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One variable equations over semigroups

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An analogue of the theorem on the existence of a primitive element for separable extensions of fields is presented for semigroups. This has two immediate consequences.

(i) A semigroup is algebraically closed with respect to equations in several variables if and only if it is closed with respect to equations in a single variable.

(ii) Any countable semigroup C is embedded in a two-generator semigroup, one of whose generators is in C.

Further, a proof is given that any free product of a semigroup of order one with one of order two is SQ-universal, that is, its factor semigroups embed all countable semigroups. The proofs are adaptations of one used by Trevor Evans, *Proc. Amer. Math. Soc.* 3 (1952), 614-620, to show that a free product of two infinite cyclic semigroups is SQ-universal.

The theorem on the existence of a primitive element for fields states that for separable extensions of fields the adjunction of several algebraic elements to a field can be accomplished with a single adjunction. An analogous result holds for groups (see [3]). The following theorem extends this to semigroups and thus answers a question posed by Professor B.H. Neumann in connection with investigations into algebraically closed semigroups.

THEOREM 1. *Let* A *be a semigroup containing a subsemigroup* B

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such that A *can be generated by* B *and a countable subset* $a_1, a_2, \ldots,$ *of elements of* A. *Then there exists an oversemigroup* A_1 *of* A *and an element* $a \in A_1$ *such that* A_1 *is generated by* B *and the single (primitive) element* a. *Further, to any prescribed integers* $n > m \geq 3$, *the oversemigroup* A_1 *can be so chosen that the primitive element* $a \in A_1$ *satisfies* $a^n = a^m$.

The proof of Theorem 1 is essentially a variation of one used by Trevor Evans in [2] to show that any countable semigroup can be embedded in a two-generator semigroup. Before demonstrating the proof, however, we first list two immediate consequences of the Theorem.

The first application of the Theorem is to equations over semigroups. An equation over a semigroup S has the form

(1) $f(x_1, x_2, \ldots, x_n ; s_1, s_2, \ldots) = g(x_1, x_2, \ldots, x_n ; s_1, s_2, \ldots)$

where f and g are words in the variables x_i and the elements $s_i \in S$. (1) is consistent over S if S can be embedded in an oversemigroup T containing elements t_i such that (1) is satisfied in T with $x_i = t_i$. S is (m, n)-algebraically closed if every consistent system of m equations in n variables can be solved in S itself. The following Corollary then follows directly from the Theorem.

COROLLARY 1. S *is* (m, n)*-algebraically closed if and only if it is* $(m, 1)$*-algebraically closed.*

The second application of the theorem extends Evans' result.

COROLLARY 2. *Let* C *be a countable semigroup. Then* C *can be embedded in a two-generator semigroup generated by any arbitrarily chosen element* $c \in C$ *and a second generator* g *which satisfies, for any prescribed integers* $n > m \geq 3$, *the relation* $g^n = g^m$.

The second Corollary follows from the Theorem by taking B as the cyclic semigroup generated by c. In particular, if C contains a zero element 0, then c can be chosen as 0. Otherwise, we may first embed C in the semigroup obtained from C by adjoining a zero element.

The proof of the Theorem is based on the following Lemma. Unless otherwise specified we use the notation of Clifford and Preston [1]. In particular, ΔS denotes the diagonal of $S \times S$, and if S is any semigroup without identity element, then S^1 denotes the semigroup obtained from S by adding an identity element 1 to S. Otherwise, $S^1 = S$. Also, $\langle x;\ x^m = x^n \rangle$ denotes the cyclic semigroup generated by x satisfying $x^m = x^n$, and $S * T$ denotes the free product of S and T.

LEMMA. *Let* S *and* T *be semigroups and* $P = S * T$. *Let* Q *be a subsemigroup of* P *having the following properties:*

(i) the subsemigroup H *of* P *generated by* S *and* Q *is their free product;*

(ii) $uv \in H$, $u \in S$ *imply that* $v \in H$;

(iii) $upv \in H$, $p \in H \backslash S$, $u, v \in P^1$ *imply that* $u, v \in H^1$.

Let β *be a congruence on* H *such that* $\beta \cap (S \times S) = \Delta S$ *and* α *be the congruence on* P *generated by* β. *Then* P/α *embeds* H/β *naturally and* $\alpha \cap (S \times S) = \Delta S$.

Proof. By Lemma 9.9 of [1], H/β is naturally embedded in P/α if and only if $\alpha \cap (H \times H) = \beta$. Thus, to prove the Lemma we must consider elements of the form

(2) $$(w_1, w_2) = (u_0 p_1 u_1 p_2 \ldots p_k u_k,\ u_0 q_1 u_1 q_2 \ldots q_k u_k)$$

where for each i, $(p_i, q_i) \in \beta$, $u_i \in P^1$, and show that $(w_1, w_2) \in H \times H$ implies $(w_1, w_2) \in \beta$.

The proof is by induction on k. First, let $(u_0 p_1 u_1,\ u_0 q_1 u_1) \in (H \times H)$. Since $\beta \cap (S \times S) = \Delta S$ we may assume $(p_1, q_1) \notin S \times S$. In particular, suppose $p_1 \notin S$. Then, by *(iii)*, $u_0 p_1 u_1 \in H$ implies that $u_0, u_1 \in H^1$ so that $(u_0 p_1 u_1,\ u_0 q_1 u_1) \in \beta$. In the general case (2), we may again assume $p_1 \in H \backslash S$. Then by *(iii)*, $u_0,\ u_1 p_2 \ldots p_k u_k \in H$. Since $u_0 q_1 \in H$ we have, by *(ii)* or *(iii)*, whichever is applicable, that $u_1 q_2 \ldots q_k u_k \in H$ so that $(u_1 p_2 \ldots p_k u_k,\ u_1 q_2 \ldots q_k u_k) \in (H \times H) \cap \alpha$. By induction, it now follows

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that $u_i \in H$ for all i so that $(w_1, w_2) \in \beta$, which proves the Lemma since $\alpha \cap (S \times S) = \beta \cap (S \times S) = \Delta S$.

Proof of Theorem 1. Let $P = B * \langle x \rangle$ and $b \in B$ be an arbitrarily chosen element of B. We define elements $h_i \in P$ in one-to-one correspondence with the generators a_i of A as follows:

$$h_i = xbx^{2+i}bx, \qquad i = 1, 2, \ldots . \tag{3}$$

Let Q be the subsemigroup of P generated by the h_i. Since an element of P is in Q if and only if it has the form $xb\ldots bx^2bx^{2+i}bx^2bx^{2+j}bx \ldots bx$, it is clear that Q is freely generated by the h_i. A similar analysis shows that H, the subsemigroup of P generated by Q and B is their free product. Thus, H satisfies *(i)* of the Lemma with S replaced by B. Part *(ii)* of the Lemma is trivial to verify, and *(iii)* follows from the special forms of the elements of $H \backslash B$. Thus, the Lemma applies. In particular, since $H = B * Q$ there is a congruence β on H with $\beta \cap (B \times B) = \Delta B$ and $H/\beta \cong A$. (That is, H can be mapped homomorphically onto A by φ, say, where φ is the identity map on B and maps each h_i onto $a_i \in A$.) Thus, if α is the congruence on P generated by β, then P/α embeds $H/\beta \cong A$ naturally and since $\alpha \cap (B \times B) = \Delta B$, B is embedded in P/α, which proves the first part of the Theorem since P/α is generated by $B\alpha$ and $x\alpha$.

For the second part of the Theorem we further embed the oversemigroup $A_1 = P/\alpha$ constructed above. Thus, consider A_1, a semigroup generated by B and $a \in A_1$. Let now $P_1 = B * \langle y; y^n = y^m \rangle$, $n > m \geq 3$. Choose $b \in B$ as above and set $h = yby^3by$. Clearly, h generates an infinite cyclic subsemigroup $Q_1 \subset P_1$. Let H_1 be generated in P_1 by B and Q_1. As above, we obtain that $H_1 = B * Q_1$ so that H_1 satisfies *(i)* of the Lemma. Part *(ii)* of the Lemma is also easy to verify here. For Part *(iii)*, we need only remark that since $n > m \geq 3$, the relation for y has no effect in deciding if an element lies in H_1 since products of elements of H_1 introduce, at most, quadratic factors of y. Thus, the Lemma is again applicable, and since $H_1 = B * Q_1$ there is, as above, a congruence

β_1 on H_1 such that $H_1/\beta_1 \cong A_1$ and $\beta_1 \cap (B \times B) = \Delta B$. Hence, if α_1 is the congruence on P_1 generated by β_1, then P_1/α_1 embeds H_1/β_1. Finally, since $\alpha_1 \cap (B \times B) = \Delta B$, we have embedded A_1 and, hence, A in P_1/α_1 which is generated by B and the congruence class containing y, which proves the Theorem.

If we drop the condition in Corollary 2 that one of the two generators of the oversemigroup of C is in C, then we obtain a slightly sharper result. For the following Theorem let $P_2 = \langle x;\ x^2 = x \rangle * T_2$, where T_2 is any semigroup of order 2 with elements t_1, t_2, say.

THEOREM 2. *Let C be a countable semigroup. Then C can be embedded in a factor semigroup of P_2.*

Proof. First embed C in a two-generator semigroup C_1 as provided by Corollary 2 or Evans [2]. Let $H_2 \subset P_2$ be the subsemigroup of P_2 generated by $h_1 = xt_1x$ and $h_2 = xt_2x$. An element $h \in H_2$ has the form $h = xs_1xs_2\ldots s_kx$, where $s_i \in T_2$. Since h is uniquely determined by the sequence $s_1, s_2, \ldots, s_k$, it is clear that H_2 is freely generated by the h_i. Thus, there is a congruence β_2 on H_2 such that $H_2/\beta_2 = C_1$. Let α_2 be the congruence of P_2 generated by β_2. Again, the proof will be completed by showing that P_2/α_2 embeds H_2/β_2 or, equivalently, that $\alpha_2 \cap (H_2 \times H_2) = \beta_2$. Thus, let $(p_i, q_i) \in \beta_2$, $i = 1, 2, \ldots$, and

$$(4) \qquad (w_1, w_2) = (u_op_1u_1p_2\ldots p_ku_k,\ u_oq_1u_1q_2\ldots q_ku_k) \in H_2 \times H_2 ,$$

where $u_i \in P_2^1$. Since $x^2 = x$ and each p_i and q_i begins and ends with an x, (w_1, w_2) can be expressed in the form

$$(5) \qquad (w_1, w_2) = (v_op_1v_1p_2\ldots p_kv_k,\ v_oq_1v_1q_2\ldots q_kv_k)$$

where, for each i, $v_i = xu_ix$ if $u_i \notin H_2^1$ and $v_i = u_i$, otherwise. In either case each $v_i \in H_2^1$ so that $(w_1, w_2) \in \beta_2$, which proves the Theorem.

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