Forum of Mathematics, Sigma (2023), **Vol. 11**:e16 1–28
doi:10.1017/fms.2023.14

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RESEARCH ARTICLE

Decompositions of moduli spaces of vector bundles and graph potentials

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Received: 23 June 2022; **Accepted:** 7 February 2023

2020 Mathematics Subject Classification: 14J33, 18G80, 14D20, 14J45

Abstract
We propose a conjectural semiorthogonal decomposition for the derived category of the moduli space of stable rank 2 bundles with fixed determinant of odd degree, independently formulated by Narasimhan. We discuss some evidence for and furthermore propose semiorthogonal decompositions with additional structure.

We also discuss two other decompositions. One is a decomposition of this moduli space in the Grothendieck ring of varieties, which relates to various known motivic decompositions. The other is the critical value decomposition of a candidate mirror Landau–Ginzburg model given by graph potentials, which in turn is related under mirror symmetry to Muñoz's decomposition of quantum cohomology. This corresponds to an orthogonal decomposition of the Fukaya category. We discuss how decompositions on different levels (derived category of coherent sheaves, Grothendieck ring of varieties, Fukaya category, quantum cohomology, critical sets of graph potentials) are related and support each other.

1. Introduction

The moduli space $\mathrm{M}_C(2,\mathcal{L})$ of stable rank 2 vector bundles with fixed determinant of odd degree on a curve C of genus $g \geq 2$ is an important and well-studied object, its history going back to the 1960s [44, 37, 43]. It is a smooth projective Fano variety of dimension $3g-3$ when considered as an algebro-geometric object, and a monotone symplectic manifold of dimension $6g-6$ when considered as a symplecto-geometric object. They also arise as moduli spaces of flat unitary $\mathrm{SU}(2)$-connections on Riemann surfaces, or character varieties of conjugacy classes.

Associated to $\mathrm{M}_C(2,\mathcal{L})$, one can study its invariants, such as its Betti numbers [22] or Chow motive [1]. The idea behind the description of the Chow motive is to find a *decomposition* which expresses it in terms of easier objects that still possess a connection to the curve C.

In this article we will discuss 3 types of decompositions, which have (conjectural) relations to each other:

- In Conjecture A, we propose a *semiorthogonal* decomposition of the derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$.
- In Theorem B, we describe the critical values and critical loci of the graph potentials from [8, 7]. This describes *orthogonal* decompositions of the Fukaya category resp. the category of matrix factorisations.

When seen as candidate Landau–Ginzburg mirrors to $\mathrm{M}_C(2,\mathcal{L})$, this description is consistent with the eigenvalue decomposition of the quantum multiplication $\mathrm{c}_1(\mathrm{M}_C(2,\mathcal{L})) *_0 -$ on quantum cohomology $\mathrm{QH}^\bullet(\mathrm{M}_C(2,\mathcal{L}))$ following Muñoz as described in Proposition 3.2.

- In Theorem C, we describe an identity in $\mathrm{K}_0(\mathrm{Var}/k)$, the Grothendieck ring of varieties, which relates to various known and expected *motivic* decompositions.

The decompositions discussed in this article for $\mathrm{M}_C(2,\mathcal{L})$ can serve as a blueprint for other decompositions of moduli spaces of sheaves, and we will discuss related work in Sections 2.1 and 4.1.

Semiorthogonal decompositions of the derived category

Semiorthogonal decompositions encode many important geometric properties of varieties (see Kuznetsov's ICM address [31] for an introduction) and currently play an important role in the geometry and construction of moduli spaces of sheaves. Hence, we propose the following conjecture.

Conjecture A. *Let C be a smooth projective curve of genus $g \geq 2$. Then there exists a semiorthogonal decomposition*

$$\begin{aligned}\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L})) = \big\langle &\mathbf{D}^{\mathrm{b}}(\mathrm{pt}), \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C), \dots \\ &\dots, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \dots \\ &\dots, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C), \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(\mathrm{pt})\big\rangle.\end{aligned} \tag{1}$$

This was conjectured independently by Narasimhan (as communicated in [34]) and the authors (see [4, Conjecture 7]). We also give a further refinements of this conjecture in Sections 2.2 and 2.3. We suggest that there exists a

- a minimal Lefschetz decomposition in the sense of [29] with respect to the generator of the Picard group $\mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}(1)$;
- a Ringel–Samokhin-type decomposition (which we will introduce), which involves a symmetry under an anti-involution.

We will explain the state of the art for these in Section 2.1.

Eigenvalue and critical value decompositions

Under mirror symmetry, $\mathrm{M}_C(2,\mathcal{L})$ corresponds to a Landau–Ginzburg model $f\colon Y \to \mathbb{A}^1$ where Y is a quasiprojective variety and f is a regular function on it. In the introduction to Section 3, we recall the motivation to describe on one hand the eigenvalues of the quantum multiplication $\mathrm{c}_1(\mathrm{M}_C(2,\mathcal{L})) *_0 -$ and on the other hand the critical values of f.

In [8] we have introduced *graph potentials*, and in [7] we explained how they can be seen as the first step in the construction of the mirror Landau–Ginzburg model. The following theorem summarises the situation and gives further evidence for Conjecture A and the claim that graph potentials are (partial) mirror Landau–Ginzburg mirrors to $\mathrm{M}_C(2,\mathcal{L})$.

Theorem B. *The critical values of graph potential for the necklace graph in Figure 4 are given by*

$$8(1-g), 8(2-g)\sqrt{-1}, 8(3-g), \dots, 0, \dots, 8(g-3), 8(g-2)\sqrt{-1}, 8(g-1). \tag{2}$$

The dimension of the critical loci with modulus $8g-8-8k$ is equal to k, where $k = 0, \dots, g-1$. For trivalent graphs with a perfect matching, the critical values form a subset of (2).

The important conclusions are that graph potentials already see all the (expected) critical values, even if they are only restrictions of the full Landau–Ginzburg mirror to certain torus charts, and that the critical loci are of the expected dimension.

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This result mirrors the eigenvalue decomposition of quantum cohomology $\mathrm{QH}^\bullet(\mathrm{M}_C(2,\mathcal{L}))$ described in Proposition 3.2, as obtained by Muñoz. It gives further evidence to the claim that graph potentials can be used as building blocks for the full Landau–Ginzburg mirror of $\mathrm{M}_C(2,\mathcal{L})$.

The Grothendieck ring of varieties

The third type of decomposition we consider is an identity for $\mathrm{M}_C(2,\mathcal{L})$ in the Grothendieck ring of varieties. This ring encodes the cut-and-paste relation, and because many invariants of varieties satisfy such a cut-and-paste relation it can be seen as the universal invariant encoding this. As such, it implies various known identities for motivic invariants of $\mathrm{M}_C(2,\mathcal{L})$, and as we explain in Corollary 4.1, it is consistent with Conjecture A. The result is the following:

Theorem C. *We have the equality*

$$[\mathrm{M}_C(2,\mathcal{L})] = \mathbb{L}^{g-1}[\mathrm{Sym}^{g-1} C] + \sum_{i=0}^{g-2}(\mathbb{L}^i + \mathbb{L}^{3g-3-2i})[\mathrm{Sym}^i C] + T \tag{3}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$*, for some class* T *such that* $(1+\mathbb{L})\cdot T = 0$.

We expect that $T = 0$, but our method of proof is not strong enough to remove this error term.

2. Semiorthogonal decompositions

In this section, we will discuss the structure of the bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$. We also suggest two semiorthogonal decompositions with additional structure, namely a Lefschetz decomposition (as introduced by Kuznetsov [29]). We introduce the notion of a Ringel–Samokhin decomposition, related to the structure of quasi-hereditary algebras, and an informal analogy between moduli spaces of bundles with flag varieties.

2.1. On the conjecture

In Conjecture A, we have formulated a conjecture giving a semiorthogonal decomposition into geometrically meaningful pieces, namely copies of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ for $i = 0,\dots,g-1$. This was conjectured independently by Narasimhan (as communicated in [34]) and the authors (see [4, Conjecture 7]). The conjecture does not specify the embedding functors.

Moreover, it was a folklore conjecture that each of the pieces $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ for $i = 1,\dots,g-1$ is indecomposable (i.e., does not admit further semiorthogonal decompositions). This was proven for $i \leq \lfloor \frac{g+3}{2} \rfloor - 1$ in [3, Corollary D], with a weaker result being given in [11, Theorem 1.3], and finally for all $i \leq g-1$ in [36, Theorem 1.4].

State of the art

The first steps (before the precise conjecture was phrased) were taken by Narasimhan [41, 42], and independently, by Fonarev–Kuznetsov [17] (for C generic), who have shown that the Fourier–Mukai functor $\Phi_{\mathcal{W}}$, where $\mathcal{W}$ is the universal vector bundle on $C \times \mathrm{M}_C(2,\mathcal{L})$, is fully faithful. Together with the exceptional objects $\mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}$ and Θ, this gives rise to *3 components* in (1).

The first and third authors have shown in [10] that it is possible to twist the embedding $\Phi_{\mathcal{W}}$ by Θ to obtain the embedding of *the four low-dimensional components* in (1) (for $g \geq 12$). The embedding of a copy of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C)$ was recently obtained by Narasimhan–Lee in [35, Theorem 1.2] (for C not hyperelliptic and $g \geq 16$).

Most importantly, in [50] and [53] a semiorthogonal decomposition that contains *all* expected components was obtained by Tevelev–Torres and Xu–Yau. The essential ingredient in the first approach is Thaddeus' wall-crossing picture, which is also used in Section 4 to prove Theorem C. The second

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approach uses Teleman's Borel–Weil–Bott theory on the moduli stack of SL_2- bundles. What remains to be proven is that the complement to the subcategory generated by these components is trivial.

Outlook

This conjecture forms a part of a greater program, which aims at finding natural semiorthogonal decompositions of the bounded derived categories of coherent sheaves on the moduli spaces of (stable) objects in the bounded derived categories of coherent sheaves on algebraic varieties.

For moduli of higher rank vector bundles there exist various motivic decompositions (see also Section 4.1), and these suggest semiorthogonal decompositions of $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(r,\mathcal{L}))$ with $\gcd(r, \deg\mathcal{L}) = 1$. In this case, $\mathrm{M}_C(r,\mathcal{L})$ is a smooth projective Fano variety of dimension $(r^2-1)(g-1)$ and index 2. The motivic decompositions suggest that derived categories of products of $\mathrm{Sym}^i C$ will play a role, and for $r = 3$, this was made precise in [20, Conjecture 1.9].

Related to this is the indecomposability of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)$ for all $i \leq g-1$ which is proven in [36, Theorem 1.4]. The semiorthogonal decomposition of the derived categories of $\mathrm{Sym}^i C$ for $i \geq g$ was obtained by Toda in [52] and studied again in [25, §1.4] and [9, Theorem D].

Another important class of moduli spaces of coherent sheaves is punctual Hilbert schemes. The study of (partial) semiorthogonal decompositions was started in [28]. It was generalized to higher dimensions in [27, 6]. Further generalizations to nested Hilbert schemes were obtained in [25, §3.1.4] and [9, Theorem E]. In a different direction, a semiorthogonal decomposition for the Hilbert square of a cubic hypersurface was obtained in [5] using the Fano variety of lines.

2.2. Lefschetz decompositions

A more structured form of Conjecture A is given by that of a Lefschetz decomposition, a notion introduced by Kuznetsov [29] in his theory of homological projective duality.

Definition 2.1. Let X be a smooth projective variety and $\mathcal{O}_X(1)$ be a line bundle on X. A *(right) Lefschetz decomposition* of $\mathbf{D}^{\mathrm{b}}(X)$ with respect to the line bundle $\mathcal{O}_X(1)$ is a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{O}_X(1), \dots, \mathcal{A}_{n-1} \otimes \mathcal{O}_X(n-1)\rangle, \tag{4}$$

such that

$$\mathcal{A}_{n-1} \subseteq \mathcal{A}_{n-2} \subseteq \dots \subseteq \mathcal{A}_0. \tag{5}$$

We say that n is the *length* of the decomposition, and $\mathcal{A}_0$ is the *starting block*.

The decomposition (4) is *rectangular* if $\mathrm{A}_{n-1} = \dots = \mathrm{A}_0$.

By [30, Lemma 2.18(i)], a Lefschetz decomposition is completely determined by its starting block, as one can inductively define $\mathrm{A}_i{:=}^{\perp}(\mathrm{A}_0 \otimes \mathrm{O}_X(-i)) \cap \mathrm{A}_{i-1}$. This allows us to define the notion of a *minimal* Lefschetz collection, by considering the (partial) inclusion order on the starting blocks for Lefschetz collections of $\mathbf{D}^{\mathrm{b}}(X)$ with respect to $\mathcal{O}_X(1)$.

In [32, Definition 1.3(iii)], the notion of *residual category* for a Lefschetz exceptional collection is introduced, to measure how far it is from being rectangular. One can generalize this to an arbitrary Lefschetz decomposition as in [32, Definition 2.6].

Definition 2.2. Let $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{O}_X(1), \dots, \mathcal{A}_{n-1} \otimes \mathcal{O}_X(n-1)\rangle$ be a Lefschetz decomposition of length n with respect to the line bundle $\mathcal{O}_X(1)$. The *residual category* $\mathcal{R}$ of this decomposition is defined as

$$\mathcal{R} := \langle \mathcal{A}_{n-1}, \mathcal{A}_{n-1} \otimes \mathcal{O}_X(1), \dots, \mathcal{A}_{n-1} \otimes \mathcal{O}_X(n-1)\rangle^{\perp}. \tag{6}$$

Remark 2.3. As discussed in [33] (in the context of Fano varieties with vanishing odd cohomology), properties of the residual category are conjecturally related to the structure of fiber over zero for the

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quantum cohomology of X. For Fano varieties with odd cohomology, one aspect of a generalization of this conjecture is the prediction that the dimension of the Hochschild homology of the residual category is equal to the length of the eigenspace associated to the eigenvalue 0. By Proposition 3.2, this eigenspace is isomorphic to $\mathrm{H}^\bullet(\mathrm{Sym}^{g-1} C)$ (of dimension $(2g+1)!/g!^2$), which, by the Hochschild–Kostant–Rosenberg decomposition, is isomorphic to $\mathrm{HH}_\bullet(\mathrm{Sym}^{g-1} C)$. Hence, the equality holds.

This brings us to the following more precise form of Conjecture A, incorporating a Lefschetz structure.

Conjecture 2.4. *Let C be a smooth projective curve of genus g. Then there exists a minimal Lefschetz decomposition of $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$ of length 2 with respect to Θ, whose first block has a semiorthogonal decomposition*

$$\mathcal{A}_0 = \langle \mathbf{D}^{\mathrm{b}}(\mathrm{pt}), \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C) \dots, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C)\rangle, \tag{7}$$

so, in particular, the residual category is $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C)$.

For $g = 2$, this conjecture is known by considering the semiorthogonal decomposition [14, Theorem 2.9] and mutating the object $\mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}(-1)$ to the right.

Proposition 2.5 (Bondal–Orlov). *Let C be a curve of genus 2. Then*

$$\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L})) = \langle \mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}, \Phi_{\mathcal{W}}(\mathbf{D}^{\mathrm{b}}(C)), \Theta\rangle \tag{8}$$

is a minimal Lefschetz decomposition, whose residual category is equivalent to $\mathbf{D}^{\mathrm{b}}(C)$.

Homological projective duality

After constructing a (minimal) Lefschetz decomposition, the next step would be to understand the homological projective dual of $\mathrm{M}_C(2,\mathcal{L})$ as in [29]. This would give the structure of the derived categories of linear sections of $\mathrm{M}_C(2,\mathcal{L})$, and in particular, for a smooth hyperplane section. By Theorem 1.1 of op. cit., we have that Conjecture 2.4 gives a semiorthogonal decomposition

$$\begin{aligned}\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}) \cap H) &= \langle \mathcal{C}_H, \mathcal{A}_1(1)\rangle \\ &= \langle \mathcal{C}_H; \mathbf{D}^{\mathrm{b}}(\mathrm{pt}), \mathbf{D}^{\mathrm{b}}(C), \dots, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C)\rangle.\end{aligned} \tag{9}$$

To indicate the complexity of the homological projective dual, observe that for $g = 2$, the category $\mathcal{C}_H$ consists of 7 exceptional objects (because it's the complement to $\mathcal{O}(1)$ in the derived category of $\mathrm{Bl}_5 \mathbb{P}^2 \cong Q \cap Q' \cap H$ in $\mathbb{P}^5$).

For $g = 3$, one can use the description from [16] of $\mathrm{M}_C(2,\mathcal{L})$ for a hyperelliptic curve C as the zero locus of $(\mathrm{Sym}^2 \mathcal{U}^\vee)^{\oplus 2}$ on $\mathrm{Gr}(g-1, 2g+2)$, together with the Borel–Weil–Bott theorem and the Koszul sequence, to compute that the Hodge diamond of a hyperplane section is given by

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 0 && 0 &&&& \\ &&& 0 && 1 && 0 &&& \\ && 0 && 3 && 3 && 0 && \\ & 0 && 0 && 2 && 0 && 0 & \\ 0 && 1 && 61 && 61 && 1 && 0 \\ & 0 && 0 && 2 && 0 && 0 & \\ && 0 && 3 && 3 && 0 && \\ &&& 0 && 1 && 0 &&& \\ &&&& 0 && 0 &&&& \\ &&&&& 1 &&&&& \end{array} \tag{10}$$

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Using the Hochschild–Kostant–Rosenberg decomposition and additivity of Hochschild homology, we see that

$$\mathrm{HH}_\bullet(\mathcal{C}_H) = k[-3] \oplus k^{64}[-1] \oplus k^5 \oplus k^{64}[1] \oplus k[3]. \tag{11}$$

There is no obvious modular interpretation for (a semiorthogonal decomposition of) $\mathcal{C}_H$, and it does not look like the homological projective dual has an easy description.

2.3. Ringel–Samokhin-type decompositions

There is another way in which we can impose further conditions on a semiorthogonal decomposition for $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$. This condition is inspired by the theory of hereditary algebras and Ringel duality, and by similar semiorthogonal decompositions for $\mathbf{D}^{\mathrm{b}}(G/B)$ obtained by Samokhin, and encodes special symmetries not found in most semiorthogonal decompositions.

Let X be a smooth projective variety, and consider a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_0, \ldots, \mathcal{A}_n \rangle \tag{12}$$

of length $n+1 \geq 2$. Let σ be an anti-equivalence of $\mathbf{D}^{\mathrm{b}}(X)$ which is moreover an involution. Let $\mathcal{B}_k$ be the image of $\mathcal{A}_{n-k}$ under this involution for $k = 0, \ldots, n$. Then we have a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{B}_0, \ldots, \mathcal{B}_n \rangle. \tag{13}$$

However, consider the left dual decomposition in the sense of [47, Definition 3.6] denoted by

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{C}_0, \ldots, \mathcal{C}_n \rangle. \tag{14}$$

This brings us to the following definition.

Definition 2.6. A decomposition (12) is of *Ringel–Samokhin type* if there exists an anti-equivalence σ of $\mathbf{D}^{\mathrm{b}}(X)$ such that for all $k = 0, \ldots, n$, the subcategory $\mathcal{C}_k$ is the image of $\mathcal{B}_k$ under σ.

Remark 2.7. In [47], Samokhin studied such exceptional collections for the derived category $\mathbf{D}^{\mathrm{b}}(G/B)$, where G/B is the full flag variety of a group of rank 2, and the anti-equivalence is given by $\mathbf{R}\mathcal{H}om(-, \omega_{G/B}^{1/2})$.

We can refine Conjecture A into a semiorthogonal decomposition of Ringel–Samokhin-type as follows.

Conjecture 2.8. *Let C be a smooth projective curve of genus $g \geq 2$. Then there exists a Ringel–Samokhin-type decomposition*

$$\begin{aligned}\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L})) = \langle \mathbf{D}^{\mathrm{b}}(\mathrm{pt}), \mathbf{D}^{\mathrm{b}}(C), \ldots, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \ldots,\\ \ldots, \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(\mathrm{pt})\rangle\end{aligned} \tag{15}$$

with the anti-equivalence given by $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_{\mathrm{M}_C(2,\mathcal{L})}(1))$.

This conjecture highlights how semiorthogonal decompositions of $\mathrm{M}_C(2,\mathcal{L})$ are expected to have strong symmetry properties, much stronger than found for most varieties exhibiting semiorthogonal decompositions.

For $g = 2$, we can illustrate this conjecture using the following general result for Ringel–Samokhin-type decompositions of length 3.

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Proposition 2.9. *Let X be a smooth projective variety. Assume that we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{L}_1, \mathcal{A}, \mathcal{L}_2 \rangle \tag{16}$$

where $\mathcal{L}_1, \mathcal{L}_2$ are exceptional line bundles. Then this decomposition is of Ringel–Samokhin-type if and only if $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ is a theta characteristic of X.

Proof. A decomposition of the form (16) is determined by the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, as the subcategory $\mathcal{A}$ consists of the objects $E \in \mathbf{D}^{\mathrm{b}}(X)$ such that

$$\mathrm{Hom}^\bullet(\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X), \mathcal{L}_1) = \mathrm{Hom}^\bullet(\mathcal{L}_2, E) = 0. \tag{17}$$

The left dual decomposition of (16) is given by

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{L}_2, \mathbf{L}_{\mathcal{L}_2}\mathcal{A}, \mathcal{L}_1 \otimes \omega_X^\vee \rangle. \tag{18}$$

However, we consider the anti-equivalence given by dualizing and tensoring with a line bundle $\mathcal{M}$ to obtain the semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{L}_2^\vee \otimes \mathcal{L}, \mathcal{A}^\vee \otimes \mathcal{L}, \mathcal{L}_1^\vee \otimes \mathcal{L} \rangle. \tag{19}$$

These two semiorthogonal decompositions agree if and only if $\mathcal{L} \cong \mathcal{L}_2^{\otimes 2}$, so that we have $(\mathcal{L}_1^\vee \otimes \mathcal{L}_2)^{\otimes 2} \cong \omega_X^\vee$, and hence $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ is a theta characteristic of X. □

By Proposition 2.5, using that $\mathrm{M}_C(2, \mathcal{L})$ is of index 2, we obtain the following corollary.

Corollary 2.10. *Conjecture 2.8 holds for $g = 2$.*

3. Eigenvalue and critical value decompositions

To understand the results in this section, we will briefly recall homological mirror symmetry for Fano varieties. If X is a (smooth projective) Fano variety, its mirror is expected to be a Landau–Ginzburg model $f\colon Y \to \mathbb{A}^1$ (where Y is a quasiprojective variety), such that we have equivalences of triangulated categories

$$\begin{aligned} \mathrm{Fuk}(X) &\cong \mathrm{MF}(Y, f), \\ \mathbf{D}^{\mathrm{b}}(X) &\cong \mathrm{FS}(Y, f). \end{aligned} \tag{20}$$

On the first line, we have that the Fukaya category of X has an *orthogonal decomposition*, indexed by the eigenvalues of $\mathrm{c}_1(X) *_0 -$ (see [49]). For the *matrix factorization category*, we have an orthogonal decomposition, indexed by the critical values of f [46, Proposition 1.14]. Hence, under the homological mirror symmetry conjecture, these sets (of eigenvalues, and critical values) are the same.

For the second line, there is no such natural (semiorthogonal) decomposition. But the philosophy behind Dubrovin's conjecture (which a priori is only formulated in the case of semisimple quantum cohomology) predicts how an orthogonal decomposition of the triangulated category $\mathrm{Fuk}(X)$ gives rise to a semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(X)$. Generalizations of this philosophy to not necessarily semisimple quantum cohomology are discussed in [48].

Hence, to understand semiorthogonal decompositions for $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2, \mathcal{L}))$ using mirror symmetry, one can study either of the following:

1. the eigenvalues of the quantum multiplication $\mathrm{c}_1(\mathrm{M}_C(2, \mathcal{L})) *_0 -$ on the quantum cohomology $\mathrm{QH}^\bullet(\mathrm{M}_C(2, \mathcal{L}))$;
2. the critical values of the potential $f\colon Y \to \mathbb{A}^1$.

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Remark 3.1. There is no worked out candidate construction for the Landau–Ginzburg mirror (Y, f) of $\mathrm{M}_C(2, \mathcal{L})$ yet, let alone a proof of homological mirror symmetry. But the graph potentials we have constructed and studied in [8, 7] can be seen as open cluster charts of Y, and gluing these tori together along the birational transformations between them (which we can do because of the compatibilities from [8, Theorems 2.12 and 2.13]) is a first step one can take in the construction of the Landau–Ginzburg model mirror to $\mathrm{M}_C(2, \mathcal{L})$. We will recall this construction in Section 3.2.

A better understanding of the critical loci would be the next step in understanding if and how these graph potentials can be glued together to obtain (part of) the Landau–Ginzburg model. In Appendix A, we discuss the dimension of the different critical loci as a first step in this program.

3.1. Muñoz's eigenvalue decomposition for quantum cohomology

The eigenvalues of quantum multiplication with c_1 have been described, albeit via an indirect route. The following summarizes this description.

Proposition 3.2 (Muñoz). *The eigenvalue decomposition for the quantum multiplication by* $c_1(\mathrm{M}_C(2, \mathcal{L}))$ *on* $\mathrm{QH}^\bullet(\mathrm{M}_C(2, \mathcal{L}))$ *is*

$$\mathrm{QH}^\bullet(\mathrm{M}_C(2, \mathcal{L})) = \bigoplus_{m=1-g}^{g-1} H_m \tag{21}$$

where

1. *the eigenvalues are* $8(1-g), 8(2-g)\sqrt{-1}, 8(3-g), \ldots, 8(g-3), 8(g-2)\sqrt{-1}, 8(g-1)$*;*
2. H_m *is isomorphic (as a vector space) to* $\mathrm{H}^\bullet(\mathrm{Sym}^{g-1-|m|} C)$.

Proof. This is a combination of various results of Muñoz. First we have [40, Proposition 20], describing the eigenvalues of multiplication by the generator of the Picard group on the instanton Floer homology of the 3-manifold given by the product of the curve (seen as a real manifold) and S^1. The (conjectural) identification as rings from [40, Theorem 1] with the quantum cohomology of $\mathrm{M}_C(2, \mathcal{L})$ is in turn given by [39, Corollary 21] after an explicit description in terms of generators and relations for both rings. Hence, we can interpret any result for instanton Floer homology as a result for quantum cohomology. In [40, Conjecture 24], the conjectural decomposition was given, and in [38, Corollary 3.7], it was proved. □

Property $\mathcal{O}$

An interesting symmetry property for the quantum cohomology of a Fano variety X of index r is obtained by considering the exceptional collection $\mathcal{O}_X, \ldots, \mathcal{O}_X(r-1)$ in $\mathbf{D}^{\mathrm{b}}(X)$. The existence of this exceptional collection can be encoded in quantum cohomology using [19, Definition 3.1.1] as follows.

Definition 3.3. Let X be a smooth projective Fano variety, of index $r \geq 1$. Define

$$T := \max\{|u| \mid u \in \mathbb{C} \text{ is an eigenvalue of } c_1(X) *_0 -\} \in \overline{\mathbb{Q}}_{\geq 0}. \tag{22}$$

Then we say that X has *property* $\mathcal{O}$ if

1. T is an eigenvalue of $c_1(X) *_0 -$, of multiplicity 1;
2. if u is another eigenvalue of $c_1(X) *_0 -$ such that $|u| = T$, then there exists a primitive rth root of unity ζ such that $u = \zeta T$.

In [19, Conjecture 3.1.2], it was conjectured that this property holds for all Fano varieties. In [15], it was checked for all homogeneous varieties G/P, and in [48, Corollary 7.7], the case of complete intersections of index $r \geq 2$ in $\mathbb{P}^n$ was checked.

Hence, by Proposition 3.2, we immediately obtain the following result.

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Corollary 3.4. *Property* $\mathcal{O}$ *holds for* $\mathrm{M}_C(2, \mathcal{L})$*, where* $T = 8(g-1)$.

3.2. Graph potentials

Now we consider the other side of the mirror, and introduce graph potentials as candidate building blocks. Whilst no construction of the homological mirror exists yet as discussed in Remark 3.1 we will study the geometry of the critical values and loci in Section 3.3. Depending on the viewpoint, this gives further evidence that graph potentials are indeed building blocks for the mirror to $\mathrm{M}_C(2, \mathcal{L})$, or it gives further evidence for Conjecture A.

Graph potentials

We briefly recall the construction from [8, §2.1]. Let $\gamma = (V, E)$ be a connected undirected trivalent graph, of genus g. Thus, $\#V = 2g-2$ and $\#E = 3g-3$. As explained in loc. cit., it is natural to define everything in terms of graph (co)homology. The relevant space is

$$\widetilde{N}_\gamma := \mathrm{C}^1(\gamma, \mathbb{Z}) \tag{23}$$

in which a vertex $v \in V$, which is adjacent to the edges $e_{v_i}, e_{v_j}, e_{v_k}$, defines cochains x_i, x_j, x_k by $x_i(e_{v_a}) = \delta_{i,v_a}$.

We will also decorate trivalent graphs, by assigning a color to every vertex (i.e., we consider $c\colon V \to \mathbb{F}_2$ where $c(v) = 0$ (resp. $c(v) = 1$) means the vertex v is uncolored (resp. colored)). An uncolored vertex is drawn as a circle ○ whilst a colored vertex corresponds to a disc ●.

Construction 3.5. *The vertex potential* $\widetilde{W}_{v,c(v)}$ *for a vertex* $v \in V$ *in a colored graph* (γ, c) *is the sum of the four monomials*

$$x_i^{(-1)^{s_i}} x_j^{(-1)^{s_j}} x_k^{(-1)^{s_k}} \tag{24}$$

where $(s_i, s_j, s_k) \in \mathbb{F}_2^{\oplus 3}$ *ranges over all sign choices such that* $s_i + s_j + s_k = c(v)$*. Here,* x_i *is the coordinate variables in* $\mathbb{Z}[\widetilde{N}_\gamma]$ *corresponding to the ith edge in an enumeration* $e_1, \ldots, e_{3g-3}$ *of the edges.*

Hence, in the variables x, y, z *there are precisely two cases:*

$$\begin{aligned} \widetilde{W}_{v,0} &= xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \\ \widetilde{W}_{v,1} &= \frac{1}{xyz} + \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x}. \end{aligned} \tag{25}$$

The global structure of the colored graph (γ, c) *then defines the graph potential as the sum of vertex potentials*

$$\widetilde{W}_{\gamma,c} := \sum_{v \in V} \widetilde{W}_{v,c(v)}. \tag{26}$$

Example 3.6. In genus $g = 2$, there are precisely two trivalent graphs: the Theta graph and the dumbbell graph [8, Figure 1]. In Figure 1, we have shown the Theta graph with the second vertex colored, and following Construction 3.5 we get that the graph potential is given by

$$\widetilde{W}_{\gamma,c} = xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} + \frac{1}{xyz} + \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x}. \tag{27}$$

More examples are given in [8, §2.1].

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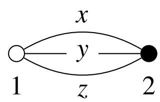

Figure 1. *Colored Theta graph in genus* $g = 2$.

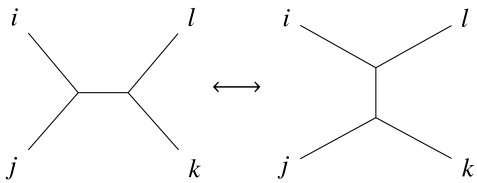

Figure 2. *Local picture of an elementary transformation of a trivalent graph.*

Elementary transformations

Trivalent graphs correspond to pair-of-pants decompositions, and for a given surface there exist many such decompositions up to isotopy. By Hatcher–Thurston [23, Appendix], they are related via certain operations, and for our purposes, we only need operation (I) of op. cit. The operation this induces on trivalent graphs is called an *elementary transformation*, and the local picture is given in Figure 2

In [8, §2.2], we have discussed how elementary transformations transform the associated graph potentials. The main results (given as Corollary 2.9 and Theorems 2.12 and 2.13 in op. cit.) state that

- up to biregular automorphism of the torus the graph potential only depends on the *parity* of the coloring;
- up to rational change of coordinates the graph potential only depends on the *genus* of the trivalent graph.

3.3. Critical value decomposition for graph potentials

Our final goal is to prove Theorem B. In this section, we will describe the critical values and discuss the conifold point of the graph potential. The computation of the critical loci is deferred to Appendix A.

Critical values

Let γ be a trivalent graph, and let c be a coloring. By [8, Corollary 2.9], we can assume that c has, at most, one colored vertex. To determine the critical values of the graph potential $\widetilde{W}_{\gamma,c}$ we need to determine solutions to the equations

$$x_i \frac{\partial}{\partial x_i} \widetilde{W}_{\gamma,c} = 0, \forall i = 1, \ldots, 3g-3. \tag{28}$$

As for the description of the behavior of the graph potential under elementary transformations in the previous section, we will study the local picture for (colored) trivalent graphs. These are given in Figure 3, where Figure 3(a) (resp. Figure 3(b)) corresponds to the case where the coloring is trivial (resp. the vertex v_2 is the unique colored vertex).

We can write the potential in the uncolored (resp. colored) case as

$$\widetilde{W}_{\gamma,0} = x\left(ab + \frac{1}{ab} + cd + \frac{1}{cd}\right) + x^{-1}\left(\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}\right) + \widetilde{W}^{\text{frozen}}_{\gamma,0} \tag{29}$$

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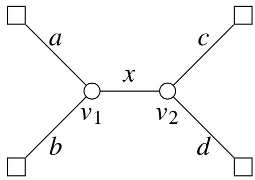
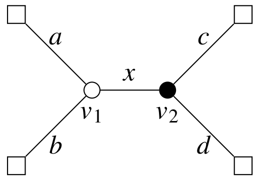


(a) Local picture of trivalent graph with c trivial



(b) Local picture of trivalent graph with c non-trivial

Figure 3. *Local pictures of colored trivalent graphs.*

resp.

$$\widetilde{W}_{\gamma,c} = x\left(ab + \frac{1}{ab} + cd + \frac{1}{cd}\right) + x^{-1}\left(\frac{a}{b} + \frac{b}{a} + cd + \frac{1}{cd}\right) + \widetilde{W}^{\text{frozen}}_{\gamma,c} \tag{30}$$

where we have split off the frozen part which does not involve the variable x. We call these expressions (without the frozen part) *edge potentials*.

Hence, we are interested in solutions to the equations $x\frac{\partial}{\partial x}\widetilde{W}_{\gamma,0} = 0$, resp. $x\frac{\partial}{\partial x}\widetilde{W}_{\gamma,c} = 0$, which can be rewritten by clearing the denominators as

$$x(1 + abcd)(ab + cd) = x^{-1}(ac + bd)(ad + bc) \tag{31}$$

resp.

$$x(a + bcd)(b + acd) = x^{-1}(c + abd)(d + abc). \tag{32}$$

Perfect matchings

The graph potential is defined as a sum over the vertex potentials associated to the vertices. But for certain graphs we can alternatively write the graph potential as a sum of edge potentials. For this we need a perfect matching (i.e., a subset $P \subseteq E$ of edges such that every vertex is contained in precisely one $e \in P$. If the genus of γ is g, then $\#P = g - 1$).

Given a perfect matching P for γ, we can rewrite the graph potential by summing over the edges in the matching and using the expressions (29), (30) for the edge potentials. Let e denote the edge between v_1 and v_2, and label the variables in Figure 3 as a_e, b_e, c_e, d_e, x_e. Then in the uncolored case we have

$$\widetilde{W}_{\gamma,0} = \sum_{e\in P} x_e\left(a_eb_e + \frac{1}{a_eb_e} + c_ed_e + \frac{1}{c_ed_e}\right) + x_e^{-1}\left(\frac{a_e}{b_e} + \frac{b_e}{a_e} + \frac{c_e}{d_e} + \frac{d_e}{c_e}\right) \tag{33}$$

whilst in the colored case we let $e_c \in E$ denote the colored vertex and omit the subscript e_c from the variables, so that

$$\begin{aligned}\widetilde{W}_{\gamma,c} = x&\left(ab + \frac{1}{ab} + cd + \frac{1}{cd}\right) + x^{-1}\left(\frac{a}{b} + \frac{b}{a} + cd + \frac{1}{cd}\right) \\ &\sum_{e\in P\setminus\{e_c\}} x_e\left(a_eb_e + \frac{1}{a_eb_e} + c_ed_e + \frac{1}{c_ed_e}\right) + x_e^{-1}\left(\frac{a_e}{b_e} + \frac{b_e}{a_e} + \frac{c_e}{d_e} + \frac{d_e}{c_e}\right).\end{aligned} \tag{34}$$

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By Petersen's theorem, a trivalent graph which is bridgeless (i.e., we cannot remove an edge to make it disconnected) has at least one perfect matching. Such a graph exists for every genus g. Using [8, Theorems 2.12 and 2.13], we have that the graph potentials for different graphs of the same genus are related via rational changes of coordinates, hence its critical values do not depend on the choice of graph.

Proposition 3.7. *Let γ be a trivalent graph. Let c be a coloring with, at most, one colored vertex. Then the following are critical values of the graph potential $\widetilde{W}_{\gamma,c}$:*

- *purely real critical values:*

$$8g - 8 - 16k \text{ for } k = 0, \dots, g-1; \tag{35}$$

- *purely imaginary critical values:*

$$(8g - 16 - 16k)\sqrt{-1} \text{ for } k = 0, \dots, g-2. \tag{36}$$

The critical value 0 is listed twice, depending on the parity of g as a purely real or a purely imaginary critical value.

Proof. If we assign ± 1 to the variables associated to the edges in the perfect matching, then it is immediate that assigning 1 to all variables associated to edges outside the perfect matching gives a solution to the equations (31), (32). If we choose k edges from $P \subseteq E$ to have value -1, evaluating (33), (34) gives the value $8g - 8 - 16k$.

If we assign $\pm\sqrt{-1}$ to the variables associated to the edges in the perfect matching, then it is immediate that assigning $\sqrt{-1}$ to all variables associated to edges outside the perfect matching gives a solution to the equations (31), (32). If we choose k edges from $P \subseteq E$ to have value $-\sqrt{-1}$, evaluating (33), (34) gives the value $(8g - 16 - 16k)\sqrt{-1}$. □

Conifold points

In [19, Remark 3.1.6], a conjecture regarding the value of T from Corollary 3.4 was suggested, relating it to critical values of the mirror Landau–Ginzburg model.

If W is a Laurent polynomial with positive coefficients such that the origin is contained in the Newton polytope (which is the convex hull of the exponents of monomials with nonzero coefficients), then it is shown in [18] that there is a *unique* critical point with strictly positive (real) coordinates. This critical point is called the *conifold point* x_{con}. We will determine this for the graph potentials.

Define $T_{\mathrm{con}} := \widetilde{W}_{\gamma,c}(x_{\mathrm{con}})$ as the value of the potential $\widetilde{W}_{\gamma,c}$ at the conifold point. The conjecture says that

$$T = T_{\mathrm{con}}. \tag{37}$$

From the description of the conifold point in Proposition 3.8, we can easily check this equality. Indeed, the graph potential $\widetilde{W}_{\gamma,c}$ is the sum over the vertex potentials $\widetilde{W}_v$ for $v \in V$, and the evaluation at the conifold point for each of these is equal to 4. There are $2(g-1)$ vertices, so $T_{\mathrm{con}} = 8(g-1)$. This agrees with the value T from Corollary 3.4.

Proposition 3.8. *Let γ be a trivalent graph, and let c be a coloring. The conifold point of the graph potential $\widetilde{W}_{\gamma,c}$ is given by $(1, \dots, 1) \in (\mathbb{C}^\times)^{3g-3}$, and the value of the graph potential at the conifold point is $8g - 8$.*

Proof. This follows from direct computation: for every vertex v we have $\mathrm{d}W_{v,c}(1,1,1) = 0$ and $W_{v,c}(1,1,1) = 4$. Hence $\mathrm{d}W_{\gamma,c} = \sum_{v \in V} \mathrm{d}W_v = 0$ and thus

$$W_{\gamma,c}(1,\dots,1) = \sum_{v \in V} W_{v,c}(1,1,1) = 4\#V. \tag{38}$$

□

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As discussed in the introduction of this section, the eigenvalues of quantum multiplication should correspond to the critical values of a suitable Landau–Ginzburg model. There is a priori no reason why it would suffice to consider a single Laurent polynomial, which is the restriction of the potential to a Zariski-open torus inside Y.

But it turns out that graph potentials already see all the critical values one expects. This is immediate by comparing Proposition 3.2 to Proposition 3.7. It should be remarked that we have not proven that there are no other critical values, but we conjecture that these are all.

One necessary feature of Landau–Ginzburg mirrors in homological mirror symmetry for Fano varieties is that the critical locus of the potential is compact and that there are no critical points at infinity. Because the critical locus contains higher-dimensional components, we see the need for multiple cluster charts, as a positive-dimensional proper variety needs multiple affine charts (so in this case at least g, and hence this grows to ∞ as the genus grows).

Remark 3.9. By Proposition 3.2, we see that the $\mathrm{H}^\bullet(\mathrm{Sym}^{g-1-k} C)$ appears as an eigenspace of the quantum multiplication with eigenvalue of absolute value $8k$. However, by Appendix A we see that the critical loci all have expected dimension (for the necklace graph). This is in agreement with the appearance of $\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1-k} C)$ in the conjectural semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2,\mathcal{L}))$. Further agreement between the two decompositions is provided by the discussion in Appendix A.

4. Decomposition in the Grothendieck ring of varieties

In this section, we compute the class of the moduli space $\mathrm{M}_C(2,\mathcal{L})$ in the Grothendieck ring of varieties $\mathrm{K}_0(\mathrm{Var}/k)$, and the main result in this section is a proof of Theorem C. This identity gives further evidence for Conjecture A by considering the appropriate motivic measure on $\mathrm{K}_0(\mathrm{Var}/k)$ as in Corollary 4.1.

4.1. The Grothendieck ring of varieties and motivic measures

Recall that the Grothendieck ring of varieties is generated by the isomorphism classes $[X]$ of algebraic varieties over k, modulo the relations $[X] = [U] + [Z]$ for $Z \hookrightarrow X$ a closed subvariety, and $U = X \setminus Z$ its complement. The product of varieties induces the multiplicative structure, such that [pt] is the unit. An important element of this ring is the class of the affine line $\mathbb{L} = [\mathbb{A}^1] = [\mathbb{P}^1] - [\mathrm{pt}]$, also called the Lefschetz class.

An alternative presentation for this ring in characteristic 0, relevant to our goal, is the Bittner presentation from [12]. It says that $\mathrm{K}_0(\mathrm{Var}/k)$ is isomorphic to the ring generated by isomorphism classes $[X]$ of smooth and proper algebraic varieties over k, modulo the relations $[\mathrm{Bl}_Z X] - [E] = [X] - [Z]$ for $Z \hookrightarrow X$ a smooth closed subvariety, and $E \to Z$ the exceptional divisor in the blowup $\mathrm{Bl}_Z X \to X$.

Via the cut-and-paste relations, one can obtain the standard identities

$$[X] = [F][Y] \tag{39}$$

if $X \to Y$ is a Zariski-locally trivial fibration with fibers F, and

$$[\mathbb{P}^n] = \sum_{i=0}^{n} \mathbb{L}^i = \frac{1-\mathbb{L}^{n+1}}{1-\mathbb{L}}. \tag{40}$$

In order to use this alternative presentation, and because we depend on other results in the literature which are only phrased for algebraically closed fields (but likely hold more generally), we will work over an arbitrary algebraically closed field k of characteristic 0.

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Motivic measures

A motivic measure is a ring morphism whose domain is the Grothendieck ring of varieties. In our setting we consider the motivic measure

$$\mu : \mathrm{K}_0(\mathrm{Var}/k) \to \mathrm{K}_0(\mathrm{dgCat}/k) \tag{41}$$

from [13, §8], obtained by using the Bittner presentation and Orlov's blowup formula, and sending $[X]$ to the class of the (unique) dg enhancement of $\mathrm{K}_0(\mathrm{dgCat}/k)$. Then the identity (3) matches up with Conjecture A.

The Grothendieck ring of dg categories $\mathrm{K}_0(\mathrm{dgCat}_k)$ precisely encodes semiorthogonal decompositions, as it is generated by the quasiequivalence classes $[\mathcal{C}]$ of smooth and proper pretriangulated dg categories, modulo the relations

$$[\mathcal{C}] = [\mathcal{A}] + [\mathcal{B}] \tag{42}$$

for every semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$.

Hence, the image of the equality (3) is consistent with Conjecture A (up to 2-torsion, as $\mathbb{L} \mapsto 1$), as explained by the following corollary.

Corollary 4.1. *We have the equality*

$$[\mathbf{D}^{\mathrm{b}}(\mathrm{M}_C(2, \mathcal{L}))] = [\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C)] + \sum_{i=0}^{g-2} 2[\mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^i C)] + T' \tag{43}$$

in $\mathrm{K}_0(\mathrm{dgCat}/k)$*, for some class* T' *such that* $2 \cdot T' = 0$.

Again we expect that $T' = 0$, but our method of proof is not strong enough to remove this error term.

This is precisely the equality induced by the semiorthogonal decomposition from Conjecture A and therefore can be seen as further evidence for it.

Comparison to other results

Motivic and cohomological invariants of $\mathrm{M}_C(2, \mathcal{L})$ have been an active topic of interest for a long time, and many tools are used for this. From the identity (3), it is possible to deduce known results on certain invariants of $\mathrm{M}_C(2, \mathcal{L})$ by taking the appropriate motivic measures. Examples of this are given by

- the Betti polynomial (with values in $\mathbb{Z}[t]$) [45],
- the Hodge–Poincaré polynomial (with values in $\mathbb{Z}[x, y]$) [1, Corollary 5.1].

It would be interesting to weaken the assumptions on the field k for (3) and to also recover point-counting realizations [1, Corollary 5.4].

With a view towards providing evidence for Conjecture A, Lee has given in [34, Theorem 1.2] an isomorphism similar to (3) which holds in any *semisimple* category of motives, such as the category of numerical motives. It is based on the isomorphism [2, Theorem 2.7] due to del Baño, which also requires this semisimplicity.

This result is stronger in the sense that it is known that the error term T vanishes. But it only works for (the Grothendieck group of) a semisimple category of motives. The motivic measure

$$\mu\colon \mathrm{K}_0(\mathrm{Var}/k) \to \mathrm{K}_0(\mathrm{Chow}/k) \tag{44}$$

which sends the class $[X]$ of a smooth projective variety to the class $[\mathfrak{h}(X)]$ of its Chow motive, allows one to obtain a strengthening (modulo vanishing of T) of [34, Theorem 1.2]. See also [2, Remark 2.8].

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4.2. The Thaddeus picture

We break up the proof of Theorem C into several steps. The main idea of the proof is to use Thaddeus' variation of GIT for moduli of stable pairs and compare the class of the variety when we cross a wall using a telescopic sum. This is the 'Thaddeus picture' alluded to in the title of this section, and it will be given in (47).

Setup

Let d be an odd integer which is greater than $2g-2$. Let $\mathcal{L}$ be a line bundle of degree d, and let $M_0^d = \mathbb{P}(\mathrm{H}^1(C, \mathcal{L}^\vee))$. For each $i \in \{1, \dots, (d-1)/2\}$, Thaddeus constructs in [51] two smooth projective varieties M_i^d and $\widetilde{M}_i^d$ such that $\widetilde{M}_i^d$ is

- a blowup of M_{i-1}^d along a projective bundle $\mathbb{P}(\mathcal{W}_i^-)$ over $\mathrm{Sym}^i C$;
- a blowup of M_i^d along a projective bundle $\mathbb{P}(\mathcal{W}_i^+)$ over $\mathrm{Sym}^i C$.

Here, $\mathcal{W}_i^-$ (respectively $\mathcal{W}_i^+$) is vector bundle on $\mathrm{Sym}^i C$ of rank i (respectively of rank $d+g-2i-1$).

This leads us to the following flip diagram:

$$\begin{array}{ccccccc} & & & E & & & \\ & \swarrow & & \downarrow & & \searrow & \\ & & & \widetilde{M}_i^d & & & \\ & & \swarrow & & \searrow & & \\ \mathbb{P}(\mathcal{W}_i^-) & \hookrightarrow & M_{i-1}^d & & M_i^d & \hookleftarrow & \mathbb{P}(\mathcal{W}_i^+) \\ & \searrow & & & & \swarrow & \\ & & & \mathrm{Sym}^i C & & & \end{array} \tag{45}$$

Summarizing the situation from [51] for all i we obtain the following theorem.

Theorem 4.2 (Thaddeus). *With the above notation:*

1. *There is a flip* $M_{i-1}^d \dashrightarrow M_i^d$ *with center* $\mathrm{Sym}^i C$ *and type* $(i, d+g-2i-1)$ *for each* $i \in \{1, \dots, (d-1)/2\}$.
2. *We have that* $M_0^d \cong \mathbb{P}^{d+g-2}$ *and* $\widetilde{M}_1 \cong M_1^d$.
3. *There is a natural map*

$$\pi : M_{(d-1)/2}^d \longrightarrow \mathrm{M}_C(2, \mathcal{L}) \tag{46}$$

with fiber $\mathbb{P}(\mathrm{H}^0(C, \mathcal{E}))$ *over a stable bundle* $\mathcal{E}$ *in* $\mathrm{M}_C(2,\mathcal{L})$. *Moreover, if* $d \geq 4g-3$, *then* π *is a projective bundle associated to a vector bundle of rank* $d + 2(1-g)$.

This yields the following picture, which we refer to as the 'Thaddeus picture':

$$\begin{array}{ccccccccccc} & \widetilde{M}_1^d & & \widetilde{M}_2^d & & & & \widetilde{M}_{(d-1)/2}^d & \\ \swarrow & & \searrow\!\!\!\!= \quad \swarrow & & \searrow & & \swarrow & & \searrow \\ M_0^d & & M_1^d & & M_2^d & \cdots & M_{(d-3)/2}^d & & M_{(d-1)/2}^d \\ & & & & & & & & \downarrow \pi \\ & & & & & & & & \mathrm{M}_C(2,\mathcal{L}) \end{array} \tag{47}$$

All morphisms in the diagram, except π, are blowups.

To minimize the dimension of the moduli spaces and the number of steps involved in the construction, we will take $d = 4g-3$. Then

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1. $\dim M_i = 5g-5$;
2. we are considering moduli spaces $M_0^{4g-3}, \ldots, M_{2g-2}^{4g-3}$;
3. the morphism M_{2g-2} is a $\mathbb{P}^{2g-2}$- fibration.

Setup for the proof

We now give some easy lemmas, as a setup for the proof in Section 4.3. The following lemma follows directly from Theorem 4.2 and (39).

Lemma 4.3. *Let d be an odd integer greater than $4g-4$. Then we have that*

$$[M^d_{(d-1)/2}] = \frac{1-\mathbb{L}^{d+2(1-g)}}{1-\mathbb{L}}[\mathrm{M}_C(2,\mathcal{L})] \tag{48}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

Using the description from Lemma 4.3 for $d = 4g-3$ and $d = 4g-1$ and computing the difference gives the following description. One could take a less optimal choice, at the cost of obtaining a larger coefficient in Theorem C.

Lemma 4.4. *We have that*

$$(1+\mathbb{L})[\mathrm{M}_C(2,\mathcal{L})] = [M^{4g-1}_{2g-1}] - \mathbb{L}^2[M^{4g-3}_{2g-2}] \tag{49}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

For every flip diagram, as in (47), one can prove the following, where only standard bookkeeping techniques are required.

Proposition 4.5. *Let d be an odd integer greater than $4g-4$. Then for $i = 1, \ldots, (d-1)/2$, the difference of the classes $[M^d_i]$ and $[M^d_{i-1}]$ satisfies*

$$[M^d_i] - [M^d_{i-1}] = \frac{\mathbb{L}}{(1-\mathbb{L})^2}\Big((1-\mathbb{L}^{d+g-2i-2})(1-\mathbb{L}^i) - (1-\mathbb{L}^{i-1})(1-\mathbb{L}^{d+g-2i-1})\Big)[\mathrm{Sym}^i C] \tag{50}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

Sketch of proof. By applying the blowup formula and the projective bundle formula twice with the appropriate codimension and relative dimension, we get the equalities

$$\begin{aligned}
[\widetilde{M}^d_i] &= [M^d_i] - [\mathbb{P}(\mathcal{W}^+_i)] + \frac{1-\mathbb{L}^i}{1-\mathbb{L}}[\mathbb{P}(\mathcal{W}^+_i)] \\
&= [M^d_i] + \frac{\mathbb{L}(1-\mathbb{L}^{i-1})}{1-\mathbb{L}}\frac{1-\mathbb{L}^{d+g-2i-2}}{1-\mathbb{L}}[\mathrm{Sym}^i C] \\
[\widetilde{M}^d_i] &= [M^d_{i-1}] - [\mathbb{P}(\mathcal{W}^-_i)] + \frac{1-\mathbb{L}^{d+g-2i-1}}{1-\mathbb{L}}[\mathbb{P}(\mathcal{W}^-_i)] \\
&= [M^d_{i-1}] + \frac{\mathbb{L}(1-\mathbb{L}^{d+g-2i-2})}{1-\mathbb{L}}\frac{1-\mathbb{L}^i}{1-\mathbb{L}}[\mathrm{Sym}^i C]
\end{aligned} \tag{51}$$

and subtracting them gives the result.

Finally, let us recall the following fundamental identity encoding the behavior of Abel–Jacobi morphisms, as discussed in [24, §3].

Proposition 4.6. *Let C be a curve of genus g. Let $e \geq 0$ and denote $a = (g-1)+e$. Then we have an equality*

$$[\mathrm{Pic}^a C][\mathbb{P}^{e-1}] = [\mathrm{Sym}^{g-1+e} C] - \mathbb{L}^e[\mathrm{Sym}^{g-1-e} C] \tag{52}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

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Observe that under our assumptions we have isomorphisms $\operatorname{Pic}^i C \cong \operatorname{Jac} C$ for all $i \in \mathbb{Z}$ by the existence of a rational point.

4.3. Proof of the decomposition

We will restrict ourselves to the cases where $d = 4g - 3$ and $4g - 1$ (i.e. the first two degrees for which the morphism π in (47) is an equidimensional projective fibration). We will compare different stages of the Thaddeus picture, using the following notation for the difference of classes of moduli of stable pairs, where $i = 0, \ldots, 2g - 2$.

$$\delta M_i := [M_i^{4g-1}] - \mathbb{L}^2 [M_i^{4g-3}]. \tag{53}$$

For notational convenience, we also set $\delta M_{-1} := 0$.

Now for $i = 0, \ldots, 2g - 2$, we define

$$X_i := \delta M_i - \delta M_{i-1}. \tag{54}$$

We can describe the classes X_i in the following way.

Proposition 4.7. *For* $i = 0, \ldots, 2g - 2$ *we have that*

$$X_i = \mathbb{L}^i (1 + \mathbb{L}) [\operatorname{Sym}^i C] \tag{55}$$

Proof. The proof follows from the definition of X_i and by applying Proposition 4.5 via the following chain of equalities.

$$\begin{aligned} X_i &= \delta M_i - \delta M_{i-1} \\ &= [M_i^{4g-1}] - \mathbb{L}^2 [M_i^{4g-3}] - [M_{i-1}^{4g-1}] + \mathbb{L}^2 [M_{i-1}^{4g-3}] \\ &= ([M_i^{4g-1}] - [M_{i-1}^{4g-1}]) - \mathbb{L}^2 ([M_i^{4g-3}] - [M_{i-1}^{4g-3}]) \end{aligned} \tag{56}$$

We now use Proposition 4.5 and rewrite X_i as

$$\begin{aligned} X_i &= \frac{\mathbb{L}}{(1-\mathbb{L})^2} \Bigg(\Big((1 - \mathbb{L}^{5g-3-2i})(1 - \mathbb{L}^i) - (1 - \mathbb{L}^{5g-2-2i})(1 - \mathbb{L}^{i-1}) \Big) \\ &\qquad - \mathbb{L}^2 \Big((1 - \mathbb{L}^{5g-5-2i})(1 - \mathbb{L}^i) - (1 - \mathbb{L}^{5g-4-2i})(1 - \mathbb{L}^{i-1}) \Big) \Bigg) [\operatorname{Sym}^i C] \\ &= \left(\frac{\mathbb{L}(1-\mathbb{L}^i)(1-\mathbb{L}^2)}{(1-\mathbb{L})^2} - \frac{\mathbb{L}(1-\mathbb{L}^2)(1-\mathbb{L}^{i-1})}{(1-\mathbb{L})^2} \right) [\operatorname{Sym}^i C] \\ &= \left(\frac{\mathbb{L}(1-\mathbb{L}^2)}{(1-\mathbb{L})^2} (\mathbb{L}^{i-1} - \mathbb{L}^i) \right) [\operatorname{Sym}^i C] \\ &= \mathbb{L}^i (1 + \mathbb{L}) [\operatorname{Sym}^i C]. \end{aligned} \tag{57}$$

□

Using Propositions 4.6 and 4.7, we obtain the following result, where we use the following polynomial in $\mathbb{L}$:

$$\mathcal{P}(i) := \mathbb{L}^{2g-2-i} (1 + \mathbb{L})(1 + \mathbb{L} + \ldots + \mathbb{L}^{g-2-i}). \tag{58}$$

Proposition 4.8. *For* $i = 0, \ldots, g - 2$*, we have that*

$$X_i + X_{2g-2-i} = [\operatorname{Sym}^i C](\mathbb{L}^i + \mathbb{L}^{3g-3-2i})(1 + \mathbb{L}) + \mathcal{P}(i)[\operatorname{Jac}(C)], \tag{59}$$

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Proof. By Proposition 4.6, we get

$$[\mathrm{Sym}^{2g-2-i} C] = \mathbb{L}^{g-i-1}[\mathrm{Sym}^i C] + [\mathbb{P}^{g-2-i}][\mathrm{Jac}(C)] \tag{60}$$

We apply Proposition 4.7 for $i \in \{0, 1, \ldots, g-2\}$ to (60) to get

$$\begin{aligned} X_{2g-2-i} &= \mathbb{L}^{2g-2-i}(1+\mathbb{L})[\mathrm{Sym}^{2g-2-i} C] \\ &= \mathbb{L}^{3g-3-2i}(1+\mathbb{L})[\mathrm{Sym}^i C] + \mathbb{L}^{2g-2-i}(1+\mathbb{L})[\mathbb{P}^{g-2-i}][\mathrm{Jac}(C)] \\ &= \mathbb{L}^{3g-3-2i}(1+\mathbb{L})[\mathrm{Sym}^i C] + \mathcal{P}(i)[\mathrm{Jac}(C)] \end{aligned} \tag{61}$$

Now by Proposition 4.7 for $i \leq g-1$, we get $X_i = \mathbb{L}^i(1+\mathbb{L})[\mathrm{Sym}^i C]$. Thus, the proof follows. □

The following proposition is an important step in the proof of Theorem C. It shows that there are no contributions of the Jacobian of C to the class of $\mathrm{M}_C(2, \mathcal{L})$.

Proposition 4.9. *We have that*

$$\left(\sum_{i=0}^{g-2} \mathcal{P}(i)\right)[\mathrm{Jac}(C)] = [M_{2g-2}^{4g-1}] - [M_{2g-1}^{4g-1}] \tag{62}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

Proof. First let us simplify the left hand side of the above equation. We have that

$$\begin{aligned} \sum_{i=0}^{g-2} \mathcal{P}(i) &= \frac{1+\mathbb{L}}{1-\mathbb{L}} \sum_{i=0}^{g-2} \mathbb{L}^{2g-2-i}(1-\mathbb{L}^{g-1-i}) \\ &= \frac{(1+\mathbb{L})}{1-\mathbb{L}} \sum_{i=0}^{g-2} \left(\mathbb{L}^{2g-2-i} - \mathbb{L}^{3g-3-2i}\right) \\ &= \frac{\mathbb{L}^g(1+\mathbb{L})}{1-\mathbb{L}} \sum_{i=0}^{g-2} (\mathbb{L}^i - \mathbb{L}^{2i+1}) \\ &= \frac{\mathbb{L}^g(1+\mathbb{L})}{1-\mathbb{L}} \left(\frac{1-\mathbb{L}^{g-1}}{1-\mathbb{L}} - \frac{\mathbb{L}(1-(\mathbb{L}^2)^{g-1})}{1-\mathbb{L}^2}\right) \\ &= \frac{\mathbb{L}^g}{(1-\mathbb{L})^2}\left((1-\mathbb{L}^{g-1})(1-\mathbb{L}^g)\right) \end{aligned} \tag{63}$$

We will be done if we can show that the above expression multiplied by the class of $\mathrm{Jac}(C)$ is equal to $[M_{2g-2}^{4g-1}] - [M_{2g-1}^{4g-1}]$. First, by Proposition 4.6 where we take $e = g$, we get

$$[\mathrm{Sym}^{2g-1} C] = [\mathbb{P}^{g-1}][\mathrm{Jac}(C)]. \tag{64}$$

Combining this with Proposition 4.5, we get

$$[M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}] = \frac{L}{(1-\mathbb{L})^3}(1-\mathbb{L}^g)\Big((1-\mathbb{L}^{g-1})(1-\mathbb{L}^{2g-1}) - (1-\mathbb{L}^g)(1-\mathbb{L}^{2g-2})\Big)[\mathrm{Jac}(C)]. \tag{65}$$

Now we use that

$$(1-\mathbb{L}^{g-1})(1-\mathbb{L}^{2g-1}) - (1-\mathbb{L}^g)(1-\mathbb{L}^{2g-2}) = -\mathbb{L}^{g-1}(1-\mathbb{L}) + \mathbb{L}^{2g-1}(1-\mathbb{L}) \tag{66}$$

Thus, from (66) we get

$$[M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}] = -\frac{\mathbb{L}(1-\mathbb{L}^g)}{(1-\mathbb{L})^3}\mathbb{L}^{g-1}(1-\mathbb{L}^{g-1})(1-\mathbb{L})[\mathrm{Jac}(C)], \tag{67}$$

and we are done. □

Now we are ready to complete the proof of Theorem C.

Proof of Theorem C. First we write the class of $\mathrm{M}_C(2,\mathcal{L})$ in terms of the difference of classes of the smooth projective varieties considered by Thaddeus. We apply Lemma 4.4 to get

$$(1+\mathbb{L})[\mathrm{M}_C(2,\mathcal{L})] = [\delta M_{2g-2}] + [M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}] \tag{68}$$

Now we can write

$$[\delta M_{2g-2}] = \sum_{i=0}^{2g-2} X_i, \tag{69}$$

where X_i's are as used in Proposition 4.8. Thus, applying Proposition 4.8 we get

$$\begin{aligned}
&(1+\mathbb{L})[\mathrm{M}_C(2,\mathcal{L})]\\
&\quad= [\delta M_{2g-2}^{4g-3}] + [M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}]\\
&\quad= \sum_{i=0}^{2g-2} X_i + [M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}]\\
&\quad= \sum_{i=0}^{g-2}(X_i + X_{2g-2-i}) + X_{g-1} + [M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}]\\
&\quad= \mathbb{L}^{g-1}(1+\mathbb{L})[\mathrm{Sym}^{g-1} C] + \sum_{i=0}^{g-2}(\mathbb{L}^i + \mathbb{L}^{3g-3-2i})(1+\mathbb{L})[\mathrm{Sym}^i C]\\
&\qquad + \left(\sum_{i=0}^{g-2}\mathcal{P}(i)\right)[\mathrm{Jac}(C)] + [M_{2g-1}^{4g-1}] - [M_{2g-2}^{4g-1}]\\
&\quad= \mathbb{L}^{g-1}(1+\mathbb{L})[\mathrm{Sym}^{g-1} C] + \sum_{i=0}^{g-2}(\mathbb{L}^i + \mathbb{L}^{3g-3-2i})(1+\mathbb{L})[\mathrm{Sym}^i C]
\end{aligned} \tag{70}$$

where the last step is by Proposition 4.9. □

4.4. Motivic zeta functions and a Harder-type formula

As an application of the results above we can give an analogue of Harder's point counting formula from [21] (see also [2, Corollary 2.11]). We do this by exhibiting an identity in the Grothendieck ring of varieties. Note however that the ground field cannot be chosen to be $\mathbb{F}_q$. Hence, Harder's formula cannot be obtained by applying the motivic measure

$$\# : \mathrm{K}_0(\mathrm{Var}/\mathbb{F}_q) \to \mathbb{Z} : [X] \mapsto \#X(\mathbb{F}_q). \tag{71}$$

For any variety X, we denote the nth symmetric $\mathrm{Sym}^n X$ to be X^n/Sym_n, where Sym_n is the symmetric group of n letters. All the symmetric powers can be put together to give Kapranov's *motivic* *zeta function*, as introduced in [26, §1.3]:

$$\mathrm{Z}_{\mathrm{Kap}}(X,t) := \sum_{n\geq 0}[\mathrm{Sym}^n X]t^n \in \mathrm{K}_0(\mathrm{Var}/k)[[t]]. \tag{72}$$

This is a universal version of the Hasse–Weil zeta function, valid for arbitrary ground fields, where the counting measure for $k = \mathbb{F}_q$ gives the usual Hasse–Weil zeta function.

The following theorem is due to Kapranov [26, Theorem 1.1.9] and shows how the motivic zeta function has properties similar to the usual Hasse–Weil zeta function.

Theorem 4.10 (Kapranov). *Let C be a smooth curve. The motivic zeta function of C is a rational function of the following form:*

$$\mathrm{Z}_{\mathrm{Kap}}(C,t) = \frac{F_{2g}(t)}{(1-t)(1-\mathbb{L}t)}, \tag{73}$$

where $F_{2g}(t)$ is a polynomial of degree $2g$. Moreover, $\mathrm{Z}_{\mathrm{Kap}}(C,t)$ (resp. $F_{2g}(t)$) satisfies the functional equation

$$\mathrm{Z}_{\mathrm{Kap}}(C,t) = \mathbb{L}^{g-1}t^{2g-2}\mathrm{Z}_{\mathrm{Kap}}\left(C, \frac{1}{\mathbb{L}t}\right) \tag{74}$$

resp.

$$F_{2g}(t) = \mathbb{L}^g t^{2g} F_{2g}\left(\frac{1}{\mathbb{L}t}\right). \tag{75}$$

We can rearrange and reinterpret the terms in the motivic zeta function as follows.

Proposition 4.11. *We have the identity*

$$\begin{aligned}\mathrm{Z}_{\mathrm{Kap}}(C,\mathbb{L}) = &\left(\sum_{i=0}^{g-2}[\mathrm{Sym}^i C](\mathbb{L}^i + \mathbb{L}^{3g-2i-3})\right) + [\mathrm{Sym}^{g-1} C]\mathbb{L}^{g-1}\\ &+ \sum_{i=0}^{g-2}[\mathrm{Jac}(C)][\mathbb{P}^{g-i-2}]\mathbb{L}^{2g-2-i} + \frac{[\mathrm{Jac}(C)]\mathbb{L}^{2g-1}}{1-\mathbb{L}}\left(\frac{1}{1-\mathbb{L}} - \frac{\mathbb{L}^g}{(1-\mathbb{L}^2)}\right)\end{aligned} \tag{76}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

Proof. Using Proposition 4.6, we have the identity[1]

$$\begin{aligned}\mathrm{Z}_{\mathrm{Kap}}(C,\mathbb{L}) &= \left(\sum_{i=0}^{g-2}[\mathrm{Sym}^i]\mathbb{L}^i\right) + [\mathrm{Sym}^{g-1} C]\mathbb{L}^{g-1} + \left(\sum_{i=0}^{g-2}[\mathrm{Sym}^{2g-i-2} C]\mathbb{L}^{2g-i-2}\right)\\ &\quad + \left(\sum_{i\geq 2g-1}[\mathrm{Sym}^i C]\mathbb{L}^i\right)\\ &= \left(\sum_{i=0}^{g-2}[\mathrm{Sym}^i]\mathbb{L}^i\right) + [\mathrm{Sym}^{g-1} C]\mathbb{L}^{g-1}\end{aligned} \tag{77}$$

[1]One avoids manipulating infinite sums of $\mathbb{L}$ by observing that $\mathrm{Z}_{\mathrm{Kap}}(C,t)$ is a rational function in t, and only finitely many copies of powers of $\mathbb{L}$ are contributing to each power of t before evaluation.

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$$+\left(\sum_{i=0}^{g-2}[\mathrm{Sym}^i\, C]\mathbb{L}^{3g-2i-3}+[\mathrm{Jac}\, C][\mathbb{P}^{g-i-2}]\mathbb{L}^{2g-i-2}\right)$$
$$+\left(\sum_{i\geq 2g-1}[\mathrm{Jac}\, C][\mathbb{P}^{i-g}]\mathbb{L}^i\right).$$

It now suffices to see that

$$\left(\sum_{i\geq 2g-1}[\mathrm{Jac}\, C][\mathbb{P}^{i-g}]\mathbb{L}^i\right)=\frac{[\mathrm{Jac}\, C]\mathbb{L}^{2g-1}}{1-\mathbb{L}}\left(\frac{1}{1-\mathbb{L}}-\frac{\mathbb{L}^g}{1-\mathbb{L}^2}\right) \tag{78}$$

which follows from an immediate verification. □

The following lemma is also an immediate verification.

Lemma 4.12. *We have the identity*

$$\sum_{i=0}^{g-2}\mathbb{L}^{2g-i-2}(1-\mathbb{L}^{g-i-1})(1-\mathbb{L}^2)+\mathbb{L}^{2g-1}(1+\mathbb{L}-\mathbb{L}^g)=\mathbb{L}^g \tag{79}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

Using Proposition 4.11 and Lemma 4.12, we can then obtain the following corollary to Theorem C. The right-hand side is an element of $\mathrm{K}_0(\mathrm{Var}/k)$ (and not some completion) by Theorem 4.10, so that the first term of the right-hand side is in fact $F_{2g}(\mathbb{L})$.

Corollary 4.13. *We have the identity*

$$(1-\mathbb{L})(1-\mathbb{L}^2)[\mathrm{M}_C(2,\mathcal{L})]=(1-\mathbb{L})(1-\mathbb{L}^2)\mathrm{Z}_{\mathrm{Kap}}(C,\mathbb{L})-\mathbb{L}^g[\mathrm{Jac}\, C] \tag{80}$$

in $\mathrm{K}_0(\mathrm{Var}/k)$.

Proof. It suffices to rewrite $[\mathbb{P}^{g-i-2}]$ as $\frac{1-\mathbb{L}^{g-i-1}}{1-\mathbb{L}}$ and then multiply both sides of (76) with $(1-\mathbb{L})(1-\mathbb{L}^2)$ and apply Lemma 4.12. □

This identity is then the analogue of Harder's formula from [2, Corollary 2.11] using the functional equation for the zeta function.

A. Dimension of critical loci

We will now determine the dimensions of the different components of the critical loci, which are part of Theorem B. The goal is the following proposition.

Proposition A.1. *Let* $\Gamma_{g,g-1}$ *denote the necklace graph of genus* g*, with the last vertex colored as in Figure 4. Let* $\widetilde{W}_g$ *denote the associated graph potential. Then the dimension of the critical locus of* $\widetilde{W}_g$ *over the critical value with absolute value* $8g-8-8k$ *for* $k=0,\ldots,g-1$ *is* $g-1$.

The proof of Proposition A.1 is a lengthy computation in order to set up an inductive description of the critical loci.

Instead of the coordinates x_i, y_i, z_i, we will work with the coordinates

$$\begin{aligned} u_i &:= x_i y_i \\ v_i &:= x_i/y_i \end{aligned} \tag{81}$$

which makes the graph potential easier to work with.

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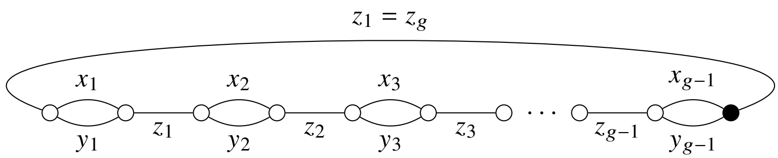

Figure 4. *Labelling of variables on the necklace graph.*

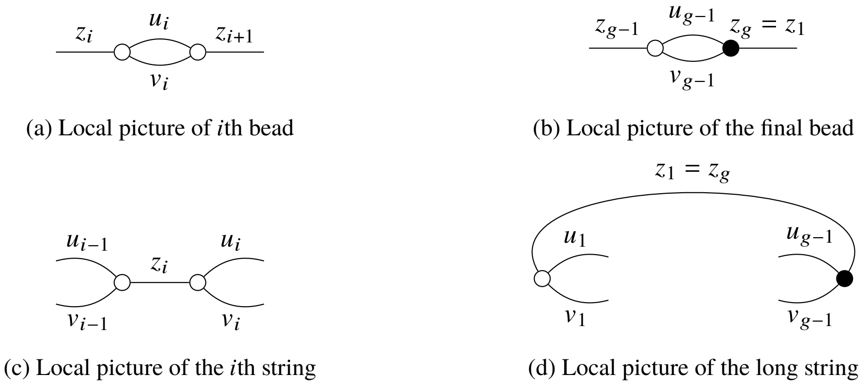

Figure 5. *Local pictures of beads and strings in the necklace graph of genus g.*

The necklace graph from Figure 4 can be decomposed in two ways, using *beads* and *strings*. In the coordinates u_i, v_i, z_i, we describe the local pictures and the labelling of the variables in Figure 5.

We will moreover use the following notation for two functions on $\mathbb{G}_{\mathrm{m}}$:

$$\begin{aligned} J^{+}(x) &:= x + x^{-1} \\ J^{-}(x) &:= x - x^{-1}, \end{aligned} \tag{82}$$

so that $x\frac{\partial}{\partial x}J^{+}(x) = J^{-}(x)$.

With this notation and the decomposition of the necklace graph, the graph potential can be written as the sum of either bead potentials or string potentials. Here the ith *bead potential* is given as

$$\widetilde{W}^{\mathrm{b}}_{g,i} := \begin{cases} z_i J^{+}(u_i) + z_i^{-1}J^{+}(v_i) + z_{i+1}J^{+}(u_i) + z_{i+1}^{-1}J^{+}(v_i) & i = 1,\dots,g-2 \\ z_{g-1}J^{+}(u_{g-1}) + z_{g-1}^{-1}J^{+}(v_{g-1}) + z_1 J^{+}(v_{g-1}) + z_1^{-1}J^{+}(u_{g-1}) & i = g-1 \end{cases} \tag{83}$$

and the ith *string potential* is given as

$$\widetilde{W}^{\mathrm{s}}_{g,i} := \begin{cases} z_1 J^{+}(v_{g-1}) + z_1^{-1}J^{+}(u_{g-1}) + z_1 J^{+}(u_1) + z_1^{-1}J^{+}(v_1) & i = 1 \\ z_i J^{+}(u_{i-1}) + z_i^{-1}J^{+}(v_{i-1}) + z_i J^{+}(u_i) + z_i^{-1}J^{+}(v_i) & i = 2,\dots,g-1 \end{cases}. \tag{84}$$

This allows us to write

$$\widetilde{W}_g = \sum_{i=1}^{g-1} \widetilde{W}^{\mathrm{b}}_{g,i} = \sum_{i=1}^{g-1} \widetilde{W}^{\mathrm{s}}_{g,i}. \tag{85}$$

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In the description of the critical locus, we therefore have the equations

$$\begin{aligned} u_i \frac{\partial}{\partial u_i}\widetilde{W}_g = u_i \frac{\partial}{\partial u_i}\widetilde{W}^{\mathrm{b}}_{g,i} = J^-(u_i)(z_i + z_{i+1}) = 0 \\ v_i \frac{\partial}{\partial v_i}\widetilde{W}_g = v_i \frac{\partial}{\partial v_i}\widetilde{W}^{\mathrm{b}}_{g,i} = J^-(v_i)(z_i + z_{i+1}) = 0 \end{aligned} \tag{86}$$

for $i = 1, \ldots, g-2$, and

$$\begin{aligned} u_{g-1} \frac{\partial}{\partial u_{g-1}}\widetilde{W}_g = u_{g-1} \frac{\partial}{\partial u_{g-1}}\widetilde{W}^{\mathrm{b}}_{g,g-1} = J^-(u_{g-1})(z_{g-1} + z_1^{-1}) = 0 \\ v_{g-1} \frac{\partial}{\partial v_{g-1}}\widetilde{W}_g = v_{g-1} \frac{\partial}{\partial v_{g-1}}\widetilde{W}^{\mathrm{b}}_{g,g-1} = J^-(v_{g-1})(z_{g-1}^{-1} + z_1) = 0. \end{aligned} \tag{87}$$

Likewise, we obtain the conditions

$$z_i \frac{\partial}{\partial z_i}\widetilde{W}_g = z_i \frac{\partial}{\partial z_i}\widetilde{W}^{\mathrm{s}}_{g,i} = z_i(J^+(u_{i-1}) + J^+(u_i)) - z_i^{-1}(J^+(v_{i-1}) + J^+(v_i)) = 0 \tag{88}$$

for $i = 2, \ldots, g-1$, and

$$z_1 \frac{\partial}{\partial z_i}\widetilde{W}_g = z_1 \frac{\partial}{\partial z_1}\widetilde{W}^{\mathrm{s}}_{g,1} = z_1(J^+(u_1) + J^+(v_{g-1})) - z_1^{-1}(J^+(v_1) + J^+(u_{g-1})) = 0. \tag{89}$$

We need to analyse the system of equations given by (86), (87), (88) and (89), so that we can describe its solutions and compute the value of the graph potential at these solutions.

The *expected dimension* of the connected component of the critical locus with critical value given by $8(g-1-k)\sqrt{-1}^{1+(-1)^{k+1}}$ (for $k = 0, \ldots, 2g-2$)) is k for $k = 0, \ldots, g-1$, and $g-k+1$ for $k = g, \ldots, 2g-2$. For $k = 0, 2g-2$ we expect isolated critical points, whilst for $k = g-1$ we obtain critical value 0 and expect a critical locus of dimension $g-1$.

Under the assumptions (88) and (89), it is possible to rewrite the string potentials by removing the z_i^{-1} so that the graph potential can be expressed as

$$2\left(z_1(J^+(u_1) + J^+(v_{g-1}) \sum_{i=2}^{g-1} z_i(J^+(u_{i-1}) + J^+(u_i)) \right). \tag{90}$$

We will get rid of the factor 2 because we are interested in the values $4(g-1-k)\sqrt{-1}^{1+(-1)^{k+1}}$ for the resulting expression.

The case $u_i^2 = v_i^2 = 1$

As a starting point, we consider the situation in which the conditions (86) and (87) are satisfied by ensuring that $J^-(u_i) = J^-(v_i) = 0$ for all $i = 1, \ldots, g-1$. We need to analyse the condition on the remaining variables z_i, as well as what the resulting critical loci and critical values are.

The condition on the variables u_i and v_i ensures that $u_i = \pm 1$ and $v_i = \pm 1$ for all $i = 1, \ldots, g-1$.

We will encode the sign choice on the variables of the ith string potential as a matrix, so that $\left(\begin{smallmatrix} s^u_{i-1} & s^u_i \\ s^v_{i-1} & s^v_i \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} s^u_1 & s^u_{g-1} \\ s^v_1 & s^v_{g-1} \end{smallmatrix}\right)$) denotes the sign choice of the varaibles u_i and v_i.

Lemma A.2. *Let $u_i^2 = v_i^2 = 1$ for all $i = 1, \ldots, g-1$. If for some string the sign choice is inadmissible (i.e., the parity of the signs in the sign matrix is odd), then at least one of the equations (88) and (89) cannot be satisfied.*

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Table 1. *Admissible sign choices and conditions (88), (89).*

	$z_1 \frac{\partial}{\partial z_i}\widetilde{W}_g$	condition on z_1	$\widetilde{W}^{\mathrm{s}}_{g,1}$	$z_i \frac{\partial}{\partial z_i}\widetilde{W}_g$	condition on z_i	$\widetilde{W}^{\mathrm{s}}_{g,i}$
$\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)$	$4J^-(z_1)$	$z_1=\pm 1$		$4J^-(z_i)$	$z_i=\pm 1$	
$\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)$	$-4J^-(z_1)$	$z_1=\pm 1$		$-4J^-(z_i)$	$z_i=\pm 1$	
$\left(\begin{smallmatrix}+&+\\-&-\end{smallmatrix}\right)$	0	z_1 free		$4J^+(z_i)$	$z_i=\pm\sqrt{-1}$	
$\left(\begin{smallmatrix}-&-\\+&+\end{smallmatrix}\right)$	0	z_1 free		$-4J^+(z_i)$	$z_i=\pm\sqrt{-1}$	
$\left(\begin{smallmatrix}+&-\\-&+\end{smallmatrix}\right)$	$4J^+(z_1)$	$z_1=\pm\sqrt{-1}$		0	z_i free	
$\left(\begin{smallmatrix}-&+\\+&-\end{smallmatrix}\right)$	$-4J^+(z_1)$	$z_1=\pm\sqrt{-1}$		0	z_i free	
$\left(\begin{smallmatrix}+&-\\+&-\end{smallmatrix}\right)$	0	z_1 free		0	z_i free	
$\left(\begin{smallmatrix}-&+\\-&+\end{smallmatrix}\right)$	0	z_1 free		0	z_i free	

Assume that, for every string, the sign choice is admissible (i.e., the parity is even). Then we have that

1. *the variables z_i are either free or are necessarily equal to $\pm 1, \pm\sqrt{-1}$;*
2. *the critical loci have the expected dimension.*

Proof. Observe that $J^+(\pm 1) = \pm 2$, so that the coefficients of the $z_i^{\pm 1}$ in (88), (89) are $-4, 0, 4$. If the sign choice is inadmissible for the ith string, then in (88) exactly one of the coefficients of $z_i^{\pm 1}$ vanishes, but the system then has no solution in $(\mathbb{C}^\times)^{3g-3}$.

If the sign choice is admissible, we can analyse the system as in Table 1. We describe $z_i \frac{\partial}{\partial z_i}\widetilde{W}_g$ and deduce the condition it imposes on z_i. This proves the first part.

The dimension of the critical loci remains to be understood. For this, we consider the expression of the necklace graph potential as the sum of string potentials. The expression in (88), (89) is equal to (84) up to a sign.

Observe that the parity condition for consecutive strings force the choices of z_i to be either all real or all imaginary. It follows that the critical loci have the expected dimension by the evaluation of the string potentials as given in Table 1. □

Inductive description

We now consider the case where there is at least one $u_i^2 \neq 1$ or $v_i^2 \neq 1$. The goal is to relate the equations of the critical locus for the necklace graph of genus g to that of the necklace graph of genus $g-2$ and $g-1$.

Hence, we first need to describe what happens for $g = 2$ and $g = 3$.

Lemma A.3. *Proposition A.1 holds for $g = 2, 3$.*

Proof. For $g = 2$, we have $\widetilde{W}_2 = J^+(z)(J^+(u) + J^+(v))$ and thus the system of partial derivatives

$$\begin{cases} 0 = \dfrac{\partial \widetilde{W}_2}{\partial u} = J^+(z)J^-(u) \\ 0 = \dfrac{\partial \widetilde{W}_2}{\partial v} = J^+(z)J^-(v) \\ 0 = \dfrac{\partial \widetilde{W}_2}{\partial z} = J^-(z)(J^+(u) + J^+(v)) \end{cases} \tag{91}$$

so that a case-by-case analysis yields

- $J^+(z) = 0$ and $J^+(u) + J^+(v) = 0$ corresponds to 1-dimensional critical loci with critical value 0;
- $J^-(z) = 0$ and $J^-(u) = J^-(v) = 0$ corresponds to 0-dimensional critical loci with critical value ± 8.

The outcome for $g = 3$ is similar, using the reduction methods from the next lemmas, except for the case where $u_1^2 \neq 1$ and $z_1 + z_2^{-1} = 0$ where the reduction methods would yield $g = 1$. If $u_1^2 \neq 1$, then $z_1 = -z_2$

and $z_1 = -z_2^{-1}$ by assumption. Then $z_2 = \pm 1$ and $z_1 = \mp 1$. Now cancelling z_1 and z_2 gives

$$2J^+(u_1) + J^+(u_2) + J^+(v_2) = 2J^+(v_1) + J^+(u_2) + J^+(v_2) \tag{92}$$

which implies $J(u_1) = J(v_1)$. Substituting this, we get $J^+(u_2) = J^+(v_2)$, and thus this critical locus has dimension 2 and critical value 0. □

We can assume that $u_1^2 \neq 1$ by the rational change of coordinates in the change of coloring [8, §2.2]. From the condition (86) for u_1, we get

$$z_1 + z_2 = 0, \tag{93}$$

and thus condition (86) for v_1 is always satisfied.

We will first analyse the case where $z_2 + z_3 = 0$ or $z_1 + z_{g-1}^{-1} = 0$. Then we will be done once we've dealt with the case $z_1 + z_{g-1}^{-1} \neq 0$ and $z_2 + z_3 \neq 0$.

Lemma A.4. *Let $u_1^2 \neq 1$. Assume that $z_2 + z_3 = 0$. Then the critical loci have the expected dimension.*

Proof. By (93) and the assumption, we have that $z_1 = z_3$. The equations (86) for $i = 1, 2$ are automatically satisfied and can be ignored.

We can rewrite (88) for $i = 2$ as

$$-z_3(J^+(u_1) + J^+(u_2)) - z_3^{-1}(J^+(v_1) + J^+(v_2)) = 0 \tag{94}$$

and (89) as

$$z_3(J^+(u_3) + J^+(v_{g-1})) - z_3^{-1}(J^+(v_1) + J^+(u_{g-1})) = 0. \tag{95}$$

By summing these equations with (88) for $i = 3$, we obtain

$$z_3(J^+(u_3) + J^+(v_{g-1})) - z_3^{-1}(J^+(v_3) + J^+(u_{g-1})) = 0. \tag{96}$$

Finally, observe that the first few terms in the expression of the graph potential from (90) read

$$\widetilde{W}_g = z_1(J^+(u_1) + J^+(v_{g-1})) + z_2(J^+(u_1) + J^+(u_2)) + z_3(J^+(u_3) + J^+(u_3)) + \dots \tag{97}$$

so that by cancellations from the equalities $z_1 = -z_2 = z_3$, it reduces to that of genus $g - 2$, and the sought-after critical values are of the form $4((g-2) - 1 - (k-2))\sqrt{-1}^{1+(-1)^{k+1}}$ for $k = 1, \dots, 2g - 3$.

Hence, we have reduced the system of equations and the graph potential for the necklace graph of genus g to that of genus $g - 2$. By the base case from Lemma A.3, the fact that the critical loci for the necklace graph of genus $g - 2$ are of the expected dimension, and that we used the two equations $z_1 + z_2 = 0$ and $z_2 + z_3 = 0$, we obtain that the critical loci for $k = 1, \dots, 2g - 3$ are also of the expected dimension $k + 2$. □

A similar analysis shows the following.

Lemma A.5. *Let $u_1^2 \neq 1$. Assume that $z_1 + z_{g-1}^{-1} = 0$. Then the critical loci have the expected dimension.*

Finally, we prove the following result.

Lemma A.6. *Let $u_1^2 \neq 1$. Assume that $z_1 + z_{g-1}^{-1} \neq 0$ and $z_2 + z_3 \neq 0$. Then the critical loci have the expected dimension.*

Proof. By (86) for $i = 2$ and (87), we obtain that $u_2^2 = v_2^2 = u_{g-1}^2 = v_{g-1}^2 = 1$. This brings us into a situation reminiscent to that of Lemma A.2, but we will only consider the sign of $\left(\begin{smallmatrix} u_2 & u_{g-1} \\ v_2 & v_{g-1} \end{smallmatrix}\right)$.

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Table 2. *Values of J^+.*

	$J^+(u_2)$	$J^+(v_{g-1})$	$J^+(v_2)$	$J^+(u_{g-1})$
$\left(\begin{smallmatrix} + & + \\ + & + \end{smallmatrix}\right)$	2	2	2	2
$\left(\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}\right)$	−2	−2	−2	−2
$\left(\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}\right)$	2	2	−2	−2
$\left(\begin{smallmatrix} - & + \\ + & - \end{smallmatrix}\right)$	−2	−2	2	2
$\left(\begin{smallmatrix} + & + \\ - & - \end{smallmatrix}\right)$	2	−2	−2	2
$\left(\begin{smallmatrix} - & - \\ + & + \end{smallmatrix}\right)$	−2	2	2	−2
$\left(\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}\right)$	2	−2	2	−2
$\left(\begin{smallmatrix} - & + \\ - & + \end{smallmatrix}\right)$	−2	2	−2	2

We can rewrite (88) for $i = 2$ as

$$z_2^2 = \frac{J^+(v_1) + J^+(v_2)}{J^+(u_1) + J^+(u_2)} \tag{98}$$

and (89) as

$$z_1^2 = \frac{J^+(v_1) + J^+(u_{g-1})}{J^+(u_1) + J^+(v_{g-1})} \tag{99}$$

so that

$$\frac{J^+(v_1) + J^+(v_2)}{J^+(u_1) + J^+(u_2)} = \frac{J^+(v_1) + J^+(u_{g-1})}{J^+(u_1) + J^+(v_{g-1})}. \tag{100}$$

In case the parity is *odd*, then either $J^+(u_2) = J^+(v_{g-1})$ and $J^+(v_2) \neq J^+(u_{g-1})$, or vice versa. But the equality of (100) forces both equalities in case one equality holds, so this situation does not arise.

In case the parity is *even*, we have listed the values of J^+ in Table 2. In the first four cases, we observe that $J^+(v_2) = J^+(v_{g-1}) = \pm 2$ and $J^+(v_2) = J^+(u_{g-1}) = \pm 2$ for the appropriate choice of signs. This implies that

$$\begin{aligned} z_1^2 &= \frac{J^+(u_1) \pm 2}{J^+(v_1) \pm 2} \\ z_2^2 &= \frac{J^+(v_1) \pm 2}{J^+(u_1) \pm 2} \end{aligned} \tag{101}$$

hence the choice of u_1 and v_1 determine z_1 and z_2 up to a sign. The equations become those for genus $g - 2$, with the additional constraint that $u_{g-1}^2 = v_{g-1}^2 = 1$. Now projecting along u_1 and v_1 reduces the equations, and we get that the dimension of the critical locus indexed by k (and genus g) is two less than that of the critical locus indexed by $k - 2$ and genus $g - 2$, hence of expected dimension.

The last four cases remain to be studied. Now the induction will not go down by two, but rather by one. Hence, we need to ensure the factor $\sqrt{-1}$ comes into play.

We claim that for $\left(\begin{smallmatrix} + & + \\ - & - \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} - & + \\ - & + \end{smallmatrix}\right)$, resp. $\left(\begin{smallmatrix} - & - \\ + & + \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}\right)$, we have that

1. $J^+(u_1) = -J^+(v_1)$ and $z_1^2 = z_2^2 = -1$;
2. $J^+(u_1) = J^+(v_1)$ and $z_1^2 = z_2^2 = 1$.

We explain this in the case of $\left(\begin{smallmatrix} + & + \\ - & - \end{smallmatrix}\right)$, the other cases being analogous. From Table 2 and (100), we obtain

$$\frac{J^+(v_1) - 2}{J^+(u_1) + 2} = \frac{J^+(v_1) + 2}{J^+(u_1) - 2}. \tag{102}$$

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This can be rewritten as

$$\frac{J^+(u_1)+J^+(v_1)}{J^+(u_1)+2}=\frac{J^+(u_1)+J^+(v_1)}{J^+(u_1)-2}. \tag{103}$$

This is only possible if $J^+(u_1)=-J^+(v_1)$. But we have that $J^+(u_1)\neq\pm 2$, so (89) implies that $z_1^2=-1$. Similarly, (88) for $i=2$ implies that $z_2^2=-1$. We will assume $z_1=z_2^{-1}=\sqrt{-1}$, and the other cases are similar.

By the sign choices, we have $J^+(u_2)=J^+(u_{g-1})=2$, $J^+(u_{g-1})=J^+(v_2)=-2$. We will now use the expression of the graph potential as a sum of bead potentials, together with the partial evaluation we can do using the equalities from above. Omitting a tedious manipulation, the presence of $z_1=z_2=\sqrt{-1}$ will ensure that the resulting Laurent polynomial is the graph potential for the necklace graph of genus $g-1$, except that it is multiplied with $\sqrt{-1}$, so that all critical values are rotated by 90 degrees. □

Acknowledgements. We want to thank Emanuele Macrì, M.S. Narasimhan, and Maxim Smirnov for interesting discussions.

This collaboration started in Bonn in January–March 2018 during the second author's visit to the "Periods in Number Theory, Algebraic Geometry and Physics" Trimester Program at the Hausdorff Center for Mathematics (HIM) and the first and third author's stay at the Max Planck Institute for Mathematics (MPIM). The remaining work was done at the Tata Institute for Fundamental Research (TIFR) during the second author's visit in December 2019–March 2020 and the first author's visit in February 2020. We would like to thank HIM, MPIM and TIFR for the very pleasant working conditions.

Conflict of Interest. The authors have no conflict of interest to declare.

Financial support. The first author was partially supported by the FWO (Research Foundation–Flanders). The third author was partially supported by the Department of Atomic Energy, India, under project no. 12-R&D-TFR-5.01-0500 and also by the Science and Engineering Research Board, India (SRG/2019/000513).

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