) = d(\beta)$, and consider two cases. If $d(\alpha) = \kappa$ we choose any $\gamma \in \mathscr{T}_X$ with $X\alpha = X\gamma$ and $\gamma \circ \gamma^{-1} = \beta \circ \beta^{-1}$ (such γ's exist since $r(\alpha) = r(\beta)$). Then $d(\gamma) = d(\alpha) = \kappa$ (and so $s(\gamma) = \kappa$ since $D(\gamma) \subseteq S(\gamma)$) and $c(\gamma) = c(\beta) = \kappa$; that is, $\gamma \in H$ and we are finished.

If on the other hand $d(\alpha)=\delta<\kappa$ then $r(\alpha)=\kappa$ and our task of finding a suitable γ is much harder. However, before accomplishing this we note in passing that this case cannot be reduced to the one already considered by restricting α,β to $Y=E(\alpha)\cup E(\beta)$. For, we might now have

$$\alpha=\begin{pmatrix} U\cup V & c_n \\ x & c_n \end{pmatrix} \qquad \beta=\begin{pmatrix} U & V & c_n \\ y & z & c_n \end{pmatrix} \tag{**}$$

where U, V and $\{c_n\}$ partition X, $|U|=|V|=\delta<\kappa$, and x, y, z are distinct elements of $U\cup V$; if this were so then $Y=U\cup V$ and $d(\alpha_1)=d(\beta_1)=\delta$ but $r(\alpha_1)\neq r(\beta_1)$.

Now let $A=X\alpha\backslash[C(\beta)\cup S(\beta)]$ and note that $|A|=\kappa$ and $a\beta=a$ for all $a\in A$. Put

$$B=X\alpha\cap[C(\beta)\cup S(\beta)]$$

$$C=D(\alpha)\cup C(\beta)\cup S(\beta)$$

$$\varepsilon=\max(|B|,|C\beta|,\aleph_0)$$

and note that $\aleph_0\leqq\varepsilon\leqq\delta<\kappa$ since $|C|=\delta$. Choose a subset D of A with $|D|=\varepsilon$ and let $E=C\cup D$, $F=B\cup D$. Now $|E\beta|=|C\beta|+|D\beta|=\varepsilon$ (since $D\beta=D$) and $|F|=\varepsilon$. Let $\theta\colon E\beta\to F$ be any bijection and define $\gamma\in\mathcal{T}_X$ by

$$\begin{aligned} x\gamma &= x \quad \text{if } x\in A\backslash D \\ &= x\beta\theta \text{ if } x\in E. \end{aligned}$$

The domain of γ is X since $E\cup(A\backslash D)$ contains $C\cup A$ which equals $X\alpha\cup D(\alpha)$. Moreover, if $x\gamma=y\gamma$ then either (1) $x=y\in A\backslash D$, or (2) $x\in A\backslash D$, $y\in E$ and $x=y\beta\theta$, or (3) $x,y\in E$ and $x\beta\theta=y\beta\theta$ (we omit the dual of (2)). If (1) occurs then $x\beta=y\beta$; if (2) occurs then $x\in F$ and so $x\in B$ (since $x\notin D$), contradicting the assumption that $x\in A$; and if (3) occurs then $x\beta=y\beta$ since θ is one-to-one. That is, $\gamma\circ\gamma^{-1}\subseteq\beta\circ\beta^{-1}$. On the other hand, if $x\beta=y\beta$ then either $x=y$ (and so $x\gamma=y\gamma$) or $x\neq y$ (in which case $x,y\in C(\beta)\subseteq E$ and so $x\gamma=x\beta\theta=y\beta\theta=y\gamma$). Hence, $\gamma\circ\gamma^{-1}=\beta\circ\beta^{-1}$. In addition,

$$X\gamma=[E\cup(A\backslash D)]\gamma=F\cup(A\backslash D)=A\cup B=X\alpha.$$

Thus, $c(\gamma)=c(\beta)=d(\alpha)=d(\gamma)$. Clearly, $S(\gamma)\subseteq E$ and $|E|=\delta$. But $D(\gamma)\subseteq S(\gamma)$ and $d(\gamma)=\delta$; thus, $s(\gamma)=\delta$ and we have found some $\gamma\in H$ such that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$.

Having characterised Green's $\mathcal{D}$ relation on H, we now consider part (c) and suppose $\beta=\lambda\alpha\mu$. Then $r(\beta)\leqq r(\alpha)$ and $d(\beta)\geqq d(\mu)$. Hence, if $d(\beta)<d(\alpha)$ then $d(\alpha\mu)=d(\alpha)$ (by Lemma 1) as well as $d(\beta)\geqq d(\alpha\mu)$, a contradiction. Therefore $d(\beta)\geqq d(\alpha)$. For the converse suppose $\alpha,\beta\in H$, $r(\beta)\leqq r(\alpha)$ and $d(\beta)\geqq d(\alpha)$. This means $\beta\in H(d(\alpha),r(\alpha)')$ which by Theorem 3 equals $H^1\alpha H^1$, and so part (c) is proved. Finally, $\alpha\mathcal{J}\beta$ implies $H^1\alpha H^1=H^1\beta H^1$ and this in turn implies $d(\alpha)=d(\beta)$ and $r(\alpha)=r(\beta)$; from the foregoing, we deduce $\alpha\mathcal{D}\beta$, and of course $\mathcal{D}\subseteq\mathcal{J}$ always.

It may be worthwhile illustrating the choice of γ for the α,β displayed in (**) above. Using the notation introduced in the second last paragraph of the proof, we have

$A=\{c_n\}$, $B=\{x\}$, $C=U\cup V$ and $\varepsilon=\aleph_0$. Then we in effect "blow-up" $r(\alpha_1)$ and $r(\beta_1)$ until they are equal by suitably enlarging the domain of α_1 and β_1. That is, we choose $D=\{c_m\}$ in A with $|M|=\varepsilon=\aleph_0$ and note that $E\beta=\{y,x\}\cup\{c_m\}$ has the same cardinal as $F=\{x\}\cup\{c_m\}$. If $\{c_p\}=A\backslash D$ then γ is the map

$$\begin{pmatrix} U & V & c_m & c_p \\ y\theta & z\theta & c_m\theta & c_p \end{pmatrix}$$

where θ is any bijection between $E\beta$ and F.

Before proceeding we note that the significance of Theorem 6 (especially part (d)) lies in the fact that it gives some hope of determining the congruences on H in a manner akin to that developed by Clifford and Preston for $\mathscr{T}_X$ in ([1], Vol. 2, Section 10.8); we shall explore this possibility in a subsequent paper.

We now consider the Rees quotient semigroup $H(\delta,\kappa')/H(\delta',\kappa')$ for $\aleph_0\leqq\delta<\kappa$. Clearly, each non-zero element of this semigroup has defect δ and rank κ. Moreover, a close perusal of the proof of Theorem 6 shows that if $d(\alpha)=d(\beta)$ and $r(\alpha)=r(\beta)$ for $\alpha,\beta\in H$ then there exists $\gamma\in H$ such that $\alpha\mathscr{L}\gamma\mathscr{R}\beta$ and $d(\gamma)=d(\alpha)$. In addition, if $\alpha\mathscr{L}\gamma$ in H then there exist $\lambda_1,\lambda_2\in H$ with $\alpha=\lambda_1\gamma$, $\gamma=\lambda_2\alpha$ and $d(\lambda_1)=d(\alpha)$, $d(\lambda_2)=d(\gamma)$; likewise, if $\gamma\mathscr{R}\beta$ in H then there exist $\mu_1,\mu_2\in H$ with $\gamma=\beta\mu_1$, $\beta=\gamma\mu_2$ and $d(\mu_1)=d(\gamma)$, $d(\mu_2)=d(\beta)$. In other words, each $H(\delta,\kappa')/H(\delta',\kappa')$ is 0-bisimple when $\delta<\kappa$. Each such semigroup is also regular since it contains non-zero idempotents and ([1], Vol. 1, Theorem 2.11) can be applied. However, none of them is completely 0-simple since they always contain non-zero non-primitive idempotents; for example, if

$$\alpha=\begin{pmatrix} A & b_m & b_n \\ a & b_m & b_n \end{pmatrix} \qquad \beta=\begin{pmatrix} A\cup\{b_m\} & b_n \\ a & b_n \end{pmatrix}$$

where $|A|=|\{b_m\}|=\delta$ and $a\in A$, then α,β are distinct idempotents satisfying $\alpha\beta=\beta\alpha=\beta$.

Unfortunately we cannot decide whether these 0-bisimple quotients in the "top half" of (*) are isomorphic to any of the quotients in the "bottom half" of (*). For, an argument similar to that applied to V_n/V_{n+1} in Section 2 can be used to show that for $\delta<\kappa$, $H(\delta,\kappa')/H(\delta',\kappa')$ has cardinal κ^δ (this is because the set of all subsets of X with cardinal δ has cardinal κ^δ: see [12], Exercise 22.25). It can also be readily shown that the non-zero group $\mathscr{H}$-classes of $H(\delta,\kappa')/H(\delta',\kappa')$ are all isomorphic to the group $\mathscr{G}(\kappa,\delta')$ of all permutations of κ letters with shift at most δ. On the other hand, for $\xi\leqq\kappa$, $I_{\xi'}/I_\xi$ has cardinal 2^κ and its non-zero group $\mathscr{H}$-classes are all isomorphic to $\mathscr{G}_\xi$. However, without GCH, we may have $2^\delta=2^\xi$ even though $\delta\neq\xi$ ([13], pp. 119 and 130).

Our final result in this section in effect determines Green's relations on $\mathscr{E}_X$.

Theorem 7. *If $\alpha,\beta\in\mathscr{E}_X$ and are related under one of Green's relations on $\mathscr{E}_X$ then $\alpha,\beta\in V$ or $\alpha,\beta\in H$.*

Proof. Suppose $\alpha\in V$, $\beta\in H$ and $\alpha=\lambda\beta$ for some $\lambda\in\mathscr{E}_X$. Then, by Lemma 1, $s(\lambda)\geqq s(\beta)$ and this means $\lambda\in H$ which in turn implies $\alpha\in H$, a contradiction. A similar argument can be applied if $\alpha=\beta\mu$ or if $\alpha=\lambda\beta\mu$ for some $\lambda,\mu\in\mathscr{E}_X$.

4. Partial Transformations

In this section we consider the way in which the results of Sections 2 and 3 can be extended to the semigroup $\mathscr{E}_X^*$ that is generated by all the idempotent partial transformations of X different from the identity (*it is clear that the identity cannot be* written as a product of idempotents in $\mathscr{P}_X$ different from the identity).

For finite X, the elements of $\mathscr{E}_X^*$ were characterised by Evseev and Podran [3] (and independently by Sullivan [15]). As one might expect (by analogy with $\mathscr{T}_X$), if $|X|=n$ then $\mathscr{E}_X^*=\{\alpha\in\mathscr{P}_X: r(\alpha)<n\}$ and moreover each $\alpha\in\mathscr{E}_X^*$ can in fact be written as a product of idempotents in $\mathscr{P}_X$ with defect 1 (note that such idempotents can equal ι_Y for some $Y\subseteq X$ with $|Y|=n-1$). Given this, it is easy to see what the ideals and Green's relations on $\mathscr{E}_X^*$ must be when X is finite.

Hence we again assume throughout this section that $|X|=\kappa\geqq\aleph_0$. It seems from a review that Evseev and Podran [4] have also investigated $\mathscr{E}_X^*$ in this case. Since we need a straight forward characterisation of the elements of $\mathscr{E}_X^*$ in order to describe the ideals of $\mathscr{E}_X^*$, we now present such a characterisation and for completeness we include a short proof based on Howie's Theorem.

However, before proceeding to do this we recall Lyapin's method of representing $\mathscr{P}_X$ as a semigroup of total transformations: namely, let $0\notin X$, put $Y=X\cup 0$ and

$$F_0=\{\beta\in\mathscr{T}_Y: 0\beta=0\},$$

and define $\theta:\mathscr{P}_X\to F_0$, $\alpha\to\alpha\theta$, where $x(\alpha\theta)=x\alpha$ if $x\in\operatorname{dom}\alpha$ and $x(\alpha\theta)=0$ otherwise. Clearly θ is an isomorphism. We extend the notions of defect, collapse and shift of $\alpha\in\mathscr{T}_X$ to elements of $\mathscr{P}_X$ as follows: for each $\alpha\in\mathscr{P}_X$, let

$$D^*(\alpha)=X\backslash X\alpha \quad \text{and} \quad d^*(\alpha)=|D^*(\alpha)|$$

$$C^*(\alpha)=C(\alpha)\cup(X\backslash\operatorname{dom}\alpha) \quad \text{and} \quad c^*(\alpha)=|C^*(\alpha)|$$

$$S^*(\alpha)=\{x\in\operatorname{dom}\alpha: x\alpha\neq x\}\cup(X\backslash\operatorname{dom}\alpha) \quad \text{and} \quad s^*(\alpha)=|S^*(\alpha)|$$

Theorem 8. *An element α of $\mathscr{P}_X$ can be written as a product of idempotents in $\mathscr{P}_X$ different from the identity if and only if either $s^*(\alpha)<\aleph_0$ and $d^*(\alpha)\neq 0$ or $s^*(\alpha)=c^*(\alpha)=d^*(\alpha)\geqq\aleph_0$.*

Proof. Suppose $\alpha\in\mathscr{E}_X^*$ and $\beta=\alpha\theta$. Then $\beta\in\mathscr{E}_Y$ and so, by Howie's Theorem, either $s(\beta)<\aleph_0$ and $d(\beta)\neq 0$ or $s(\beta)=c(\beta)=d(\beta)\geqq\aleph_0$. Since $Y\backslash Y\beta=X\backslash X\alpha$, $C(\beta)=C^*(\alpha)$ and $S(\beta)=S^*(\alpha)$, this produces the desired result. Now suppose $\alpha\in\mathscr{P}_X$ and α satisfies $s^*(\alpha)=c^*(\alpha)=d^*(\alpha)\geqq\aleph_0$. Then $\beta=\alpha\theta\in F_0$ and β satisfies the corresponding condition in Howie's Theorem. Suppose

$$\beta=\begin{pmatrix} B_i & C & y_j & a_n \\ x_i & 0 & z_j & a_n \end{pmatrix}$$

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where each B_i contains at least 2 elements and $y_j \neq z_j$ for each j. Choose $b_i \in B_i$ and let

$$\lambda = \begin{pmatrix} B_i & C & y_j & a_n \\ b_i & 0 & y_j & a_n \end{pmatrix} \qquad \mu = \begin{pmatrix} b_i & 0 & y_j & a_n & D \\ x_i & 0 & z_j & a_n & d \end{pmatrix}$$

where $D = (C \backslash 0) \cup (C(\beta) \backslash \{b_i\})$ and $d \in D$. Then $\beta = \lambda\mu$ where $\lambda^2 = \lambda \in F_0$. Now since $c(\beta)$ equals either $|\cup B_i|$ or $|C|$, we have $|D| = c(\beta)$. Thus, since $|I| \leqq c(\beta)$ and $|J| \leqq s(\beta)$, we have $c(\mu) = s(\mu) = c(\beta)$. Moreover, $D(\mu) = D(\beta) \backslash d$ and so $d(\mu) = s(\beta)$. Therefore, by Howie's Theorem, $\mu | X$ is a product of idempotents in $\mathscr{T}_X$, each of which can be extended in an obvious way to an idempotent in F_0. By applying the isomorphism θ^{-1}, we obtain a product of idempotents in $\mathscr{P}_X$ that equals α.

Since the proof for the case when $s^*(\alpha) < \aleph_0$ and $d^*(\alpha) \neq 0$ can be carried through in an entirely similar manner, we omit the details. However we note that in this case β cannot look like

$$\begin{pmatrix} 0 & y_j & a_n \\ 0 & z_j & a_n \end{pmatrix}$$

where $|J| < \aleph_0$ for this would mean β is a permutation, contradicting $d(\beta) \neq 0$. In other words, either $|C| \geqq 2$ or $I \neq \square$: this fact can be used to ensure that defects are non-zero.

We can now write $\mathscr{E}_X^* = V^* \cup H^*$ where

$$V^* = \{\alpha \in \mathscr{P}_X : s^*(\alpha) < \aleph_0 \quad \text{and} \quad d^*(\alpha) \neq 0\}$$

$$H^* = \{\alpha \in \mathscr{P}_X : s^*(\alpha) = d^*(\alpha) = c^*(\alpha) \geqq \aleph_0\}$$

and $\square \in H^*$. Using the results of Section 2 and the isomorphism θ, it is a simple matter to check that the ideals of V^* take the form

$$V_n^* = \{\alpha \in V^* : d^*(\alpha) \geqq n\}$$

where $n \geqq 1$, and the ideals of H^* equal

$$H^*(\delta, \xi) = \{\alpha \in H^* : d^*(\alpha) \geqq \delta \quad \text{and} \quad r(\alpha) < \xi\}$$

for some δ, ξ satisfying $\aleph_0 \leqq \delta \leqq \kappa$ and $1 \leqq \xi \leqq \kappa'$. In addition, Green's relations on V^* and on H^* are precisely what one would expect given the results of Section 3.

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