THE REFERENCE FRAMES AND A TRANSFORMATION OF THE SPHERICAL FUNCTIONS

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ABSTRACT

The use of spherical functions in dynamical problems is very common. As a rule, they arise in perturbing functions. It is well known that passing from one reference frame to another is accompanied by a double transformation of the perturbation function. That is why problems lose their simplicity and elegance. The problem of two solid bodies is a typical example in this respect.

In the present paper the questions connected with the transformation of the spherical functions when passing from one reference frame to another frame are considered. Traditional functions are generally unsuitable as they introduce a series of difficulties in the problems. That is why complex spherical functions are used. The transformation of spherical functions due to rotation of the coordinate frame is made by means of the Wigner's functions. When translating the frame the Clebsch-Gordon's coefficients are used.

1. INTRODUCTION

In dynamical problems different modifications of the right-handed Cartesian-coordinate system $\{0,x,y,z\}$ and connected with it, spherical reference frame $\{r,\theta,\lambda\}$, are used:

$$x = r\sin\theta\cos\lambda,\ y = r\sin\theta\cos\lambda,\ z = r\cos\theta, \tag{1}$$

in which the traditional spherical functions are defined:

$$Y_{nm}(\theta,\lambda) = P_n^m(\cos\theta)\left\{\begin{matrix}\cos m\lambda\\ \sin m\lambda\end{matrix}\right\}. \tag{2}$$

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In (2) $P_n^m(\cos\theta)$ are Associated Legendre functions.

The expression of the perturbation function by means of (2) leads to some difficulties and complicated transformations when passing from one reference frame to another. The Kaula's transformation (1961) is the best example of that. There the tesseral harmonics and Kepler orbital elements are connected within the two body problem. When refining the analysis of the planet's gravitational potential one can meet the same difficulty (Shkodrov, 1975).

Above all these difficulties relate the transformation of (2) connected with the translation or rotation of the reference system $\{r,\theta,\lambda\}$, as well as the choice of proper Euler angles. In the mathematical apparatus of Quantum mechanics these difficulties are eliminated almost completely. At a large extent one can realise it by replacing (1) with the new complex frame, defined as follows:

$$x_{+1} = \frac{1}{\sqrt{2}} r\sin\theta e^{i\lambda}, \quad x_0 = r\cos\theta, \quad x_{-1} = \frac{1}{\sqrt{2}} r\sin\theta e^{-i\lambda}, \tag{3}$$

which is known as a cyclical frame. The spherical functions (2) in (3) have the following form:

$$\mathscr{D}_{nm}(\theta,\lambda) = \bar{P}_n^m(\cos\theta)e^{im\lambda}, \quad \overset{*}{\mathscr{D}}_{nm}(\theta,\lambda) = \bar{P}_n^m(\cos\theta)e^{-im\lambda}, \tag{4}$$

where (*) means the complex-conjugated spherical function, and $\bar{P}_n^m(\cos\theta)$ are normalised Associated Legendre functions.

The formalism, based on (3) and (4), does not hide the qualitative picture of the dynamical problems, and leads to greater elegance. The only difficulty in this case is probably the novelty of the formalism proposed, but it seems to be not so bad, as complex quantities are usual things in Celestial mechanics and have been used in it's apparatus since it was formed as a scientific discipline.

Here we have to note that in this case our efforts to use this formalism is just contrary to that of B. Jeffreys (1965). Our wish is to use it in dynamical problems in the form close to that in Quantum mechanics.

2. TRANSFORMATION OF THE COMPLEX SPHERICAL FUNCTIONS UNDER TRANSLATION OF THE REFERENCE SYSTEM

Let the reference system $S_1\{G_1,x_1,y_1,z_1\}$ pass into the system $S_2\{G_2,x_2,y_2,z_2\}$, due to the translation of the vector $\vec{R}$. As it is known that under this transformation, the coordinates of the point $P(\vec{r}_1)$, given in S_1, are transformed according to

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$$\vec{r}_2 = \vec{r}_1 - \vec{R}, \tag{5}$$

where $\vec{r}_2$ is the radius-vector of P in S_2. In order to obtain the rule for transformation of the spherical functions

$$[\mathscr{D}_{n_1m_1}(S_1) \Rightarrow \mathscr{D}_{n_2m_2}(S_2)]$$

as a result of the translation (5) we shall use the relation

$$\cos\theta_2 = \frac{r_1}{r_2}\cos\theta_1 - \frac{R}{r_2}\cos\Theta, \tag{6}$$

bearing in mind that r_1,θ_1,λ_1 and r_2,θ_2,λ_2 are spherical coordinates of the point P connected with the reference system S_1 and S_2, respectively, and R,Θ,Λ are spherical coordinates of G_2 in S_1, defined by $\vec{R}$. From (6) we obtain:

$$\mathscr{D}_{10}(\theta_2,\lambda_2) = \sum_{k=0}^{1}(-1)^{k+1}\left(\frac{R}{r_2}\right)\left(\frac{r_1}{R}\right)^k \mathscr{D}_{k0}(\theta_1,\lambda_1)\,\mathscr{D}_{1-k,0}(\Theta,\Lambda). \tag{7}$$

The general rule for transformation of the spherical functions $\mathscr{D}_{nm}(\theta,\lambda)$ in this case is obtained from (5), (6) and (7) by the method of induction. It is:

$$\begin{aligned}\mathscr{D}_{n_2m_2}(\theta_2,\lambda_2) = \sum_{k=0}^{n_2}\sum_{(p,q)}(-1)^{n_2+k}C^{n_2m_2}_{kpn_2-kq}\left(\frac{R}{r_2}\right)^{n_2}\left(\frac{r_1}{R}\right)^k \\ \times\ \mathscr{D}_{kp}(\theta_1,\lambda_1)\,\mathscr{D}_{n_2-kq}(\Theta,\Lambda),\end{aligned} \tag{8}$$

where $C^{n_2m_2}_{kpn_2-kq}$ are the Clebsch-Gordon coefficients. The summation in (8) is made for all integer values of p and q, for which $C^{n_2m_2}_{kpn_2-kq} \neq 0$.

3. TRANSFORMATION OF $\mathscr{D}_{nm}(\theta,\lambda)$ UNDER ROTATION OF THE REFERENCE SYSTEM

In order to obtain the rule for transformation of $\mathscr{D}_{nm}(\theta,\lambda)$ due to rotation of reference frame let us define the Euler angles α, β and γ. After (Edmonds, 1957) we call α a rotation about the z-axis, bringing the frame of axes from the initial position S into the position S', and β a rotation about the y-axis of the frame S' called the line of nodes. The resulting position of the frame of axes is denoted by S'', and γ a rotation about the z-axis of the frame S''. That is the final position of the frame. For that selection of Euler angles, the spherical functions $\mathscr{D}_{nm}(\theta,\lambda)$ are transformated as follows:

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$$\mathscr{D}_{nm'}(\theta_1,\lambda_1) = \sum_{m=-n}^{n} \mathscr{D}_{nm}(\theta,\lambda)\ D^n_{mm'}(\alpha,\beta,\gamma), \tag{9}$$

where θ,λ are the spherical coordinates in initial reference frame S, and θ_1,λ_1 are the spherical coordinates in the new rotating system S_1. The angles θ,λ and θ_1,λ_1 are related by:

$$\begin{aligned} \cos\theta_1 &= \cos\theta\cos\beta + \sin\theta\sin\beta\cos(\lambda-\alpha), \\ \mathrm{ctg}(\lambda'+\gamma) &= \mathrm{ctg}(\lambda-\alpha)\cos\beta - \mathrm{ctg}\theta\sin\beta\sin^{-1}(\lambda-\alpha). \end{aligned} \tag{10}$$

The inverse transformation ($S \Leftarrow S_1$) is:

$$\mathscr{D}_{nm}(\theta,\lambda) = \sum_{m'=-n}^{n} \mathscr{D}_{nm'}(\theta',\lambda')D^{n*}_{mm'}(\alpha,\beta,\gamma). \tag{11}$$

Wigner's D-functions, or generalised spherical functions, are complex functions of three real arguments. They have the following structure:

$$D^n_{mm'}(\alpha,\beta,\gamma) = e^{-im\alpha}d^n_{mm'}(\beta)e^{-im'\gamma}, \tag{12}$$

where the real function $d^n_{mm'}(\beta)$ may be defined:

$$\begin{aligned} d^n_{mm'}(\beta) = {} & \left[\frac{(n+m')!(n-m')!}{(n+m)!(n-m)!}\right]^{\frac{1}{2}} \sum_{\sigma}\binom{n+m}{n-m-\sigma}\binom{n-m}{\sigma}(-1)^{n-m-\sigma} \\ & \times\ \cos^{2\sigma-m'+m}\left(\frac{\beta}{2}\right)\sin^{2n-2\sigma-m'-m}\left(\frac{\beta}{2}\right) \end{aligned} \tag{13}$$

In a particular case when m=0, or m'=0 we have:

$$\begin{aligned} D^n_{om'}(\alpha,\beta,\gamma) &= (-1)^{m'}(2n+1)^{-\frac{1}{2}}\mathscr{D}^*_{nm'}(\beta,\gamma), \\ D^n_{mo}(\alpha,\beta,\gamma) &= (2n+1)^{-\frac{1}{2}}\mathscr{D}^*_{nm}(\beta,\alpha). \end{aligned} \tag{14}$$

If m=m'=0, then

$$D^n_{oo}(\alpha,\beta,\gamma) = P_n(\cos\beta). \tag{15}$$

4. CLEBSCH-GORDON COEFFICIENTS

The Clebsch-Gordon coefficients are of great importance for our coordinate transformations. In the theory of groups the representation of the elements of a group by linear functions of an irreducible representation is realized by means of these coefficients. They are real and satisfy the conditions:

$$C^{j_1+j_2m_1+m_2}_{j_1m_1j_2m_2} > 0,\ C^{jm}_{j_1m_1j_2-j_1} > 0,\ C^{j_1+j_2j_1+j_2}_{j_1j_1j_2j_2} = 1. \tag{16}$$

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and differ from zero if $m_1+m_2 = n$, $j+j_1+j_2$ is even.

The following Clebsch-Gordon series:

$$\mathscr{D}_{n_1m_1}(\theta,\lambda)\,\mathscr{D}_{n_2m_2}(\theta,\lambda) = \sum_{\ell=\max\{n_1+n_2,m_1+m_2\}}^{n_1+n_2} \times\; C^{\ell m_1+m_2}_{n_1m_1n_2m_2}\, C^{\ell o}_{n_1on_2o}\, \mathscr{D}_{\ell m_1+m_2}(\theta,\lambda) \tag{17}$$

may be of great importance for the dynamical problems. The Clebsch-Gordon series may be obtained using the recurrence relations of spherical functions (Shkodrov, 1975).

5. CONCLUSION

The formalism briefly stated here considerably facilitates the transformation of the perturbation function when passing from one frame to another. The fact, that the functions of Wigner and the Clebsch-Gordon coefficients are perfectly mastered in Quantum mechanics, is an additional advantage. The attempt in (Shkodrov, Gechev, 1979, 1980; Shkodrov, 1980, 1980a) to use this mathematical formalism in different problems of Celestial mechanics showed, that it brings elegance and makes the solutions simpler. Its effectiveness is specially evident in the problem of two solid bodies. Due to this formalism a simple representation of the force function is obtained. In the two planet problem and in the two body problem the use of (7) leads to all known expressions for the transformations of the perturbation function.

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