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An extension of the stochastic sewing lemma and applications to fractional stochastic calculus

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Abstract

We give an extension of Lê's stochastic sewing lemma. The stochastic sewing lemma proves convergence in L_m of Riemann type sums $\sum_{[s,t]\in\pi} A_{s,t}$ for an adapted two-parameter stochastic process A, under certain conditions on the moments of $A_{s,t}$ and of conditional expectations of $A_{s,t}$ given $\mathcal{F}_s$. Our extension replaces the conditional expectation given $\mathcal{F}_s$ by that given $\mathcal{F}_v$ for $v < s$, and it allows to make use of asymptotic decorrelation properties between $A_{s,t}$ and $\mathcal{F}_v$ by including a singularity in $(s - v)$. We provide three applications for which Lê's stochastic sewing lemma seems to be insufficient. The first is to prove the convergence of Itô or Stratonovich approximations of stochastic integrals along fractional Brownian motions under low regularity assumptions. The second is to obtain new representations of local times of fractional Brownian motions via discretization. The third is to improve a regularity assumption on the diffusion coefficient of a stochastic differential equation driven by a fractional Brownian motion for pathwise uniqueness and strong existence.

Contents

1. Introduction and the main theorem

In analysis and probability theory, we often consider the convergence of sums

$$\sum_{[s,t]\in\pi} A_{s,t}. \tag{1.1}$$

Here, π is a partition of an interval $[0, T]$, and we consider the limit of

$$|\pi| := \max_{[s,t]\in\pi} |t - s| \to 0.$$

For instance, if $A_{s,t} := f(s)(t-s)$, then we consider a Riemann sum approximation of $\int_0^T f(s)\,ds$, and if $A_{s,t} := X_s(W_t - W_s)$, where W is a Brownian motion and X is an adapted process, then we consider the Itô approximation of the stochastic integral $\int_0^T X_r\,dW_r$.

Gubinelli [17], inspired by Lyons' results on almost multiplicative functionals in the theory of rough paths [30], showed that if

$$\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}, \quad 0 \le s < u < t \le T, \tag{1.2}$$

satisfies $|\delta A_{s,u,t}| \lesssim |t-s|^{1+\varepsilon}$ for some $\varepsilon > 0$, then the sums (1.1) converge. This result is now called the *sewing lemma*, named so in the work of Feyel and de La Pradelle [13]. This lemma is so powerful that many applications and many extensions are known. For instance, it can be used to define rough integrals (see [17] and the monograph [14] of Friz and Hairer).

When $(A_{s,t})_{s \le t}$ is random, and when we want to prove the convergence of the sums (1.1), the above sewing lemma is often not sufficient. For instance, if $A_{s,t} := (W_t - W_s)^2$, the sums converge to the quadratic variation of the Brownian motion. However, we only have

$$|\delta A_{s,u,t}(\omega)| \lesssim_{\varepsilon,\omega} |t-s|^{1-\varepsilon}$$

almost surely for every $\varepsilon > 0$, and hence, we cannot apply the sewing lemma.

Lê [24] proved a stochastic version of the sewing lemma (*stochastic sewing lemma*): if a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is given, such that

○ $A_{s,t}$ is $\mathcal{F}_t$-measurable and
○ for some $\varepsilon_1, \varepsilon_2 > 0$ and $m \in [2, \infty)$, we have for every $s < u < t$,

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]\|_{L_m(\mathbb{P})} \lesssim |t-s|^{1+\varepsilon_2}, \tag{1.3}$$

$$\|\delta A_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim |t-s|^{\frac{1}{2}+\varepsilon_1}, \tag{1.4}$$

then the sums (1.1) converge in $L_m(\mathbb{P})$. As usual, the Banach space $L_m(\mathbb{P})$ is equipped with the norm

$$\|X\|_{L_m(\mathbb{P})} := \left(\int_\Omega |X|^m\,d\mathbb{P} \right)^{\frac{1}{m}}.$$

If $A_{s,t} := (W_t - W_s)^2$, then we have $\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s] = 0$ and (1.4) is satisfied with $\varepsilon_1 = \frac{1}{2}$. Therefore, we can prove the convergence of (1.1) in $L_m(\mathbb{P})$. The stochastic sewing lemma has been already shown to be very powerful in the original work [24] of Lê, and an increasing number of papers are appearing that take advantage of the lemma.

However, there are situations where Lê's stochastic sewing lemma seems insufficient. For instance, consider

$$A_{s,t} := |B_t - B_s|^{\frac{1}{H}}, \tag{1.5}$$

where B is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. It is well-known that the sums (1.1) converge to $c_H T$ in $L_m(\mathbb{P})$. Although we have the estimate (1.4), we fail to obtain the estimate (1.3) unless $H = \frac{1}{2}$.

To get an idea on how Lê's stochastic sewing lemma should be modified for this problem, observe the following trivial fact:

$$\mathbb{E}[\delta A_{s,u,t}] = 0.$$

This suggests that we consider estimates that interpolate $\mathbb{E}[\delta A_{s,u,t}]$ and $\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_s]$. In fact, we can obtain the following estimates:

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v]\|_{L_m(\mathbb{P})} \lesssim_H \left(\frac{t-s}{s-v} \right)^{1-H} (t-s), \quad 0 \le v < s < u < t \le T. \tag{1.6}$$

We can prove (1.6), for instance, by applying Picard's result [37, Lemma A.1] on the asymptotic independence of fractional Brownian increments, or more directly by doing a similar calculation as in Section 4. This discussion motivates the following main theorem of our paper.

Theorem 1.1. *Suppose that we have a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ and a family of $\mathbb{R}^d$-valued random variables $(A_{s,t})_{0 \le s \le t \le T}$, such that $A_{s,s} = 0$ for every $s \in [0,T]$ and such that $A_{s,t}$ is $\mathcal{F}_t$-measurable. We define $\delta A_{s,u,t}$ by (1.2). Furthermore, suppose that there exist constants*

$$m \in [2, \infty), \quad \Gamma_1, \Gamma_2, M \in [0, \infty), \quad \alpha, \beta_1, \beta_2 \in [0, \infty),$$

such that the following conditions are satisfied.

○ *For every $0 \le t_0 < t_1 < t_2 < t_3 \le T$, we have*

$$\|\mathbb{E}[\delta A_{t_1,t_2,t_3} | \mathcal{F}_{t_0}]\|_{L_m(\mathbb{P})} \le \Gamma_1 (t_1 - t_0)^{-\alpha} (t_3 - t_1)^{\beta_1}, \quad \text{if } M(t_3 - t_1) \le t_1 - t_0, \tag{1.7}$$

$$\|\delta A_{t_0,t_1,t_2}\|_{L_m(\mathbb{P})} \le \Gamma_2 (t_2 - t_0)^{\beta_2}. \tag{1.8}$$

○ *We have*

$$\beta_1 > 1, \quad \beta_2 > \frac{1}{2}, \quad \beta_1 - \alpha > \frac{1}{2}. \tag{1.9}$$

Then, there exists a unique, up to modifications, $\mathbb{R}^d$-valued stochastic process $(\mathcal{A}_t)_{t \in [0,T]}$ with the following properties.

○ *$\mathcal{A}_0 = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$-measurable, and $\mathcal{A}_t$ belongs to $L_m(\mathbb{P})$.*
○ *There exist nonnegative constants C_1, C_2, and C_3, such that*

$$\|\mathbb{E}[\mathcal{A}_{t_2} - \mathcal{A}_{t_1} - A_{t_1,t_2} | \mathcal{F}_{t_0}]\|_{L_m(\mathbb{P})} \le C_1 |t_1 - t_0|^{-\alpha} |t_2 - t_1|^{\beta_1}, \tag{1.10}$$

$$\|\mathcal{A}_{t_2} - \mathcal{A}_{t_1} - A_{t_1,t_2}\|_{L_m(\mathbb{P})} \le C_2 |t_2 - t_1|^{\beta_1 - \alpha} + C_3 |t_2 - t_1|^{\beta_2}, \tag{1.11}$$

where $t_2 - t_1 \le M^{-1}(t_1 - t_0)$ is assumed for the inequality (1.10).

In fact, we can choose C_1, C_2, and C_3 so that

$$C_1 \lesssim_{\beta_1} \Gamma_1, \quad C_2 \lesssim_{\alpha, \beta_1, \beta_2, M} \kappa_{m,d} \Gamma_1, \quad C_3 \lesssim_{\alpha, \beta_1, \beta_2, M} \kappa_{m,d} \Gamma_2,$$

where $\kappa_{m,d}$ is the constant of the Burkholder-Davis-Gundy inequality (see (1.14)). Furthermore, for $\tau \in [0, T]$, if we set

$$A_\tau^\pi := \sum_{[s,t] \in \pi} A_{s,t}, \quad \text{where } \pi \text{ is a partition of } [0, \tau],$$

then the family $(A_\tau^\pi)_\pi$ converges to $\mathcal{A}_\tau$ in $L_m(\mathbb{P})$ as $|\pi|$ tends to 0.

Remark 1.2. We discuss the optimality of the condition (1.9). By considering a deterministic $(A_{s,t})$, we see that the condition $\beta_1 > 1$ is necessary. To see that the conditions $\beta_2 > \frac{1}{2}$ and $\beta_1 - \alpha > \frac{1}{2}$ are necessary, let B^1 and B^2 be two independent one-dimensional fractional Brownian motions with Hurst parameter $\frac{1}{4}$ (see Definition 3.1), and we set $A_{s,t} := B_s^1 B_{s,t}^2$. It is well-known since the work [10] of Coutin and Qian that the iterated integral $\int B^1 dB^2$ does not exist, and, therefore, the Riemann sum with respect to $(A_{s,t})$ should not converge. In fact, the family $(A_{s,t})$, with filtration $(\mathcal{F}_t)$ generated by (B^1, B^2), satisfies (1.7) and (1.8) with

$$\alpha = \frac{3}{2}, \quad \beta_1 = 2, \quad \beta_2 = \frac{1}{2}.$$

To see this, we observe $\delta A_{t_1,t_2,t_3} = -B^1_{t_1,t_2} B^2_{t_2,t_3}$, and

$$\|\delta A_{t_1,t_2,t_3}\|_{L_m(\mathbb{P})} \lesssim_m (t_3 - t_1)^{\frac{1}{2}}.$$

To compute the conditional expectation, we observe

$$\mathbb{E}[\delta A_{t_1,t_2,t_3}|\mathcal{F}_{t_0}] = -\mathbb{E}[B^1_{t_1,t_2}|\mathcal{F}_{t_0}]\mathbb{E}[B^2_{t_2,t_3}|\mathcal{F}_{t_0}],$$

and by the estimate (3.4), we have

$$\|\mathbb{E}[\delta A_{t_1,t_2,t_3}|\mathcal{F}_{t_0}]\|_{L_m(\mathbb{P})} \lesssim_m (t_1 - t_0)^{-\frac{3}{2}} (t_3 - t_1)^2.$$

Remark 1.3. The proof shows that if

$$1 + \alpha - \beta_1 < 2\alpha\beta_2 - \alpha, \tag{1.12}$$

then we have $C_2 \lesssim_{\alpha,\beta_1,\beta_2,M} \Gamma_1$, and we can omit the factor $\kappa_{m,d}$. This is similar to [24], where C_2 also does not depend on $\kappa_{m,d}$. If $\alpha = 0$ and $M = 0$, Theorem 1.1 recovers Lê's stochastic sewing lemma [24, Theorem 2.1]. If $\alpha = 0$ and $M > 0$, it recovers a lemma [16, Lemma 2.2] by Gerencsér.

Recently, Gerencsér's stochastic sewing lemma is called *shifted* stochastic sewing lemma. In the follow-up works, we continue to refer to Theorem 1.1 by the same name.

Remark 1.4. The proof shows that there exists $\varepsilon = \varepsilon(\alpha, \beta_1, \beta_2) > 0$, such that

$$\|\mathcal{A}_\tau - A^\pi_\tau\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M,m,d,T} (\Gamma_1 + \Gamma_2)|\pi|^\varepsilon$$

for every $\tau \in [0, T]$ and every partition π of $[0, \tau]$. A similar remark holds in the setting of Corollary 2.7.

Remark 1.5. As in another work [26] of Lê, it should be possible to extend Theorem 1.1 so that the stochastic process $(A_{s,t})_{s,t \in [0,T]}$ takes values in a certain Banach space.

Remark 1.6. A multidimensional version of the sewing lemma is the *reconstruction theorem* [18, Theorem 3.10] of Hairer. A stochastic version of the reconstruction theorem was obtained by Kern [21]. It could be possible to extend Theorem 1.1 in the multidimensional setting, but we will not pursue it in this paper.

The proof of Theorem 1.1 is given in Section 2. If $A_{s,t}$ is given by (1.5), then we can apply Theorem 1.1 with

$$\alpha = 1 - H, \quad \beta_1 = 2 - H, \quad \beta_2 = 1.$$

However, the application of Theorem 1.1 goes beyond this simple problem of $\frac{1}{H}$-variation of the fractional Brownian motion. Indeed, in Section 3, we prove the convergence of Itô and Stratonovich approximations to the stochastic integrals

$$\int_0^T f(B_s)\, \mathrm{d}B_s \quad \text{and} \quad \int_0^T f(B_s) \circ \mathrm{d}B_s$$

with $H > \frac{1}{2}$ in Itô's case and with $H > \frac{1}{6}$ in Stratonovich's case, under rather general assumptions on the regularity of f, in fact, $f \in C^2_b(\mathbb{R}^d, \mathbb{R}^d)$ works for all $H > \frac{1}{6}$. In Section 4, we obtain new representations of local times of fractional Brownian motions via discretization.

Finally, we remark that one of the most interesting applications of Lê's stochastic sewing lemma lies in the phenomenon of *regularization by noise* (see, e.g. [24], Athreya et al. [2], [16], and Anzeletti et al. [1]). In these works, they consider the stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t)\, \mathrm{d}t + \mathrm{d}Y_t \tag{1.13}$$

with an *additive* noise Y, which is often a fractional Brownian motion. It is interesting that, although in absence of noise the coefficient b needs to belong to C^1 for well-posedness, the presence of noise enables us to prove the certain well-posedness of (1.13) under much weaker assumption, in fact, b can be even a distribution; hence the name *regularization by noise*. In Section 5 of our paper, we are interested in a related but different problem. Indeed, we are interested in improving the regularity of the diffusion coefficient rather than the drift coefficient. We consider the Young SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t$$

driven by a fractional Brownian motion B with Hurst parameter $H \in (\frac{1}{2}, 1)$. The pathwise theory of Young's differential equation requires that the regularity of σ better than $1/H$ for uniqueness, and this condition is sharp for general drivers B of the same regularity as the fractional Brownian motion. We will improve this regularity assumption for pathwise uniqueness and strong existence. Again, a stochastic sewing lemma (Lemma 5.5), which is a variant of Theorem 1.1, will play a key role.

Notation

We write $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{N} := \{1, 2, \ldots\}$. Given a function $f : [S, T] \to \mathbb{R}^d$, we write $f_{s,t} := f_t - f_s$. We denote by $\kappa_{m,d}$ the best constant of the discrete Burkholder-Davis-Gundy (BDG) inequality for $\mathbb{R}^d$-valued martingale differences [6]. Namely, if we are given a filtration $(\mathcal{F}_n)_{n=1}^\infty$ and a sequence $(X_n)_{n=1}^\infty$ of $\mathbb{R}^d$-valued random variables, such that X_n is $\mathcal{F}_n$-measurable for every $n \geq 1$ and $\mathbb{E}[X_n | \mathcal{F}_{n-1}] = 0$ for every $n \geq 2$, then

$$\|\sum_{n=1}^\infty X_n\|_{L_m(\mathbb{P})} \leq \kappa_{m,d} \|\sum_{n=1}^\infty X_n^2\|_{L_{\frac{m}{2}}(\mathbb{P})}^{\frac{1}{2}}. \tag{1.14}$$

Rather than (1.14), we mostly use the inequality

$$\|\sum_{n=1}^\infty X_n\|_{L_m(\mathbb{P})} \leq \kappa_{m,d} \Big(\sum_{n=1}^\infty \|X_n\|_{L_m(\mathbb{P})}^2\Big)^{\frac{1}{2}} \tag{1.15}$$

for $m \geq 2$, which follows from (1.14) by Minkowski's inequality. We write $A \lesssim B$ or $A = O(B)$ if there exists a nonnegative constant C, such that $A \leq CB$. To emphasize the dependence of C on some parameters $a, b, \ldots$, we write $A \lesssim_{a,b,\ldots} B$.

2. Proof of the main theorem

The overall strategy of the proof is the same as that of the original work [24] of Lê. Namely, we combine the argument of the deterministic sewing lemma ([17], [13], and Yaskov [40]) with the discrete BDG inequality [6]. However, the proof of Theorem 1.1 requires more labor at a technical level. Some proofs will be postponed to Appendix A.

As in [24], the following lemma, which originates from [40], will be needed. It allows us to replace general partitions by dyadic partitions.

Lemma 2.1 [24, Lemma 2.14]. *Under the setting of Theorem 1.1, let*

$$0 \leq t_0 < t_1 < \cdots < t_{N-1} < t_N \leq T.$$

Then, we have

$$A_{t_0, t_N} - \sum_{i=1}^N A_{t_{i-1}, t_i} = \sum_{n \in \mathbb{N}_0} \sum_{i=0}^{2^n - 1} R_i^n, \tag{2.1}$$

where

$$R_i^n := \delta A_{s_1^{n,i}, s_2^{n,i}, s_3^{n,i}} + \delta A_{s_1^{n,i}, s_3^{n,i}, s_4^{n,i}}, \tag{2.2}$$

and

$$n \in \mathbb{N}_0, \quad i \in \{0, 1, \ldots, 2^n - 1\}, \quad s_j^{n,i} \in [t_0 + \frac{i(t_N - t_0)}{2^n}, t_0 + \frac{(i+1)(t_N - t_0)}{2^n}],$$

and where $R_i^n = 0$ for all sufficiently large n.

The next two lemmas (Lemmas 2.2 and 2.3) correspond to the estimates [24, (2.50) and (2.51)], respectively.

Lemma 2.2. *Under the setting of Theorem 1.1, let*

$$0 \le s < t_0 < t_1 < \cdots < t_{N-1} < t_N \le T, \quad t_N - t_1 \le \frac{t_0 - s}{M}.$$

Then,

$$\|\mathbb{E}[A_{t_0, t_N} - \sum_{i=1}^{N} A_{t_{i-1}, t_i} | \mathcal{F}_s]\|_{L_m(\mathbb{P})} \lesssim_{\beta_1} \Gamma_1 |t_0 - s|^{-\alpha} |t_N - t_0|^{\beta_1}.$$

Proof. In view of the decomposition (2.1), the triangle inequality gives

$$\|\mathbb{E}[A_{t_0, t_N} - \sum_{i=1}^{N} A_{t_{i-1}, t_i} | \mathcal{F}_s]\|_{L_m(\mathbb{P})} \le \sum_{n \in \mathbb{N}_0} \sum_{i=0}^{2^n - 1} \|\mathbb{E}[R_i^n | \mathcal{F}_s]\|_{L_m(\mathbb{P})}.$$

By (1.7) and (2.2),

$$\|\mathbb{E}[R_i^n | \mathcal{F}_s]\|_{L_m(\mathbb{P})} \le 2\Gamma_1 (t_0 - s)^{-\alpha} (2^{-n} |t_N - t_0|)^{\beta_1} = 2\Gamma_1 2^{-n\beta_1} |t_0 - s|^{-\alpha} |t_N - t_0|^{\beta_1}.$$

Therefore, recalling $\beta_1 > 1$ from (1.9), the claim follows.   □

The following lemma is the most important technical ingredient for the proof of Theorem 1.1.

Lemma 2.3. *Under the setting of Theorem 1.1, let*

$$0 \le t_0 < t_1 < \cdots < t_{N-1} < t_N \le T.$$

Then,

$$\|A_{t_0, t_N} - \sum_{i=1}^{N} A_{t_{i-1}, t_i}\|_{L_m(\mathbb{P})} \lesssim_{\alpha, \beta_1, \beta_2, M} \kappa_{m,d} \Gamma_1 |t_N - t_0|^{\beta_1 - \alpha} + \kappa_{m,d} \Gamma_2 |t_N - t_0|^{\beta_2}.$$

Under (1.12), we can replace $\kappa_{m,d} \Gamma_1$ by Γ_1.

Proof under (1.12). To simplify the proof, here, we assume (1.12), that is, that the additional technical condition $1 + \alpha - \beta_1 < 2\alpha\beta_2 - \alpha$ holds. The proof in the general setting will be given in Appendix A.

We, again, use the representation (2.1). We fix a large $n \in \mathbb{N}$ and set $\mathcal{F}_k^n := \mathcal{F}_{t_0 + \frac{k}{2^n}(t_N - t_0)}$. Fix an integer $L = L_n \in [M+1, 2^n]$, which will be chosen later. We have

$$\sum_{i=0}^{2^n-1} R_i^n = \sum_{l=0}^{L-1} \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \left(R_{Lj+l}^n - \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \mathbb{1}_{\{j \geq 1\}} \right) + \sum_{l=0}^{L-1} \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n].$$

$$(2.3)$$

We estimate the first term of (2.3). By the BDG inequality together with Minkowski's inequality (see (1.15)), we have

$$\| \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \left(R_{Lj+l}^n - \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \mathbb{1}_{\{j \geq 1\}} \right) \|_{L_m(\mathbb{P})}^2$$

$$\leq \kappa_{m,d}^2 \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \| R_{Lj+l}^n - \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \mathbb{1}_{\{j \geq 1\}} \|_{L_m(\mathbb{P})}^2$$

$$\leq 4\kappa_{m,d}^2 \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \| R_{Lj+l}^n \|_{L_m(\mathbb{P})}^2.$$

Using (1.8) and (2.2) and noting that we include more terms in the sum by requiring $j \leq 2^n/L$ only instead of $Lj + l \leq 2^n - 1$, we get

$$\sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \| R_{Lj+l}^n \|_{L_m(\mathbb{P})}^2 \leq 4\Gamma_2^2 2^{-n(2\beta_2-1)} L^{-1} |t_N - t_0|^{2\beta_2}.$$

Therefore,

$$\| \sum_{l=0}^{L-1} \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \left(R_{Lj+l}^n - \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \mathbb{1}_{\{j \geq 1\}} \right) \|_{L_m(\mathbb{P})} \lesssim \kappa_{m,d} \Gamma_2 L^{\frac{1}{2}} 2^{-n(\beta_2-\frac{1}{2})} |t_N - t_0|^{\beta_2}.$$

We next estimate the second term of (2.3). The triangle inequality yields

$$\| \sum_{l=0}^{L-1} \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \|_{L_m(\mathbb{P})} \leq \sum_{l=0}^{L-1} \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \| \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \|_{L_m(\mathbb{P})}.$$

By (1.7),

$$\| \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \|_{L_m(\mathbb{P})} \leq \Gamma_1 (L-1)^{-\alpha} 2^{-(\beta_1-\alpha)n} |t_N - t_0|^{\beta_1-\alpha}.$$

Therefore,

$$\| \sum_{l=0}^{L-1} \sum_{\substack{j \geq 0: \\ Lj+l < 2^n}} \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \|_{L_m(\mathbb{P})} \lesssim_\alpha \Gamma_1 L^{-\alpha} 2^{-(\beta_1-\alpha-1)n} |t_N - t_0|^{\beta_1-\alpha}.$$

In conclusion,

$$\| \sum_{i=0}^{2^n-1} R_i^n \|_{L_m(\mathbb{P})} \lesssim_\alpha \Gamma_1 L^{-\alpha} 2^{-(\beta_1-\alpha-1)n} |t_N - t_0|^{\beta_1-\alpha} + \kappa_{m,d} \Gamma_2 L^{\frac{1}{2}} 2^{-n(\beta_2-\frac{1}{2})} |t_N - t_0|^{\beta_2}. \qquad (2.4)$$

We wish to choose $L = L_n$ so that (2.4) is summable with respect to n. We therefore set $L_n := \lfloor 2^{\delta n} \rfloor$, where

$$\alpha\delta + \beta_1 - \alpha - 1 > 0, \quad 0 < \delta < \min\{2\beta_2 - 1, 1\}. \tag{2.5}$$

Such a δ exists exactly under the additional technical assumption (1.12), namely, if $1 + \alpha - \beta_1 < 2\alpha\beta_2 - \alpha$. Then, (2.4) yields

$$\| \sum_{n:2^{n\delta} \geq M+2} \sum_{i=0}^{2^n-1} R_i^n \|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2} \Gamma_1 |t_N - t_0|^{\beta_1 - \alpha} + \kappa_{m,d}\Gamma_2 |t_N - t_0|^{\beta_2}.$$

To estimate the contribution coming from the small n with $2^{n\delta} < M + 2$, we apply (1.8) which yields

$$\| \sum_{i=0}^{2^n-1} R_i^n \|_{L_m(\mathbb{P})} \leq 2\Gamma_2 \sum_{i=0}^{2^n-1} 2^{-n\beta_2} |t_N - t_0|^{\beta_2} = \Gamma_2 2^{1+n(1-\beta_2)} |t_N - t_0|^{\beta_2}.$$

Thus, we conclude

$$\| \sum_{n \in \mathbb{N}_0} \sum_{i=0}^{2^n-1} R_i^n \|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M} \Gamma_1 |t_N - t_0|^{\beta_1 - \alpha} + \kappa_{m,d}\Gamma_2 |t_N - t_0|^{\beta_2},$$

where the fact $\kappa_{m,d} \geq 1$ is used. $\qquad\square$

Lemma 2.4. *Under the setting of Theorem 1.1, let π, π' be partitions of $[0, T]$, such that π refines π'. Suppose that we have*

$$\min_{[s,t] \in \pi'} |s - t| \geq \frac{|\pi'|}{3}. \tag{2.6}$$

Then, there exists $\varepsilon \in (0, 1)$, such that

$$\|A_T^{\pi'} - A_T^{\pi}\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M,m,d,T} (\Gamma_1 + \Gamma_2)|\pi'|^{\varepsilon}.$$

Sketch of the proof. Here, we give a sketch of the proof under (1.12). The complete proof is given in Appendix A. The argument is similar to Lemma 2.3.

Write

$$\pi' =: \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$$

and

$$\{[s,t] \in \pi \mid t_j \leq s < t \leq t_{j+1}\} =: \{t_j = t_0^j < t_1^j < \cdots < t_{N_j-1}^j < t_{N_j}^j = t_{j+1}\}.$$

We set $L := \lfloor |\pi'|^{-\delta} \rfloor$, where δ satisfies (2.5). We set

$$Z_j^l := A_{t_{jL+l}, t_{jL+l+1}} - \sum_{k=1}^{N_{jL+l}} A_{t_{k-1}^{jL+l}, t_k^{jL+l}}.$$

As in Lemma 2.3, we consider the decomposition $A_T^{\pi'} - A_T^{\pi} = A + B$, where

$$A := \sum_{l<L} \sum_{j:Lj<N-l} \left\{ Z_j^l - \mathbb{E}[Z_j^l | \mathcal{F}_{t_{(j-1)L+l+1}}] \right\}, \quad B := \sum_{l<L} \sum_{j:Lj<N-l} \mathbb{E}[Z_j^l | \mathcal{F}_{t_{(j-1)L+l+1}}].$$

We estimate A by using the BDG inequality, Lemma 2.3, and (2.6), to obtain

$$\|A\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M,m,d,T} L^{\frac{1}{2}}(\Gamma_1|\pi'|^{\beta_1-\alpha-\frac{1}{2}} + \Gamma_2|\pi'|^{\beta_2-\frac{1}{2}}).$$

We estimate B by using the triangle inequality, Lemma 2.2, and (2.6), to obtain

$$\|B\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,M,m,d,T} \Gamma_1 L^{-\alpha k}|\pi'|^{\beta_1-\alpha-1}.$$

As in Lemma 2.3, we choose $L := \lfloor|\pi'|^{-\delta}\rfloor$ with δ satisfying (2.5). We then obtain the claimed estimate.

Remark 2.5. In the setting of Lemma 2.4, assume that the adapted process $(\mathcal{A}_t)_{t\in[0,T]}$ satisfies (1.10) and (1.11). Then we obtain for some $\varepsilon > 0$:

$$\|A_T^{\pi'} - \mathcal{A}_T\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M,m,d,T} (\Gamma_1 + \Gamma_2)|\pi'|^{\varepsilon}.$$

Indeed, it suffices to replace $A_{t_{jL+l},t_{jL+l+1}}$ by $\mathcal{A}_{t_{jL+l},t_{jL+l+1}}$ in the previous proof.

Lemma 2.6. *Let π be a partition of $[0,T]$. Then, there exists a partition π' of $[0,T]$, such that π refines π', $|\pi'| \leq 3|\pi|$ and*

$$\min_{[s,t]\in\pi'}|t-s| \geq \frac{|\pi'|}{3}.$$

Proof. We write $\pi = \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$. We set $k_0 := -1$, and for $l \in \mathbb{N}$, we inductively set

$$k_l := \inf\{j > k_{l-1} \mid t_{j+1} - t_{k_{l-1}+1} \geq |\pi|\}, \quad \text{where } \inf\varnothing := N.$$

Set $L := \sup\{l \mid k_l < N\}$. Then, we define

$$s_j := \begin{cases} t_{k_j+1} & \text{if } j < L, \\ t_N & \text{if } j = L. \end{cases}$$

By construction, $\pi' = \{s_j\}_{j=1}^L$ satisfies the claimed properties: $s_{j+1} - s_j \leq 2|\pi|$ if $j < L - 2$, and $s_L - s_{L-1} \leq 3|\pi|$, so $|\pi'| \leq 3|\pi|$; moreover, $\min_{[s,t]\in\pi'}|t-s| \geq |\pi| \geq 3^{-1}|\pi'|$. □

Proof of Theorem 1.1. We will not write down dependence on $\alpha, \beta_1, \beta_2, M, m, d, T$. We first prove the convergence of $(A_\tau^\pi)_\pi$. Without loss of generality, we assume $\tau = T$. Let π_1, π_2 be partitions of $[0,T]$. By Lemma 2.6, there exist partitions π_1', π_2', such that for $j \in \{1,2\}$, the partition π_j refines π_j', $|\pi_j'| \leq 3|\pi_j|$ and

$$\min_{[s,t]\in\pi_j'}|t-s| \geq 3^{-1}|\pi_j'|.$$

Lemma 2.4 shows that for some $\varepsilon > 0$, we have

$$\|A_T^{\pi_j} - A_T^{\pi_j'}\|_{L_m(\mathbb{P})} \lesssim (\Gamma_1 + \Gamma_2)|\pi_j|^{\varepsilon}.$$

Therefore, by the triangle inequality,

$$\|A_T^{\pi_1} - A_T^{\pi_2}\|_{L_m(\mathbb{P})} \lesssim \|A_T^{\pi_1'} - A_T^{\pi_2'}\|_{L_m(\mathbb{P})} + (\Gamma_1 + \Gamma_2)(|\pi_1|^{\varepsilon} + |\pi_2|^{\varepsilon}). \tag{2.7}$$

Let π refine both π_1' and π_2'. Lemma 2.4 implies that

$$\|A_T^{\pi_1'} - A_T^{\pi_2'}\|_{L_m(\mathbb{P})} \leq \|A_T^{\pi_1'} - A_T^{\pi}\|_{L_m(\mathbb{P})} + \|A_T^{\pi_2'} - A_T^{\pi}\|_{L_m(\mathbb{P})} \lesssim (\Gamma_1 + \Gamma_2)(|\pi_1|^{\varepsilon} + |\pi_2|^{\varepsilon}). \tag{2.8}$$

The estimates (2.7) and (2.8) show

$$\|A_T^{\pi_1} - A_T^{\pi_2}\|_{L_m(\mathbb{P})} \lesssim (\Gamma_1 + \Gamma_2)(|\pi_1|^{\varepsilon} + |\pi_2|^{\varepsilon}).$$

Thus, $\{A_T^{\pi}\}_{\pi}$ forms a Cauchy net in $L_m(\mathbb{P})$. We denote the limit by $\mathcal{S}_T$. We next prove that $(\mathcal{S}_t)_{t \in [0,T]}$ satisfies (1.10) and (1.11). Let $t_0 < t_1 < t_2$ be such that $M(t_2 - t_1) \leq t_1 - t_0$. Let $\pi_n = \{t_1 + k2^{-n}(t_2 - t_1) : k = 0, \ldots, 2^n\}$ be the nth dyadic partition of $[t_1, t_2]$, and we write

$$A_{t_1,t_2}^n := \sum_{[s,t] \in \pi_n} A_{s,t}.$$

We have

$$\mathbb{E}[\mathcal{S}_{t_1,t_2} - A_{t_1,t_2}|\mathcal{F}_{t_0}] = \lim_{n \to \infty} \mathbb{E}[A_{t_1,t_2}^n - A_{t_1,t_2}|\mathcal{F}_{t_0}] \quad \text{in } L_m(\mathbb{P}). \tag{2.9}$$

By Lemma 2.2,

$$\|\mathbb{E}[A_{t_1,t_2} - A_{t_1,t_2}^n|\mathcal{F}_{t_0}]\|_{L_m(\mathbb{P})} \lesssim_{\beta_1} \Gamma_1|t_1 - t_0|^{-\alpha}|t_2 - t_1|^{\beta_1}.$$

In this estimate, we can replace A_{t_1,t_2}^n by $\mathcal{S}_{t_1,t_2}$ in view of (2.9). Similarly, by Lemma 2.3, we obtain

$$\|\mathcal{S}_{t_1,t_2} - A_{t_1,t_2}\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M} \kappa_{m,d}\Gamma_1|t_2 - t_1|^{\beta_1-\alpha} + \kappa_{m,d}\Gamma_2|t_2 - t_1|^{\beta_2}.$$

Under (1.12), we can replace $\kappa_{m,d}\Gamma_1$ by Γ_1.

Finally, let us prove the uniqueness of $\mathcal{A}$. Let $(\tilde{\mathcal{A}}_t)_{t \in [0,T]}$ be another adapted process satisfying $\tilde{\mathcal{A}}_0 = 0$, (1.10) and (1.11). It suffices to show $\mathcal{A}_T = \tilde{\mathcal{A}}_T$ almost surely. Let π_n be the nth dyadic partition of $[0, T]$. By Remark 2.5, we have

$$\|\mathcal{A}_T - \tilde{\mathcal{A}}_T\|_{L_m(\mathbb{P})} \leq \|\mathcal{A}_T - A_T^{\pi^n}\|_{L_m(\mathbb{P})} + \|A_T^{\pi^n} - \tilde{\mathcal{A}}_T\|_{L_m(\mathbb{P})} \lesssim 2^{-n\varepsilon}T^{\varepsilon}.$$

Since $n \in \mathbb{N}$ is arbitrary, we must have $\mathcal{A}_T = \tilde{\mathcal{A}}_T$ almost surely. $\qquad\square$

As in [2, Theorem 4.1] of Athreya et al., we will give an extension of Theorem 1.1 that allows singularity at $t = 0$, which will be needed in Section 4.

Corollary 2.7. *Suppose that we have a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ and a family of $\mathbb{R}^d$-valued random variables $(A_{s,t})_{0 \leq s \leq t \leq T}$, such that $A_{s,s} = 0$ for every $s \in [0, T]$ and such that $A_{s,t}$ is $\mathcal{F}_t$-measurable. Furthermore, suppose that there exist constants*

$$m \in [2, \infty), \quad \Gamma_1, \Gamma_2, \Gamma_3, M \in [0, \infty), \quad \alpha, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2 \in [0, \infty),$$

such that the following conditions are satisfied.

$\circ$ *For every $0 \leq t_0 < t_1 < t_2 < t_3 \leq T$, we have*

$$\|\mathbb{E}[\delta A_{t_1,t_2,t_3}|\mathcal{F}_{t_0}]\|_{L_m(\mathbb{P})} \leq \Gamma_1 t_1^{-\gamma_1}(t_1 - t_0)^{-\alpha}(t_3 - t_1)^{\beta_1}, \tag{2.10}$$

$$\|\delta A_{t_0,t_1,t_2}\|_{L_m(\mathbb{P})} \leq \Gamma_2 t_0^{-\gamma_2}(t_2 - t_0)^{\beta_2}, \tag{2.11}$$

$$\|\delta A_{t_0,t_1,t_2}\|_{L_m(\mathbb{P})} \leq \Gamma_3(t_2 - t_0)^{\beta_3}, \tag{2.12}$$

where $M(t_3 - t_1) \leq t_1 - t_0$ is assumed for (2.10) and $t_0 > 0$ is assumed for (2.11).

$\circ$ *We have*

$$\beta_1 > 1, \quad \beta_2 > \frac{1}{2}, \quad \beta_1 - \alpha > \frac{1}{2}, \quad \gamma_1, \gamma_2 < \frac{1}{2}, \quad \beta_3 > 0. \tag{2.13}$$

Then, there exists a unique, up to modifications, $\mathbb{R}^d$-valued stochastic process $(\mathcal{A}_t)_{t \in [0,T]}$ with the following properties.

○ $\mathcal{A}_0 = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$-measurable and $\mathcal{A}_t$ belongs to $L_m(\mathbb{P})$.
○ There exist nonnegative constants $C_1, \ldots, C_6$, such that

$$\|\mathbb{E}[\mathcal{A}_{t_2} - \mathcal{A}_{t_1} - A_{t_1, t_2} | \mathcal{F}_{t_0}]\|_{L_m(\mathbb{P})} \leq C_1 t_1^{-\gamma_1} |t_1 - t_0|^{-\alpha} |t_2 - t_1|^{\beta_1}, \tag{2.14}$$

$$\|\mathcal{A}_{t_2} - \mathcal{A}_{t_1} - A_{t_1, t_2}\|_{L_m(\mathbb{P})} \leq C_2 t_1^{-\gamma_1} |t_2 - t_1|^{\beta_1 - \alpha} + C_3 t_1^{-\gamma_2} |t_2 - t_1|^{\beta_2}, \tag{2.15}$$

$$\|\mathcal{A}_{t_2} - \mathcal{A}_{t_1} - A_{t_1, t_2}\|_{L_m(\mathbb{P})} \leq C_4 |t_2 - t_1|^{\beta_1 - \alpha_1 - \gamma_1} + C_5 |t_2 - t_1|^{\beta_2 - \gamma_2} + C_6 |t_2 - t_1|^{\beta_3}, \tag{2.16}$$

where $t_2 - t_1 \leq M^{-1}(t_1 - t_0)$ is assumed for the inequality (2.14) and $t_1 > 0$ is assumed for the inequality (2.15).

In fact, we can choose $C_1, \ldots, C_6$ so that

$$C_1 \lesssim_{\beta_1} \Gamma_1, \quad C_2 \lesssim_{\alpha, \beta_1, \beta_2, M} \kappa_{m,d} \Gamma_1, \quad C_3 \lesssim_{\alpha, \beta_1, \beta_2, M} \kappa_{m,d} \Gamma_2,$$
$$C_4 \lesssim_{\alpha, \beta_1, \gamma_1, M} \kappa_{m,d} \Gamma_1, \quad C_5 \lesssim_{\beta_2, \gamma_2, M} \kappa_{m,d} \Gamma_2, \quad C_6 \lesssim_{\beta_3, M} \kappa_{m,d} \Gamma_3.$$

Furthermore, for $\tau \in [0, T]$, if we set

$$A_\tau^\pi := \sum_{[s,t] \in \pi} A_{s,t}, \quad \text{where } \pi \text{ is a partition of } [0, \tau],$$

then the family $(A_\tau^\pi)_\pi$ converges to $\mathcal{A}_\tau$ in $L_m(\mathbb{P})$ as $|\pi| \to 0$.

The proof is given in Appendix A.

3. Integration along fractional Brownian motions

The goal of this section is to prove the convergence of Itô and Stratonovich approximations of

$$\int_0^t f(B_s) \, dB_s \quad \text{and} \quad \int_0^t f(B_s) \circ dB_s$$

along a multidimensional fractional Brownian motion B with Hurst parameter H, using Theorem 1.1. For Itô's case, we let $H \in (\frac{1}{2}, 1)$, and for Stratonovich's case, we let $H \in (\frac{1}{6}, \frac{1}{2})$.

Definition 3.1. Let $(\mathcal{F})_{t \in \mathbb{R}}$ be a filtration. We say that a process B is an $(\mathcal{F}_t)$-*fractional Brownian motion with Hurst parameter* $H \in (0, 1)$ if

○ a two-sided d-dimensional $(\mathcal{F}_t)$-Brownian motion $(W_t)_{t \in \mathbb{R}}$ is given;
○ a random variable $B(0)$ is a (not necessarily centered) $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued Gaussian random variable and is independent of $(W_t)_{t \in \mathbb{R}}$;
○ we set

$$K(t, s) := K_H(t, s) := [(t - s)_+^{H - \frac{1}{2}} - (-s)_+^{H - \frac{1}{2}}],$$

then we have the *Mandelbrot–Van Ness representation* ([31])

$$B_t = B(0) + \int_{\mathbb{R}} K_H(t, s) \, dW_s. \tag{3.1}$$

If B has the representation (3.1), then

$$\mathbb{E}[(B_t^i - B_s^i)(B_t^j - B_s^j)] = \delta_{ij} c_H |t - s|^{2H}, \quad c_H := \frac{3/2 - H}{2H} \mathcal{B}(2 - 2H, H + 1/2),$$

where we write $B = (B^i)_{i=1}^d$ in components and $\mathcal{B}$ is the Beta function

$$\mathcal{B}(\alpha, \beta) := \int_0^1 t^{\alpha - 1} (1 - t)^{\beta - 1} \, dt.$$

Regarding the expression of the constant c_H, see [38, Appendix B]. In particular, we have

$$\mathbb{E}[(B_s^i - B(0)^i)(B_t^i - B(0)^i)] = \frac{c_H}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \tag{3.2}$$

In this section, we always write B for an $(\mathcal{F}_t)$-fractional Brownian motion. An advantage of the representation (3.1) is that given $v < s$, we have the decomposition

$$B_s - B(0) = \int_{-\infty}^{v} K(s, r)\, \mathrm{d}W_r + \int_{v}^{s} K(s, r)\, \mathrm{d}W_r,$$

where the second term $\int_{v}^{s} K(s, r)\, \mathrm{d}W_r$ is independent of $\mathcal{F}_v$. Later, we will need to estimate the correlation of

$$\int_{v}^{s} K(s, r)\, \mathrm{d}W_r, \quad s > v.$$

We note that for $s \leq t$

$$\mathbb{E}[\int_{v}^{s} K(s, r)\, \mathrm{d}W_r^i \int_{v}^{t} K(t, r)\, \mathrm{d}W_r^j] = \delta_{ij} \int_{v}^{s} K(s, r)K(t, r)\, \mathrm{d}r.$$

Lemma 3.2. *Let $H \neq \frac{1}{2}$. Let $0 \leq v < s \leq t$ be such that $t - s \leq s - v$. Then,*

$$\int_{v}^{s} K(s, r)K(t, r)\, \mathrm{d}r = \frac{1}{2H}(s - v)^{2H} + \frac{1}{2}(s - v)^{2H-1}(t - s) - \frac{c_H}{2}(t - s)^{2H} + g_H(v, s, t),$$

where we have

$$|g_H(v, s, t)| \lesssim_H (s - v)^{2H-2}(t - s)^2$$

uniformly over such v, s, t.

Proof. See Appendix A.   □

We apply Theorem 1.1 to construct a stochastic integral

$$\int_{0}^{T} f(B_s)\, \mathrm{d}B_s, \quad H \in (1/2, 1)$$

as the limit of Riemann type approximations. An advantage of the stochastic sewing lemma is that we do not need any regularity of f. We denote by $L_\infty(\mathbb{R}^d, \mathbb{R}^d)$ the space of bounded measurable maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. We write

$$x \cdot y := \sum_{i=1}^{d} x^i y^i, \quad x = (x^i)_{i=1}^{d}, \quad y = (y^i)_{i=1}^{d}$$

for the inner product of $\mathbb{R}^d$.

Proposition 3.3. *Let $H \in (1/2, 1)$ and $f \in L_\infty(\mathbb{R}^d, \mathbb{R}^d)$. Then, for any $\tau \in [0, T]$ and $m \in [2, \infty)$, the sequence*

$$\sum_{[s,t] \in \pi} f(B_s) \cdot (B_t - B_s), \quad \text{where } \pi \text{ is a partition of } [0, \tau],$$

converges in $L_m(\mathbb{P})$ for every $m < \infty$ as $|\pi| \to 0$. Furthermore, if we denote the limit by $\int_0^\tau f(B_r)\,\mathrm{d}B_r$ and if we write

$$\int_s^t f(B_r)\,\mathrm{d}B_r := \int_0^t f(B_r)\,\mathrm{d}B_r - \int_0^s f(B_r)\,\mathrm{d}B_r,$$

then for every $0 \le s < t \le T$,

$$\|\int_s^t f(B_r)\,\mathrm{d}B_r\|_{L_m(\mathbb{P})} \lesssim_{d,H,m} \|f\|_{L_\infty(\mathbb{R}^d)}|t-s|^H.$$

Remark 3.4. We can replace $f(B_s)$ by $f(B_u)$ for any $u \in [s,t]$. It is well-known that the sums converge to the Young integral if $f \in C^\gamma(\mathbb{R})$ with $\gamma > H^{-1}(1-H)$. Yaskov [40, Theorem 3.7] proves that the sums converge in some $L_p(\mathbb{P})$-space if f is of bounded variation.

Proof. We will not write down dependence on d, H, and m. The filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ is generated by the Brownian motion W appearing in the Mandelbrot–Van Ness representation (3.1). We will apply Theorem 1.1 with $A_{s,t} := f(B_s) \cdot (B_t - B_s)$. Let $m \ge 2$. We have

$$\|A_{s,t}\|_{L_m(\mathbb{P})} \lesssim \|f\|_{L_\infty}|t-s|^H.$$

To estimate conditional expectations, let $0 \le v < s < t$ be such that $t - s \le s - v$ and set

$$Y_s := \int_{-\infty}^v K(s,r)\,\mathrm{d}W_r, \quad \tilde{B}_s := \int_v^s K(s,r)\,\mathrm{d}W_r.$$

We write $y_s := Y_s$, if conditioned under $\mathcal{F}_v$. Namely, we write, for instance

$$\mathbb{E}[g(y_s, \tilde{B}_s)] := \mathbb{E}[g(Y_s, \tilde{B}_s)|\mathcal{F}_v] = \mathbb{E}[g(y, \tilde{B}_s)]|_{y=Y_s}.$$

We are going to compute $\mathbb{E}[A_{s,t}|\mathcal{F}_v]$. Conditionally on $\mathcal{F}_v$, we have the Wiener chaos expansion [34, Theorem 1.1.1]

$$f(B_s) = f(y_s + \tilde{B}_s) = a_0(s) + \sum_{i=1}^d a_i(s)\tilde{B}_s^i + \tilde{B}_s^\perp,$$

where $\tilde{B}_s^\perp$ is orthogonal in $L_2(\mathbb{P})$ to the subspace spanned by the constant 1 and

$$(\tilde{B}_r^i)_{i=1,\dots,d;r\ge v}.$$

Note that

$$a_0(s) = \mathbb{E}[f(y_s + \tilde{B}_s)],$$
$$a_i(s) = \mathbb{E}[(\tilde{B}_s^i)^2]^{-1}\mathbb{E}[f(y_s + \tilde{B}_s)\tilde{B}_s^i] \stackrel{\text{Lem. } 3.2}{=} 2H(s-v)^{-2H}\mathbb{E}[f(y_s + \tilde{B}_s)\tilde{B}_s^i].$$

Then, by the orthogonality of the Wiener chaos decomposition,

$$\mathbb{E}[A_{s,t}|\mathcal{F}_v] = a_0(s) \cdot Y_{s,t} + \sum_{i=1}^d a_i(s) \cdot \mathbb{E}[\tilde{B}_s^i\tilde{B}_{s,t}].$$

Hence, for $u \in (s,t)$,

$$\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v] = A_{s,u,t}^0 + \sum_{i=1}^d A_{s,u,t}^i,$$

where

$$A^0_{s,u,t} := a_0(s) \cdot Y_{s,t} - a_0(s) \cdot Y_{s,u} - a_0(u) \cdot Y_{u,t} = (a_0(s) - a_0(u)) \cdot Y_{u,t},$$
$$A^i_{s,u,t} := a_i(s) \cdot \mathbb{E}[\tilde{B}^i_s \tilde{B}_{s,t}] - a_i(s) \cdot \mathbb{E}[\tilde{B}^i_s \tilde{B}_{s,u}] - a_i(u) \cdot \mathbb{E}[\tilde{B}^i_u \tilde{B}_{u,t}]$$
$$= [a_i(s) \cdot e_i]\mathbb{E}[\tilde{B}^i_s \tilde{B}^i_{s,t}] - [a_i(s) \cdot e_i]\mathbb{E}[\tilde{B}^i_s \tilde{B}^i_{s,u}] - [a_i(u) \cdot e_i]\mathbb{E}[\tilde{B}^i_u \tilde{B}^i_{u,t}].$$

Here, e_i is the ith unit vector of $\mathbb{R}^d$. We first estimate $A^0_{s,u,t}$, for which we begin with estimating $a_0(s) - a_0(u)$. We set

$$F(m, \sigma) := \mathbb{E}[f(m + \sigma X)], \quad m \in \mathbb{R}^d, \ \sigma \in (0, \infty),$$

where X has the standard normal distribution in $\mathbb{R}^d$. Note that

$$a_0(s) = F(Y_s, (2H)^{-\frac{1}{2}}(s-v)^H),$$

and similarly for $a_0(u)$, we have

$$\partial_{m^i} F(m, \sigma) = \frac{1}{(2\pi)^{\frac{d}{2}} \sigma^{d+2}} \int_{\mathbb{R}^d} x^i e^{-\frac{|x|^2}{2\sigma^2}} f(x + m) \, dx,$$
$$\partial_\sigma F(m, \sigma) = \frac{-d}{(2\pi)^{\frac{d}{2}} \sigma^{d+1}} \int_{\mathbb{R}^d} f(m + x) e^{-\frac{|x|^2}{2\sigma^2}} \, dx + \frac{1}{(2\pi)^{\frac{d}{2}} \sigma^{d+3}} \int_{\mathbb{R}^d} |x|^2 f(m + x) e^{-\frac{|x|^2}{2\sigma^2}} \, dx.$$

Therefore,

$$|\partial_m F(m, \sigma)| + |\partial_\sigma F(m, \sigma)| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)} \sigma^{-1}.$$

This yields

$$|a_0(s) - a_0(u)| \le |F(Y_s, (2H)^{-\frac{1}{2}}(s-v)^H) - F(Y_u, (2H)^{-\frac{1}{2}}(s-v)^H)|$$
$$+ |F(Y_u, (2H)^{-\frac{1}{2}}(s-v)^H) - F(Y_u, (2H)^{-\frac{1}{2}}(u-v)^H)|$$
$$\lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H}|Y_{s,u}| + \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H}(|u-v|^H - |s-v|^H)$$
$$\lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H}|Y_{s,u}| + \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-1}(t-s).$$

Therefore,

$$|A^0_{s,u,t}| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H}|Y_{s,u}||Y_{u,t}| + \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-1}(t-s)|Y_{u,t}|. \tag{3.3}$$

The random variable $Y_{s,u}$ is Gaussian and

$$\mathbb{E}[|Y_{s,u}|^2] = d \int_{-\infty}^v (K(s,r) - K(u,r))^2 \, dr = d \int_{s-v}^\infty ((u-s+r)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}})^2 \, dr$$
$$\lesssim (u-s)^2 \int_{s-v}^\infty r^{2H-3} \, dr \lesssim (s-v)^{2H-2}(u-s)^2. \tag{3.4}$$

We have a similar estimate for $Y_{u,t}$. Therefore,

$$\|A^0_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{H-2}(t-s)^2 \quad \text{if } t-s \le v-s.$$

Now we move to estimate $A^i_{s,u,t}$. By Lemma 3.2, we have

$$\mathbb{E}[\tilde{B}^i_s \tilde{B}^i_{s,t}] = \int_v^s K(s,r)K(t,r)\,\mathrm{d}r - \int_v^s K(s,r)K(s,r)\,\mathrm{d}r = \frac{1}{2}(s-v)^{2H-1}(t-s) + O((t-s)^{2H}).$$

Therefore, if we write $a^i_i(s) := a_i(s) \cdot e_i$,

$$A^i_{s,u,t} = \frac{1}{2}\big[a^i_i(s)(s-v)^{2H-1} - a^i_i(u)(u-v)^{2H-1}\big](t-u) + O((|a^i_i(s)| + |a^i_i(u)|)|t-s|^{2H}).$$

If we set

$$G_i(m,\sigma) := \sigma^{-1}\mathbb{E}[f^i(m+\sigma X)X^i], \quad m \in \mathbb{R}^d, \ \sigma \in (0,\infty),$$

then $a^i_i(s) = G_i(Y_s, (2H)^{-\frac{1}{2}}(s-v)^H)$ and similarly for $a^i_i(u)$. Since

$$G_i(m,\sigma) = (2\pi)^{-\frac{d}{2}}\sigma^{-d-2}\int_{\mathbb{R}^d} f^i(y)(y^i - m^i)e^{-\frac{|y-m|^2}{2\sigma^2}}\,\mathrm{d}y,$$

we have

$$(2\pi)^{\frac{d}{2}}\sigma^2 \partial_{m^j} G_i(m,\sigma) = \int_{\mathbb{R}^d} f^i(m+\sigma x)[-\delta_{ij} + x^i x^j]e^{-\frac{|x|^2}{2}}\,\mathrm{d}x$$

$$(2\pi)^{\frac{d}{2}}\sigma^2 \partial_\sigma G_i(m,\sigma) = \int_{\mathbb{R}^d} f^i(m+\sigma x)x^i[-(d+2) + |x|^2]e^{-\frac{|x|^2}{2}}\,\mathrm{d}x.$$

Therefore,

$$|G_i(m,\sigma)| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}\sigma^{-1},$$

$$|\partial_m G_i(m,\sigma)| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}\sigma^{-2}, \quad |\partial_\sigma G_i(m,\sigma)| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}\sigma^{-2}$$

and thus

$$|a^i_i(s)| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H},$$

$$|a^i_i(s) - a^i_i(u)| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-2H}\big(|Y_{s,u}| + (u-v)^H - (s-v)^H\big)$$

$$\lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-2H}\big(|Y_{s,u}| + (s-v)^{H-1}(u-s)\big).$$

This yields

$$|A^i_{s,u,t}| \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}\big[(s-v)^{-1}(t-s)|y_{s,u}| + (s-v)^{H-2}(t-s)^2$$

$$+ (s-v)^{H-2}(t-s)^2 + (s-v)^{-H}(t-s)^{2H}\big]$$

and

$$\|A^i_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}\big[(s-v)^{H-2}(t-s)^2 + (s-v)^{-H}(t-s)^{2H}\big] \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H}(t-s)^{2H} \tag{3.5}$$

if $t - s \leq s - v$.

Therefore, by (3.3) and (3.5),

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v]\|_{L_m(\mathbb{P})} \lesssim \|f\|_{L_\infty(\mathbb{R}^d)}(s-v)^{-H}(t-s)^{2H}$$

if $t - s \leq s - v$. Hence, $(A_{s,t})$ satisfies the assumption of Theorem 1.1 with

$$\alpha = H, \quad \beta_1 = 2H, \quad \beta_2 = H, \quad M = 1. \qquad \square$$

Next, we consider the case $H \in (\frac{1}{6}, \frac{1}{2})$. The following result reproduces [35, Theorem 3.5], with a more elementary proof and with improvement of the regularity of f. More precisely, the cited result requires $f \in C^6$ while here $f \in C^\gamma$ with $\gamma > \frac{1}{2H} - 1$ is sufficient and thus, in particular, $f \in C^2$ works for all $H \in (\frac{1}{6}, \frac{1}{2})$. We denote by $C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$ the space of γ-Hölder maps from $\mathbb{R}^d$ to $\mathbb{R}^d$, with the norm

$$\|f\|_{C^\gamma} := \|f\|_{L_\infty(\mathbb{R}^d)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$$

if $\gamma \in (0, 1)$ and

$$\|f\|_{C^\gamma} := \|f\|_{L_\infty(\mathbb{R}^d)} + \sum_{i=1}^{d} \|\partial_i f\|_{C^{\gamma-1}}$$

if $\gamma \in (1, 2)$.

Proposition 3.5. *Let $H \in (\frac{1}{6}, \frac{1}{2})$, $\gamma > \frac{1}{2H} - 1$ and $f \in C^\gamma(\mathbb{R}^d, \mathbb{R}^d)$. If $H \leq \frac{1}{4}$ and $d > 1$, assume furthermore that*

$$\partial_i f^j = \partial_j f^i, \quad \forall i, j \in \{1, \dots, d\}. \tag{3.6}$$

Then, for every $m \in [2, \infty)$ and $\tau \in [0, T]$, the family of Stratonovich approximations

$$\sum_{[s,t] \in \pi} \frac{f(B_s) + f(B_t)}{2} \cdot B_{s,t}, \quad \text{where } \pi \text{ is a partition of } [0, \tau],$$

converges in $L_m(\mathbb{P})$ as $|\pi| \to 0$. Moreover, if we denote the limit by $\int_0^\tau f(B_r) \circ \mathrm{d}B_r$ and if we write

$$\int_s^t f(B_r) \circ \mathrm{d}B_r := \int_0^t f(B_r) \circ \mathrm{d}B_r - \int_0^s f(B_r) \circ \mathrm{d}B_r,$$

then for every $0 \leq s < t \leq T$, we have

$$\left\| \int_s^t f(B_r) \circ \mathrm{d}B_r - \frac{f(B_s) + f(B_t)}{2} \cdot B_{s,t} \right\|_{L_m(\mathbb{P})} \lesssim_{d,H,m,\gamma} \|f\|_{C^\gamma} |t - s|^{(\gamma+1)H}.$$

Proof. We will not write down dependence on d, H, m, and γ. The filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ is generated by the Brownian motion W appearing in the Mandelbrot–Van Ness representation (3.1). We can assume

$$\gamma < \mathbb{1}_{\{H > \frac{1}{4}\}} + 2\mathbb{1}_{\{H \leq \frac{1}{4}\}}.$$

We will apply Theorem 1.1 with

$$A_{s,t} := (f(B_s) + f(B_t)) \cdot B_{s,t}.$$

We first claim

$$\|\delta A_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\gamma} |t - s|^{(\gamma+1)H}. \tag{3.7}$$

Observe

$$\delta A_{s,u,t} = f(B)_{u,t} \cdot B_{s,u} + f(B)_{u,s} \cdot B_{u,t}.$$

If $H > \frac{1}{4}$, the claim (3.7) follows from the estimates

$$|f(B)_{u,t}| \leq \|f\|_{C^\gamma} |B_{u,t}|^\gamma, \quad |f(B)_{u,s}| \leq \|f\|_{C^\gamma} |B_{u,s}|^\gamma.$$

If $H \leq \frac{1}{4}$, then $\gamma > 1$, and we have

$$\delta A_{s,u,t} = \left(f(B)_{u,t} - \sum_{j=1}^{d} \partial_j f(B_u) B_{u,t}^j \right) \cdot B_{s,u} + \left(f(B)_{u,s} - \sum_{j=1}^{d} \partial_j f(B_u) B_{u,s}^j \right) \cdot B_{u,t},$$

where (3.6) is used. Then, the claim (3.7) follows, again, from the Hölder estimate of f. Note that the condition $\gamma > \frac{1}{2H} - 1$ is equivalent to $(\gamma + 1)H > \frac{1}{2}$.

The rest of the proof consists of estimating the conditional expectation $\mathbb{E}[\delta A_{s,u,t} | \mathcal{F}_v]$. Let $t - s \leq s - v$. We will use the same notation as in the proof of Proposition 3.3. We have

$$\mathbb{E}[\delta A_{s,u,t} | \mathcal{F}_v] = D_{s,u,t}^0 + \sum_{i=1}^{d} D_{s,u,t}^i,$$

where

$$
\begin{aligned}
D_{s,u,t}^0 &:= (a_0(s) + a_0(t)) \cdot Y_{s,t} - (a_0(s) + a_0(u)) \cdot Y_{s,u} - (a_0(u) + a_0(t)) \cdot Y_{u,t} \\
&= (a_0(t) - a_0(u)) \cdot Y_{s,u} + (a_0(s) - a_0(u)) \cdot Y_{u,t},
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
D_{s,u,t}^i &:= \mathbb{E}[(a_i^i(s) \tilde{B}_s^i + a_i^i(t) \tilde{B}_t^i) \tilde{B}_{s,t}^i | \mathcal{F}_v] \\
&\quad - \mathbb{E}[(a_i^i(s) \tilde{B}_s^i + a_i^i(u) \tilde{B}_u^i) \tilde{B}_{s,u}^i | \mathcal{F}_v] - \mathbb{E}[(a_i^i(u) \tilde{B}_u^i + a_i^i(t) \tilde{B}_t^i) \tilde{B}_{u,t}^i | \mathcal{F}_v].
\end{aligned}
$$

We first estimate $D_{s,u,t}^0$. Suppose that $H > \frac{1}{4}$. Recall

$$
\begin{aligned}
\partial_{m^i} F(m, \sigma) &= \frac{1}{(2\pi)^{\frac{d}{2}} \sigma^{d+2}} \int_{\mathbb{R}^d} x^i e^{-\frac{|x|^2}{2\sigma^2}} [f(x + m) - f(m)] \, dx, \\
\partial_\sigma F(m, \sigma) &= \frac{-d}{(2\pi)^{\frac{d}{2}} \sigma^{d+1}} \int_{\mathbb{R}^d} [f(m + x) - f(m)] e^{-\frac{|x|^2}{2\sigma^2}} \, dx \\
&\quad + \frac{1}{(2\pi)^{\frac{d}{2}} \sigma^{d+3}} \int_{\mathbb{R}^d} |x|^2 [f(m + x) - f(m)] e^{-\frac{|x|^2}{2\sigma^2}} \, dx.
\end{aligned}
$$

Therefore,

$$|\partial_{m^i} F(m, \sigma)| + |\partial_\sigma F(m, \sigma)| \lesssim \|f\|_{C^\gamma} \sigma^{\gamma-1}.$$

This yields

$$|D_{s,u,t}^0| \lesssim \|f\|_{C^\gamma} \left[(s - v)^{(\gamma-1)H} |Y_{s,u}| |Y_{u,t}| + (s - v)^{\gamma H - 1} (t - s) (|Y_{s,u}| + |Y_{u,t}|) \right]. \tag{3.9}$$

Therefore, by (3.4),

$$\|D_{s,u,t}^0\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\gamma} (s - v)^{(\gamma+1)H-2} (t - s)^2. \tag{3.10}$$

Now suppose that $H \leq \frac{1}{4}$. To simplify notation, we write $I(m, \sigma) := F(m, (2H)^{-\frac{1}{2}}\sigma)$. Since (3.6) gives $\partial_{m^i} I^j = \partial_{m^j} I^i$ for every i, j, we have

$$
\begin{aligned}
D^0_{s,u,t} &= [I(Y_s, (u-v)^H) - I(Y_u, (u-v)^H) - \sum_{i=1}^d \partial_{m^i} I(Y_u, (u-v)^H)Y^i_{u,s}] \cdot Y_{u,t} \\
&+ [I(Y_t, (u-v)^H) - I(Y_u, (u-v)^H) - \sum_{i=1}^d \partial_{m^i} I(Y_u, (u-v)^H)Y^i_{u,t}] \cdot Y_{s,u} \\
&+ [I(Y_s, (s-v)^H) - I(Y_s, (u-v)^H)] \cdot Y_{u,t} + [I(Y_t, (t-v)^H) - I(Y_t, (u-v)^H)] \cdot Y_{s,u}.
\end{aligned}
$$

Since

$$
\partial_{m^i} \partial_{m^j} F(m, \sigma) = \frac{1}{(2\pi)^{\frac{d}{2}} \sigma^{d+2}} \int_{\mathbb{R}^d} x^i e^{-\frac{|x|^2}{2\sigma^2}} [\partial_j f(x+m) - \partial_j f(m)] \, \mathrm{d}x,
$$

we have

$$
|I(Y_s, (u-v)^H) - I(Y_u, (u-v)^H) - \sum_{i=1}^d \partial_{m^i} I(Y_u, (u-v)^H)Y^i_{u,s}| \lesssim \|f\|_{C^\gamma} (s-v)^{(\gamma-2)H} |Y_{s,u}|^2.
$$

Notice

$$
\begin{aligned}
\partial_\sigma F(m, \sigma) &= \frac{-d}{(2\pi)^{\frac{d}{2}} \sigma^{d+1}} \int_{\mathbb{R}^d} [f(m+x) - f(m) - \sum_{i=1}^d \partial_i f(m)x^i] e^{-\frac{|x|^2}{2\sigma^2}} \, \mathrm{d}x \\
&+ \frac{1}{(2\pi)^{\frac{d}{2}} \sigma^{d+3}} \int_{\mathbb{R}^d} |x|^2 [f(m+x) - f(m) - \sum_{i=1}^d \partial_i f(m)x^i] e^{-\frac{|x|^2}{2\sigma^2}} \, \mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
|\partial_\sigma F(m, \sigma)| \lesssim \|f\|_{C^\gamma} \sigma^{\gamma-1}.
$$

This yields

$$
|I(Y_s, (s-v)^H) - I(Y_s, (u-v)^H)| \lesssim \|f\|_{C^\gamma} (s-v)^{\gamma H-1}(t-s).
$$

Hence, we obtain the estimate (3.10) when $H \leq \frac{1}{4}$.

We move to estimate $D^i_{s,u,t}$. By using the identity,

$$
\mathbb{E}[(\tilde{B}^i_a + \tilde{B}^i_b)\tilde{B}^i_{a,b}] = \mathbb{E}[(\tilde{B}^i_b)^2] - \mathbb{E}[(\tilde{B}^i_a)^2],
$$

we obtain

$$
\begin{aligned}
D^i_{s,u,t} &= (a^i_i(t) - a^i_i(u))\mathbb{E}[\tilde{B}^i_t \tilde{B}^i_{s,t}] + (a^i_i(s) - a^i_i(u))\mathbb{E}[\tilde{B}^i_s \tilde{B}^i_{s,t}] \\
&- (a^i_i(s) - a^i_i(u))\mathbb{E}[\tilde{B}^i_s \tilde{B}^i_{s,u}] - (a^i_i(t) - a^i_i(u))\mathbb{E}[\tilde{B}^i_t \tilde{B}^i_{u,t}].
\end{aligned} \tag{3.11}
$$

Since the other terms can be estimated similarly, we only estimate $(a^i_i(t) - a^i_i(u))\mathbb{E}[\tilde{B}^i_t \tilde{B}^i_{s,t}]$. By Lemma 3.2,

$$
|\mathbb{E}[\tilde{B}_t \tilde{B}_{s,t}]| \lesssim |t-s|^{2H}.
$$

Now we estimate $|a_i^i(t) - a_i^i(u)|$. Recall $a_i^i(s) = G_i(Y_s, (2H)^{-\frac{1}{2}}(s-v)^H)$,

$$(2\pi)^{\frac{d}{2}}\sigma^2\partial_{m^j}G_i(m,\sigma) = -\delta_{ij}\int_{\mathbb{R}^d}[f^i(m+\sigma x) - f^i(m)]e^{-\frac{|x|^2}{2}}\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^d}[f^i(m+\sigma x) - f^i(m)]x^i x^j e^{-\frac{|x|^2}{2}}\,\mathrm{d}x,$$
$$(2\pi)^{\frac{d}{2}}\sigma^2\partial_\sigma G_i(m,\sigma) = -(d+2)\int_{\mathbb{R}^d}[f^i(m+\sigma x) - f^i(m)]x^i e^{-\frac{|x|^2}{2}}\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^d}[f^i(m+\sigma x) - f^i(m)]x^i|x|^2 e^{-\frac{|x|^2}{2}}\,\mathrm{d}x.$$

If $H \leq \frac{1}{4}$, we can replace $f^i(m+\sigma x) - f^i(m)$ by

$$f^i(m+\sigma x) - f^i(m) - \sum_{k=1}^d \partial_k f^i(m)\sigma x^k.$$

Therefore,

$$|\partial_{m^j}G_i(m,\sigma)| + |\partial_\sigma G_i(m,\sigma)| \lesssim \|f\|_{C^\gamma}\sigma^{\gamma-2}.$$

This yields

$$|a_i^i(t) - a_i^i(u)| \lesssim \|f\|_{C^\gamma}(s-v)^{(\gamma-2)H}(|Y_{u,t}| + (s-v)^{H-1}(t-s))$$

and hence

$$\|a_i^i(t) - a_i^i(u)\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\gamma}(s-v)^{(\gamma-1)H-1}(t-s).$$

Therefore, we obtain

$$\|D_{s,u,t}^i\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\gamma}(s-v)^{(\gamma-1)H-1}(t-s)^{1+2H}. \tag{3.12}$$

By (3.10) and (3.12), we conclude

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v]\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\gamma}[(s-v)^{(1+\gamma)H-2}(t-s)^2 + (s-v)^{(\gamma-1)H-1}(t-s)^{1+2H}]$$
$$\lesssim \|f\|_{C^\gamma}(s-v)^{(\gamma-1)H-1}(t-s)^{1+2H}$$

if $t - s \leq s - v$. Therefore, we can apply Theorem 1.1 with

$$\alpha = 1 - (\gamma-1)H, \quad \beta_1 = 1 + 2H, \quad \beta_2 = (\gamma+1)H, \quad M = 1. \qquad \square$$

4. Local times of fractional Brownian motions

In this section, we set $d = 1$, and we are interested in local times of fractional Brownian motions. In case of a Brownian motion W, or, more generally, semimartingales as discussed in Łochowski et al. [28], there are three major methods to construct its local time.

1. *Via occupation measure.* The local time $L_T^W(\cdot)$ of W is defined as the density with respect to the Lebesgue measure of

$$A \mapsto \int_0^T \mathbb{1}_A(W_s)\,\mathrm{d}s.$$

Heuristically,

$$L_T^W(a) = \int_0^T \delta(W_s - a)\, ds,$$

where δ is Dirac's delta function concentrated at 0.

2. *Via discretization.* The local time $L_T^W(a)$ is defined by

$$L_T^W(a) := \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} |W_t - a| \mathbb{1}_{(\min\{W_s, W_t\}, \max\{W_s, W_t\})}(a),$$

where π is a partition of $[0, T]$ and the convergence is in probability. This representation of the local time is often used in the pathwise stochastic calculus (see Wuermli [39], Perkowski and Prömel [36], Davis et al. [12], Cont and Perkowski [9], and Kim [23]).

3. *Via numbers of interval crossing.* For $n \in \mathbb{N}$, we set $\tau_0^n := 0$ and inductively

$$\tau_k^n := \inf\{t > \tau_{k-1}^n \mid W_t \in 2^{-n}\mathbb{Z} \setminus \{W_{\tau_{k-1}^n}\}\}.$$

Then, the local time $L_T^W(a)$ is defined by

$$L_T^W(a) := \lim_{n \to \infty} 2^{-n} \sum_{k \in \mathbb{Z}} \mathbb{1}_{[\frac{k}{2^n}, \frac{k+1}{2^n})}(a) \#\{\ell \in \mathbb{N}_0 \mid \{W_{\tau_\ell^n}, W_{\tau_{\ell+1}^n}\} = \{k2^{-n}, (k+1)2^{-n}\}, \tau_{k+1}^n \le T\},$$

where the convergence holds almost surely. See the monograph [32] for the Brownian motion. For general semimartingales, see El Karoui [22], Lemieux [27], and [28].

In case of a fractional Brownian motion, the construction of its local time via the method 1 is well-known, see the survey [15] and the monograph [5]. In contrast, there are few results in the literature in which the local time of a fractional Brownian motion is constructed via the method 2 or 3. Because of this, the construction of the local time via the method 3 was stated as a conjecture in [9]. We are aware of only two results in this direction. One is the work [4] of Azaïs, who proves Corollary 4.8 below. The other is the work [33] of Mukeru, who proves that the local time $L_T(a)$ of a fractional Brownian motion with Hurst parameter less than $\frac{1}{2}$ is represented as

$$\lim_{n \to \infty} 2^{n(2H-1)} \sum_{k \le \lfloor T 2^n \rfloor} 2|B_{k2^{-n}} - a| \mathbb{1}_{\{(B_{k2^{-n}} - a)(B_{(k-1)2^{-n}} - a) < 0\}} \quad \text{almost surely.}$$

Our goal in this section is to give new representations of the local times of fractional Brownian motions in the spirit of the method (b) along deterministic partitions. The representation in Corollary 4.9 is compatible with [9, Definition 3.1].

Theorem 4.1. *Let B be an $(\mathcal{F}_t)$-fractional Brownian motion with Hurst parameter $H \ne \frac{1}{2}$, in the sense of Definition 3.1. Let $m \in [2, \infty)$, $\gamma \in [0, \infty)$, and $a \in \mathbb{R}$. If $H > \frac{1}{2}$, assume that m satisfies*

$$\frac{1}{m} > 1 - \frac{1}{2H}. \tag{4.1}$$

Then, as $|\pi| \to 0$, where π is a partition of $[0, T]$, the family of

$$\sum_{[s,t] \in \pi, B_s < a < B_t} (t - s)^{1-(1+\gamma)H} |B_t - B_s|^\gamma$$

converges in $L_m(\mathbb{P})$ to $\mathfrak{c}_{H,\gamma} L_T(a)$, where $L_T(a)$ is the local time of B at level a and

$$\mathfrak{c}_{H,\gamma} := c_H^{\frac{1+\gamma}{2}} \int_0^\infty x^{\gamma+1} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, \mathrm{d}x.$$

Furthermore, we have

$$\lim_{|\pi| \to 0} \mathbb{E}\left[\int_{\mathbb{R}} \left| \mathfrak{c}_{H,\gamma} L_T(x) - \sum_{[s,t] \in \pi, B_s < x < B_t} (t-s)^{1-(1+\gamma)H} |B_t - B_s|^\gamma \right|^m \mathrm{d}x \right] = 0. \qquad (4.2)$$

Remark 4.2. A similar result holds for a Brownian motion ($H = \frac{1}{2}$). However, we omit a proof since it is easier but requires a special treatment.

Remark 4.3. We can similarly prove

$$\lim_{|\pi| \to 0} \sum_{[s,t] \in \pi, B_s > a > B_t} (t-s)^{1-(1+\gamma)H} |B_t - B_s|^\gamma = \mathfrak{c}_{H,\gamma} L_T(a).$$

Consequently,

$$\lim_{|\pi| \to 0} \sum_{[s,t] \in \pi, \min\{B_s, B_t\} < a < \max\{B_s, B_t\}} (t-s)^{1-(1+\gamma)H} |B_t - B_s|^\gamma = 2\mathfrak{c}_{H,\gamma} L_T(a),$$

where the convergence is in $L_m(\mathbb{P})$.

Proof. We will not write down dependence on H, γ, and m. Without loss of generality, we can assume $\mathbb{E}[B(0)] = 0$. To apply Theorem 1.1 (for $H < \frac{1}{2}$) or Corollary 2.7 (for $H > \frac{1}{2}$), respectively, we set

$$A_{s,t} := A_{s,t}(a) := (t-s)^{1-(1+\gamma)H} |B_t - B_s|^\gamma \mathbb{1}_{\{B_s < a < B_t\}}.$$

If we set $\mathcal{A}_t := \mathfrak{c}_{H,\gamma} L_t$, it suffices to show that the estimates (1.10) and (1.11) are satisfied for $H < \frac{1}{2}$, and that the estimates (2.14), (2.15), and (2.16) are satisfied for $H > \frac{1}{2}$. Since the proof is rather long, we split the main arguments into three lemmas.

Lemma 4.4. *We have*

$$\|A_{s,t}\|_{L_m(\mathbb{P})} \lesssim_T \begin{cases} |t-s|^{1-H} f_H(a), & \text{for all } H \in (0,1), \\ (\mathbb{E}[B(0)^2] + s^{2H})^{-\frac{1}{2m}} |t-s|^{1-H+\frac{H}{m}} f_H(a), & \text{if } H > \frac{1}{2}, \end{cases}$$

where in either case there exists a constant $c = c(H) > 0$, such that

$$|f_H(a)| \leq \exp\left(- \frac{ca^2}{m(\mathbb{E}[B(0)^2] + T^{2H})} \right), \quad \forall a \in \mathbb{R}. \qquad (4.3)$$

Remark 4.5. Due to (4.1), the exponent $1 + \frac{H}{m} - H$ is greater than $\frac{1}{2}$.

Proof. We have

$$\|A_{s,t}\|_{L_m(\mathbb{P})} \leq (t-s)^{1-(1+\gamma)H} \||B_t - B_s|^{2\gamma}\|_{L_m(\mathbb{P})}^{\frac{1}{2}} \mathbb{P}(B_s < a < B_t)^{\frac{1}{2m}}$$

$$\lesssim |t-s|^{1-H} e^{-\frac{a^2}{4mc_H(\mathbb{E}[B(0)^2]+T^{2H})}}.$$

Now we consider the case $H > \frac{1}{2}$.

Set

$$\chi_0 := (\mathbb{E}[B(0)^2] + s^{2H})^{\frac{1}{2}}, \quad \chi_1 := \frac{1}{2\chi_0}(t^{2H} - s^{2H} - |t-s|^{2H}), \quad \chi_2^2 := |t-s|^{2H} - \chi_1^2.$$

Since $H > \frac{1}{2}$, $\chi_1 \geq 0$, and

$$|\chi_1| \lesssim \chi_0^{-1} T^{2H-1} |t-s|, \quad 0 \leq \chi_2 \leq |t-s|^H.$$

Then, if X and Y are two independent standard normal distributions on $\mathbb{R}$, we have

$$(B_s, B_{s,t}) = \sqrt{c_H}(\chi_0 Y, \chi_1 Y + \chi_2 X) \quad \text{in law.}$$

Therefore, if we set

$$\hat{A}_{s,t} := |B_t - B_s|^\gamma \mathbb{1}_{\{B_s < a < B_t\}} \tag{4.4}$$

$$\|\hat{A}_{s,t}\|_{L_m(\mathbb{P})}^m \lesssim \chi_1^{m\gamma} \mathbb{E}[|Y|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a, a - \chi_0 Y < \chi_1 Y + \chi_2 X\}}] + \chi_2^{m\gamma} \mathbb{E}[|X|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a, a - \chi_0 Y < \chi_1 Y + \chi_2 X\}}].$$

We first estimate $\mathbb{E}[|Y|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a, a - \chi_0 Y < \chi_1 Y + \chi_2 X\}}]$. Using the estimate

$$\mathbb{P}(X > x) \lesssim e^{-\frac{x^2}{2}} \quad \text{for } x > 0,$$

we have

$$\mathbb{P}(X > \chi_2^{-1}(a - \chi_0 Y - \chi_1 Y)|Y) \lesssim e^{-\frac{1}{8}[\chi_2^{-1}(a-\chi_0 Y)]^2} + \mathbb{1}_{\{\chi_1 Y < \frac{1}{2}(a-\chi_0 Y)\}}. \tag{4.5}$$

Then,

$$\mathbb{E}[e^{-\frac{1}{8}[\chi_2^{-1}(a-\chi_0 Y)]^2}|Y|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a\}}] = \frac{1}{\sqrt{2\pi}} \frac{\chi_2}{\chi_0} \int_0^\infty \left|\frac{\chi_2 y - a}{\chi_0}\right|^{m\gamma} e^{-\frac{1}{8}|y|^2} e^{-\frac{1}{2}(\frac{\chi_2 y - a}{\chi_0})^2} \, dy$$

$$\lesssim \frac{\chi_2}{\chi_0} \int_0^\infty e^{-\frac{1}{8}|y|^2} e^{-\frac{1}{4}(\frac{\chi_2 y - a}{\chi_0})^2} \, dy \lesssim \frac{\chi_2}{\chi_0} e^{-\frac{a^2}{4(\chi_0^2 + 2\chi_2^2)}},$$

where in the third line we applied

$$\sup_{z \in \mathbb{R}} |z|^{m\gamma} e^{-\frac{|z|^2}{4}} < \infty.$$

And,

$$\mathbb{E}[|Y|^{m\gamma} \mathbb{1}_{\{\chi_1 Y < \frac{1}{2}(a-\chi_0 Y)\}}] = \int_{\frac{a}{\chi_0 + 2\chi_1} < y < \frac{a}{\chi_0}} |y|^{m\gamma} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \, dy \lesssim \frac{\chi_1}{\chi_0} e^{-\frac{a^2}{4\chi_0^2}}.$$

Therefore,

$$\mathbb{E}[|Y|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a, a - \chi_0 Y < \chi_1 Y + \chi_2 X\}}] \lesssim \frac{\chi_2}{\chi_0} e^{-\frac{a^2}{4(\chi_0^2 + 2\chi_2^2)}} + \frac{\chi_1}{\chi_0} e^{-\frac{a^2}{4\chi_0^2}},$$

We now estimate $\mathbb{E}[|X|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a, a - \chi_0 Y < \chi_1 Y + \chi_2 X\}}]$. Similarly to (4.5), we have

$$\mathbb{E}[|X|^{m\gamma} \mathbb{1}_{\{X > \chi_2^{-1}(a-\chi_0 Y - \chi_1 Y)\}}|Y] \lesssim e^{-\frac{1}{16}[\chi_2^{-1}(a-\chi_0 Y)]^2} + \mathbb{1}_{\{\chi_1 Y < \frac{1}{2}(a-\chi_0 Y)\}}$$

and similarly

$$\mathbb{E}[|X|^{m\gamma} \mathbb{1}_{\{\chi_0 Y < a, a - \chi_0 Y < \chi_1 Y + \chi_2 X\}}] \lesssim \frac{\chi_2}{\chi_0} e^{-\frac{a^2}{4(\chi_0^2 + 2\chi_2^2)}} + \frac{\chi_1}{\chi_0} e^{-\frac{a^2}{4\chi_0^2}}.$$

Therefore, we conclude

$$\|\hat{A}_{s,t}\|_{L_m(\mathbb{P})} \lesssim \left(\frac{\chi_1 + \chi_2}{\chi_0}\right)^{\frac{1}{m}} (\chi_1^\gamma + \chi_2^\gamma) e^{-\frac{a^2}{4m(\chi_0^2 + \chi_2^2)}} \lesssim_T \chi_0^{-\frac{1}{m}} |t - s|^{(\gamma + \frac{1}{m})H} e^{-\frac{a^2}{4m(\mathbb{E}[B(0)^2] + T^{2H})}},$$

which completes the proof of the lemma. □

Recall the Mandelbrot–Van Ness representation (3.1), and recall that W is an $(\mathcal{F}_t)_{t \in \mathbb{R}}$-Brownian motion.

Lemma 4.6. *Let $v < s < t$, and set*

$$Y_s := B(0) + \int_{-\infty}^{v} K(s, r) \, dW_r, \quad \sigma_s^2 := \mathbb{E}[(\int_v^s K(s, r) \, dW_r)^2] = \frac{1}{2H}|s - v|^{2H}.$$

If $\frac{t-s}{s-v}$ is sufficiently small, then

$$\mathbb{E}[A_{s,t}|\mathcal{F}_v] = \mathfrak{c}_{H,\gamma} \frac{e^{-\frac{(Y_s - a)^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s}(t - s) + R,$$

where for some $c = c(H, m) > 0$,

$$\|R\|_{L_m(\mathbb{P})} \lesssim (\mathbb{E}[B(0)^2] + s^{2H})^{-\frac{1}{2m}} e^{-c(\mathbb{E}[B(0)]^2 + T^{2H})^{-1} a^2} \left(\frac{t-s}{s-v}\right)^{\min\{1, 2H\} - \frac{H}{m}} |t - s|^{1 - H + \frac{H}{m}}.$$

Proof. As in the proof of Proposition 3.3, for $s > v$, we set

$$Y_s := B(0) + \int_{-\infty}^{v} K(s, r) \, dW_r, \quad \tilde{B}_s := \int_v^s K(s, r) \, dW_r,$$

and we write $y_s := Y_s$ under the conditioning of $\mathcal{F}_v$. Then, recalling $\hat{A}_{s,t}$ from (4.4), we have

$$\mathbb{E}[\hat{A}_{s,t}|\mathcal{F}_v] = \mathbb{E}[|y_{s,t} + \tilde{B}_{s,t}|^\gamma \mathbb{1}_{\{y_s + \tilde{B}_s < a, \ y_{s,t} + \tilde{B}_{s,t} > a - y_s - \tilde{B}_s\}}]. \tag{4.6}$$

To compute, we set

$$\sigma_{s,t}^2 := \mathbb{E}[\tilde{B}_{s,t}^2], \quad \sigma_s^2 := \mathbb{E}[\tilde{B}_s^2], \quad \rho_{s,t} := \mathbb{E}[\tilde{B}_s \tilde{B}_{s,t}].$$

By Lemma 3.2,

$$\sigma_{s,t}^2 = c_H|t - s|^{2H} + O(|s - v|^{2H-2}|t - s|^2), \quad \sigma_s^2 = \frac{1}{2H}|s - v|^{2H}$$

and

$$\rho_{s,t} = \frac{1}{2}|s - v|^{2H-1}|t - s| - \frac{c_H}{2}|t - s|^{2H} + O(|s - v|^{2H-2}|t - s|^2).$$

We have the decomposition

$$\tilde{B}_{s,t} = \sigma_s^{-2} \rho_{s,t} \tilde{B}_s + (\tilde{B}_{s,t} - \sigma_s^{-2} \rho_{s,t} \tilde{B}_s),$$

where the second term is independent of $\tilde{B}_s$. If we set

$$\kappa_{s,t}^2 := \sigma_{s,t}^2 - \sigma_s^{-2} \rho_{s,t}^2 = c_H|t - s|^{2H} + O(|s - v|^{2H-2}|t - s|^2) + O(|s - v|^{-2H}|t - s|^{4H}), \tag{4.7}$$

and if we write X and Y for two independent standard normal distributions, then the quantity (4.6) equals to

$$
\begin{aligned}
\mathbb{E}[|y_{s,t} + \sigma_s^{-1}\rho_{s,t}Y &+ \kappa_{s,t}X|^\gamma \mathbb{1}_{\{y_s+\sigma_s Y<a\}}\mathbb{1}_{\{y_{s,t}+\sigma_s^{-1}\rho_{s,t}Y+\kappa_{s,t}X>a-y_s-\sigma_s Y\}}] \\
&= \kappa_{s,t}^\gamma \mathbb{E}[|X+p|^\gamma \mathbb{1}_{\{y_s+\sigma_s Y<a\}}\mathbb{1}_{\{X>q\}}],
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
p &:= p(Y) := \kappa_{s,t}^{-1}(y_{s,t} + \sigma_s^{-1}\rho_{s,t}Y), \\
q &:= q(Y) := \kappa_{s,t}^{-1}(a - y_s - \sigma_s Y - y_{s,t} - \sigma_s^{-1}\rho_{s,t}Y).
\end{aligned}
$$

For a while, assume $\gamma > 0$. Using the estimate

$$
|(1+\varepsilon)^\gamma - 1| \lesssim \varepsilon, \quad \text{if } |\varepsilon| \le 1,
$$

we have

$$
\begin{aligned}
&\mathbb{E}[|X+p|^\gamma \mathbb{1}_{\{X>q\}}|Y] \\
&= \int_q^\infty |x|^\gamma \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx\,\mathbb{1}_{\{q\ge 2|p|\}} + O\Big(|p|\int_q^\infty |x|^{\gamma-1}e^{-\frac{x^2}{2}}\,dx\Big)\mathbb{1}_{\{q\ge 2|p|\}} + O((1+|p|)^\gamma)\mathbb{1}_{\{q<2|p|\}}.
\end{aligned}
$$

We set

$$
I_\gamma(y) := \int_{|y|}^\infty |x|^\gamma \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx.
$$

We have for $y > 0$

$$
I_\gamma'(y) = -|y|^\gamma \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}},
$$

and if $q \ge 2|p|$ and $y \in [q-|p|, q+|p|]$, then

$$
|I_\gamma'(y)| \lesssim e^{-\frac{1}{4}(q-|p|)^2} \le e^{-\frac{1}{16}q^2}.
$$

Therefore, if $q \ge 2|p|$, we have

$$
|I_\gamma(q) - I_\gamma(p+q)| \lesssim e^{-\frac{q^2}{16}}|p|.
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}[|X+p|^\gamma \mathbb{1}_{\{X>q\}}|Y] \\
&= I_\gamma(p+q) + O(|p|I_{\gamma-1}(q))\mathbb{1}_{\{q\ge 2|p|\}} + O(|p|e^{-\frac{q^2}{16}}) + O((1+|p|)^\gamma)\mathbb{1}_{\{q<2|p|\}} \\
&= I_\gamma(p+q) + O(|p|e^{-\frac{q^2}{16}}) + O((1+|p|)^\gamma)\mathbb{1}_{\{q<2|p|\}}.
\end{aligned}
\tag{4.9}
$$

When $\gamma = 0$, we have

$$
\mathbb{E}[\mathbb{1}_{\{X>q\}}|Y] = I_0(q) = I_0(p+q) + O(|p|e^{-\frac{q^2}{16}}) + O(1)\mathbb{1}_{\{q<2|p|\}},
$$

and thus (4.9) holds for $\gamma = 0$. We estimate the expectation (with respect to Y) of each term.

We have

$$\mathbb{E}[I_\gamma(p(Y) + q(Y))\mathbb{1}_{\{y_s + \sigma_s Y \le a\}}] = \frac{\kappa_{s,t}}{\sqrt{2\pi}\sigma_s} \int_0^\infty I_\gamma(z) e^{-\frac{(\kappa_{s,t}z + y_s - a)^2}{2\sigma_s^2}} \, dz.$$

By using the estimate

$$|e^{-(z-\eta)^2} - e^{-\eta^2}| \le 3|\eta| e^{-\frac{\eta^2}{4}} |z| + 2\mathbb{1}_{\{|\eta| \le 2|z|\}}, \tag{4.10}$$

we obtain

$$\mathbb{E}[I_\gamma(p(Y) + q(Y))\mathbb{1}_{\{y_s + \sigma_s Y \le 0\}}]$$

$$= \frac{\kappa_{s,t} e^{-\frac{(y_s-a)^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s} \int_0^\infty I_\gamma(z) \, dz + O\left(\sigma_s^{-2}\kappa_{s,t}^2 e^{-\frac{(y_s-a)^2}{16\sigma_s^2}}\right) + O\left(\sigma_s^{-1}\kappa_{s,t} e^{-\frac{(y_s-a)^2}{8\kappa_{s,t}^2}}\right)$$

$$= \frac{\kappa_{s,t} e^{-\frac{(y_s-a)^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s} \int_0^\infty |x|^{\gamma+1} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx + O\left(\sigma_s^{-2}\kappa_{s,t}^2 e^{-\frac{(y_s-a)^2}{16\sigma_s^2}}\right) + O\left(\sigma_s^{-1}\kappa_{s,t} e^{-\frac{(y_s-a)^2}{8\kappa_{s,t}^2}}\right).$$

Next, we estimate the second term of (4.9). We have for $n \in \{0, 1\}$,

$$\mathbb{E}[|Y|^n e^{-\frac{q(Y)^2}{16}}] = \frac{\kappa_{s,t}}{\sigma_s + \sigma_s^{-1}\rho_{s,t}} \int_{\mathbb{R}} \left|\frac{\kappa_{s,t}z + a - y_{s,t} - y_s}{\sigma_s + \sigma_s^{-1}\rho_{s,t}}\right|^n e^{-\frac{z^2}{16}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\kappa_{s,t}z + a - y_{s,t} - y_s}{\sigma_s + \sigma_s^{-1}\rho_{s,t}}\right)^2} \, dz$$

$$\lesssim \frac{\kappa_{s,t}}{\sigma_s + \sigma_s^{-1}\rho_{s,t}} \left(\frac{\kappa_{s,t} + |a - y_t|}{\sigma_s + \sigma_s^{-1}\rho_{s,t}}\right)^n$$

$$\times \left[e^{-\frac{(y_t-a)^2}{2(\sigma_s + \sigma_s^{-1}\rho_{s,t})^2}} + e^{-\frac{(y_t-a)^2}{16(\sigma_s + \sigma_s^{-1}\rho_{s,t})^2}} \frac{\kappa_{s,t}}{\sigma_s + \sigma_s^{-1}\rho_{s,t}} + e^{-\frac{(y_t-a)^2}{32\kappa_{s,t}^2}}\right],$$

where we applied (4.10) to get the last inequality. Therefore, we obtain

$$\mathbb{E}[|p(Y)|e^{-\frac{q(Y)^2}{16}}] \lesssim \left(\frac{|y_{s,t}|}{\sigma_s + \sigma_s^{-1}\rho_{s,t}} + \sigma_s^{-1}|\rho_{s,t}|\frac{\kappa_{s,t} + |y_t - a|}{(\sigma_s + \sigma_s^{-1}\rho_{s,t})^2}\right) e^{-\frac{(y_t-a)^2}{32(\sigma_s + \sigma_s^{-1}\rho_{s,t})^2}}.$$

Finally, we estimate the third term of (4.9). Suppose that $\frac{t-s}{s-v}$ is so small that $|\sigma_s^{-2}\rho_{s,t}| \le \frac{1}{24}$, and then we have

$$\mathbb{1}_{\{y_s + \sigma_s Y \le a\}}\mathbb{1}_{\{q(Y) \le 2|p(Y)|\}} \le \mathbb{1}_{\{|a - y_s - \sigma_s Y| \le 3|y_{s,t} + \sigma_s^{-1}\rho_{s,t} Y|\}}$$

$$\le \mathbb{1}_{\{|\sigma_s^{-1}(a-y_s) - Y| \le 6\sigma_s^{-1}|y_{s,t}| + 6\sigma_s^{-3}|\rho_{s,t}||a-y_s|\}}.$$

Hence,

$$\mathbb{E}[(1 + p(Y))^\gamma \mathbb{1}_{\{y_s + \sigma_s Y \le a\}}\mathbb{1}_{\{q(Y) \le 2|p(Y)|\}}]$$

$$\lesssim (1 + \kappa_{s,t}^{-1}(|y_{s,t}| + \sigma_s^{-1}\rho_{s,t}))^\gamma$$

$$\times \left(e^{-\frac{(y_s-a)^2}{5\sigma_s^2}} + \mathbb{1}_{\{|\sigma_s^{-1}|y_s - a|| \le 12\sigma_s^{-1}|y_{s,t}| + 12\sigma_s^{-3}|\rho_{s,t}||y_s-a|\}}\right)(\sigma_s^{-1}|y_{s,t}| + \sigma_s^{-3}|\rho_{s,t}||y_s - a|),$$

and

$$\mathbb{1}_{\{|\sigma_s^{-1}(y_s-a)| \le 12\sigma_s^{-1}|y_{s,t}| + 12\sigma_s^{-3}|\rho_{s,t}||y_s-a|\}} \le \mathbb{1}_{\{|y_s-a| \le 24|y_{s,t}|\}}.$$

This gives the estimate of the third term.

In summary, recalling that $\mathbb{E}[\hat{A}_{s,t}|\mathcal{F}_v]$ equals to (4.8), we obtain

$$\mathbb{E}[\hat{A}_{s,t}|\mathcal{F}_v] = \frac{\kappa_{s,t}^{\gamma+1} e^{-\frac{(Y_s-a)^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s} \int_0^\infty |x|^{\gamma+1} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx + R_1,$$

where

$$|R_1| \lesssim \sigma_s^{-2} \kappa_{s,t}^{\gamma+2} e^{-\frac{(Y_s-a)^2}{16\sigma_s^2}} + \sigma_s^{-1} \kappa_{s,t}^{\gamma+1} e^{-\frac{(Y_s-a)^2}{8\kappa_{s,t}^2}}$$
$$+ \kappa_{s,t}^\gamma \left(\frac{|Y_{s,t}|}{\sigma_s + \sigma_s^{-1}\rho_{s,t}} + \sigma_s^{-1}|\rho_{s,t}| \frac{\kappa_{s,t} + |Y_t - a|}{(\sigma_s + \sigma_s^{-1}\rho_{s,t})^2} \right) e^{-\frac{(Y_t-a)^2}{32(\sigma_s+\sigma_s^{-1}\rho_{s,t})^2}}$$
$$+ (\kappa_{s,t}^\gamma + (|Y_{s,t}| + \sigma_s^{-1}\rho_{s,t})^\gamma) \left(e^{-\frac{(Y_s-a)^2}{5\sigma_s^2}} + \mathbb{1}_{\{|Y_s-a|\le 24|Y_{s,t}|\}} \right)$$
$$\times (\sigma_s^{-1}|Y_{s,t}| + \sigma_s^{-3}|\rho_{s,t}||Y_s - a|).$$

Let us estimate $\|R_1\|_{L_m(\mathbb{P})}$. Recall that

$$\kappa_{s,t} \lesssim |t-s|^H, \qquad \sigma_s \lesssim |s-v|^H,$$

and

$$|\rho_{s,t}| \lesssim \begin{cases} |s-v|^{2H-1}|t-s|, & H > \frac{1}{2}, \\ |t-s|^{2H}, & H < \frac{1}{2}. \end{cases}$$

We have the estimate (3.4) of $Y_{s,t}$. Since

$$\mathbb{E}[Y_s^2] = \mathbb{E}[B(0)^2] + c_H s^{2H} - \frac{1}{2H}|s-v|^{2H} \gtrsim \mathbb{E}[B(0)^2] + s^{2H} =: \chi_s^2,$$

there is a constant $c = c(H) > 0$, such that

$$\mathbb{E}[|Y_s - a|^n e^{-\frac{(Y_s-a)^2}{\sigma^2}}] \lesssim_n \chi_s^{-1} \sigma^{n+1} e^{-\frac{ca^2}{\chi_s^2+\sigma^2}}, \tag{4.11}$$

$$\mathbb{E}[|Y_{s,t}|^n e^{-\frac{(Y_t-a)^2}{\sigma^2}}] \lesssim_n \chi_s^{-1} \sigma |v-s|^{n(H-1)}|t-s|^n e^{-\frac{ca^2}{\chi_s^2+\sigma^2}}. \tag{4.12}$$

Therefore, for some constant $c_1 = c(H, m) > 0$,

$$\left\| \sigma_s^{-2} \kappa_{s,t}^{\gamma+2} e^{-\frac{(Y_s-a)^2}{2\sigma_s^2}} \right\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} \sigma_s^{-2+\frac{1}{m}} \kappa_{s,t}^{\gamma+2} e^{-\frac{c_1a^2}{\chi_s^2}} \lesssim \chi_s^{-\frac{1}{m}} |s-v|^{-(2-\frac{1}{m})H}|t-s|^{(\gamma+2)H} e^{-\frac{c_1a^2}{\chi_s^2}},$$

$$\left\| \kappa_{s,t}^\gamma \frac{|Y_{s,t}|}{\sigma_s + \sigma_s^{-1}\rho_{s,t}} e^{-\frac{(Y_t-a)^2}{32(\sigma_s+\sigma_s^{-1}\rho_{s,t})^2}} \right\|_{L_m(\mathbb{P})}$$
$$\lesssim \chi_s^{-\frac{1}{m}} \kappa_{s,t}^\gamma \sigma_s^{-1+\frac{1}{m}} |s-v|^{H-1}|t-s| e^{-\frac{c_1a^2}{\chi_s^2}} \lesssim \chi_s^{-\frac{1}{m}} |s-v|^{\frac{H}{m}-1}|t-s|^{\gamma H+1} e^{-\frac{c_1a^2}{\chi_s^2}},$$

$$\left\| \kappa_{s,t}^\gamma \sigma_s^{-1}|\rho_{s,t}| \frac{\kappa_{s,t} + |Y_t - a|}{(\sigma_s + \sigma_s^{-1}\rho_{s,t})^2} e^{-\frac{(Y_t-a)^2}{32(\sigma_s+\sigma_s^{-1}\rho_{s,t})^2}} \right\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} \kappa_{s,t}^\gamma \sigma_s^{-2+\frac{1}{m}} |\rho_{s,t}| e^{-\frac{c_1a^2}{\chi_s^2}}$$
$$\lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1a^2}{\chi_s^2}} \begin{cases} |s-v|^{-1+\frac{H}{m}}|t-s|^{\gamma H+1}, & H > \frac{1}{2}, \\ |s-v|^{-2H+\frac{H}{m}}|t-s|^{(\gamma+2)H}, & H < \frac{1}{2}, \end{cases}$$

$$\left\| \kappa_{s,t}^{\gamma} e^{-\frac{(Y_s-a)^2}{5\sigma_s^2}} (\sigma_s^{-1}|Y_{s,t}| + \sigma_s^{-3}|\rho_{s,t}||Y_s-a|) \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \chi_s^{-1} \kappa_{s,t}^{\gamma} (\sigma_s^{\frac{1}{m}-1}|s-v|^{H-1}|t-s| + \sigma_s^{-2+\frac{1}{m}}|\rho_{s,t}|) e^{-\frac{c_1 a^2}{\chi_s^2}}$$

$$\lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} \begin{cases} |s-v|^{\frac{H}{m}-1}|t-s|^{2-H}, & H > \frac{1}{2}, \\ (|s-v|^{\frac{H}{m}-1}|t-s|^{2-H} + |s-v|^{(-2+\frac{1}{m})H}|t-s|^{1+H}), & H < \frac{1}{2}, \end{cases}$$

$$\left\| |Y_{s,t}|^{\gamma} e^{-\frac{(Y_s-a)^2}{5\sigma_s^2}} (\sigma_s^{-1}|Y_{s,t}| + \sigma_s^{-3}|\rho_{s,t}||Y_s-a|) \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \sigma_s^{-1} \left\| |Y_{s,t}|^{\gamma+1} e^{-\frac{(Y_s-a)^2}{5\sigma_s^2}} \right\|_{L_m(\mathbb{P})} + \sigma_s^{-2}\rho_{s,t} \left\| |Y_{s,t}|^{\gamma} e^{-\frac{(Y_s-a)^2}{6\sigma_s^2}} \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} \Big[\sigma_s^{\frac{1}{m}-1}|s-v|^{(\gamma+1)(H-1)}|t-s|^{(\gamma+1)}$$

$$+ \sigma_s^{-2+\frac{1}{m}}|\rho_{s,t}||v-s|^{\gamma(H-1)}|t-s|^{\gamma} \Big]$$

$$\lesssim_{H,m} \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}}$$

$$\times \begin{cases} |s-v|^{(\gamma+\frac{1}{m})H-(\gamma+1)}|t-s|^{\gamma+1}, & H > \frac{1}{2}, \\ (|s-v|^{(\gamma+\frac{1}{m})H-(\gamma+1)}|t-s|^{\gamma+1} + |s-v|^{(-2+\frac{1}{m}+\gamma)H-\gamma}|t-s|^{\gamma+2H}), & H < \frac{1}{2}, \end{cases}$$

$$\left\| (\sigma_s^{-1}\rho_{s,t})^{\gamma} e^{-\frac{Y_s^2}{5\sigma_s^2}} (\sigma_s^{-1}|Y_{s,t}| + \sigma_s^{-3}|\rho_{s,t}||Y_s|) \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} \Big[\sigma_s^{\frac{1}{m}-1-\gamma}|\rho_{s,t}|^{\gamma}|s-v|^{H-1}|t-s| + \sigma_s^{-2+\frac{1}{m}-\gamma}|\rho_{s,t}|^{\gamma+1} \Big]$$

$$\lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}}$$

$$\times \begin{cases} |s-v|^{(\gamma+\frac{1}{m})H-(\gamma+1)}|t-s|^{\gamma+1}, & H > \frac{1}{2}, \\ |s-v|^{(\frac{1}{m}-\gamma)H-1}|t-s|^{1+2H\gamma} + |s-v|^{(-2+\frac{1}{m}-\gamma)H}|t-s|^{2H(\gamma+1)}, & H < \frac{1}{2}, \end{cases}$$

and finally

$$\left\| (\kappa_{s,t}^{\gamma} + (|Y_{s,t}| + \sigma_s^{-1}\rho_{s,t})^{\gamma}) \mathbb{1}_{\{|Y_s-a|\leq 24|Y_{s,t}|\}} (\sigma_s^{-1}|Y_{s,t}| + \sigma_s^{-3}|\rho_{s,t}||Y_s|) \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \sigma_s^{-1} (\kappa_{s,t}^{\gamma} + \sigma_s^{-\gamma}|\rho_{s,t}|^{\gamma}) \left\| Y_{s,t} \mathbb{1}_{\{|Y_s-a|\leq 24|Y_{s,t}|\}} \right\|_{L_m(\mathbb{P})}$$

$$+ \sigma_s^{-1} \left\| |Y_{s,t}|^{\gamma+1} \mathbb{1}_{\{|Y_s-a|\leq 24|Y_{s,t}|\}} \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \sigma_s^{-1} (\kappa_{s,t}^{\gamma} + \sigma_s^{-\gamma}|\rho_{s,t}|^{\gamma}) \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} |s-v|^{(H-1)(1+\frac{1}{m})}|t-s|^{1+\frac{1}{m}}$$

$$+ \sigma_s^{-1} \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} |s-v|^{(H-1)(1+\gamma+\frac{1}{m})}|t-s|^{1+\gamma+\frac{1}{m}}$$

$$\lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} \Big[|s-v|^{\frac{H}{m}-(1+\frac{1}{m})}|t-s|^{1+\frac{1}{m}+\gamma H} + |s-v|^{(\gamma+\frac{1}{m})H-(1+\gamma+\frac{1}{m})}|t-s|^{1+\gamma+\frac{1}{m}}$$

$$+ \mathbb{1}_{\{H<\frac{1}{2}\}} |s-v|^{(\frac{1}{m}-\gamma)H-(1+\frac{1}{m})}|t-s|^{1+2H\gamma+\frac{1}{m}} \Big].$$

After this long calculation, we conclude

$$\|R_1\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} \left(\frac{t-s}{s-v} \right)^{\min\{1,2H\}-\frac{H}{m}} |t-s|^{(\gamma+\frac{1}{m})H} \tag{4.13}$$

if $\frac{t-s}{s-v}$ is sufficiently small.

By (4.7), we have

$$
\frac{\kappa_{s,t}^{\gamma+1} e^{-\frac{(y_s-a)^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s} \int_0^\infty |x|^{\gamma+1} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx = \mathfrak{c}_{H,\gamma} \frac{e^{-\frac{(y_s-a)^2}{2\sigma_s^2}} |t-s|^{(\gamma+1)H}}{\sqrt{2\pi}\sigma_s} + R_2,
$$

where

$$
|R_2| \lesssim_H \frac{e^{-\frac{(y_s-a)^2}{2\sigma_s^2}}}{\sigma_s} \left(\frac{t-s}{s-v}\right)^{2\min\{H,1-H\}} |t-s|^{(\gamma+1)H}. \tag{4.14}
$$

Therefore,

$$
\|R_2\|_{L_m(\mathbb{P})} \lesssim_H \chi_s^{-\frac{1}{m}} e^{-\frac{c_1 a^2}{\chi_s^2}} \begin{cases} |s-v|^{(\frac{1}{m}+1)H-2}|t-s|^{2+(\gamma-1)H}, & H > \frac{1}{2}, \\ |s-v|^{(\frac{1}{m}-3)H}|t-s|^{(\gamma+3)H}, & H < \frac{1}{2}. \end{cases}
$$

This completes the proof of the lemma. ☐

Lemma 4.7. *We have*

$$
\|L_t(a) - L_s(a)\|_{L_m(\mathbb{P})} \lesssim_T \begin{cases} |t-s|^{1-H} f_H(a), & \text{for all } H \in (0,1), \\ (\mathbb{E}[B(0)^2] + s^{2H})^{-\frac{1}{2m}} |t-s|^{1-H+\frac{H}{m}} f_H(a), & \text{if } H > \frac{1}{2}, \end{cases} \tag{4.15}
$$

where $f_H(a)$ satisfies the estimate (4.3). Moreover, if $\frac{t-s}{s-v}$ is sufficiently small, then

$$
\mathbb{E}[L_t(a) - L_s(a)|\mathcal{F}_v] = \frac{e^{-\frac{|Y_s-a|^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s}(t-s) + \tilde{R},
$$

where for some $c = c(H,m) > 0$,

$$
\|\tilde{R}\|_{L_m(\mathbb{P})} \lesssim (\mathbb{E}[B(0)^2] + s^{2H})^{-\frac{1}{2m}} e^{-c(\mathbb{E}[B(0)^2]+T^{2H})^{-1}a^2} \left(\frac{t-s}{s-v}\right)^{\frac{1}{m}+(1-\frac{1}{m})H} |t-s|^{1-H+\frac{H}{m}}. \tag{4.16}
$$

Proof. The estimate in (4.15) follows from [3, (3.38)]. However, since this is not entirely obvious, we sketch here an alternative derivation, which is motivated by [7]. In view of the formal expression $L_t(a) = \int_0^t \delta_a(B_r) \, dr$, we set

$$
\bar{A}_{s,t}(a) := \int_s^t \mathbb{E}[\delta_a(B_r)|\mathcal{F}_s] \, dr := \int_s^t \sqrt{\frac{H}{\pi c_H (r-s)^{2H}}} e^{-\frac{H(Y_s-a)^2}{c_H(r-s)^{2H}}} \, dr.
$$

We note $\mathbb{E}[\delta \bar{A}_{s,u,t}(a)|\mathcal{F}_s] = 0$. By Lê's stochastic sewing lemma [24], to prove (4.15), it suffices to show

∘ the estimate

$$
\|e^{-\frac{H(Y_s-a)^2}{c_H(r-s)^{2H}}}\|_{L_m(\mathbb{P})} \lesssim_T \begin{cases} f_H(a), & H < \frac{1}{2}, \\ (\mathbb{E}[B(0)^2] + s^{2H})^{-\frac{1}{2m}} |r-s|^{\frac{H}{m}} f_H(a), & H > \frac{1}{2}, \end{cases}
$$

∘ and the identity $L_t(a) = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \bar{A}_{u,v}(a)$, where π is a partition of $[0,t]$.

The first point is essentially given in Lemma 4.4. The second point follows from the identity

$$
\int_\mathbb{R} \left(\sum_{[u,v]\in\pi} \bar{A}_{u,v}(a)\right) f(a) \, da = \sum_{[u,v]\in\pi} \int_u^v \mathbb{E}[f(B_r)|\mathcal{F}_u] \, dr.
$$

Thus, we now focus on the estimate (4.16). We have the identity

$$\mathbb{E}[L_t - L_s | \mathcal{F}_v] = \int_s^t \frac{e^{-\frac{|Y_r - a|^2}{2\sigma_r^2}}}{\sqrt{2\pi}\sigma_r} \, dr.$$

Indeed, we can convince ourselves of the validity of the identity from the formal expression

$$L_t - L_s = \int_s^t \delta(B_r - a) \, dr.$$

We have the decomposition

$$\frac{e^{-\frac{|Y_r - a|^2}{2\sigma_r^2}}}{\sigma_r} - \frac{e^{-\frac{|Y_s - a|^2}{2\sigma_s^2}}}{\sigma_s} = \left[\frac{e^{-\frac{|Y_r - a|^2}{2\sigma_r^2}}}{\sigma_r} - \frac{e^{-\frac{|Y_r - a|^2}{2\sigma_r^2}}}{\sigma_s} \right]$$
$$+ \left[\frac{e^{-\frac{|Y_r - a|^2}{2\sigma_r^2}}}{\sigma_s} - \frac{e^{-\frac{|Y_r - a|^2}{2\sigma_s^2}}}{\sigma_s} \right] + \left[\frac{e^{-\frac{|Y_r - a|^2}{2\sigma_s^2}}}{\sigma_s} - \frac{e^{-\frac{|Y_s - a|^2}{2\sigma_s^2}}}{\sigma_s} \right] =: R_3 + R_4 + R_5.$$

By (4.11), we obtain

$$\|R_3\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} \sigma_r^{\frac{1}{m}} |\sigma_r^{-1} - \sigma_s^{-1}| \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} |s - v|^{-1-H} |r - s|^{1+\frac{H}{m}}.$$

We have

$$\sigma_s |R_4| = e^{-\frac{|Y_r|^2}{2\sigma_r^2}} |1 - e^{-\frac{Y_r^2}{2}(\frac{1}{\sigma_s} - \frac{1}{\sigma_r})}| \lesssim |\sigma_s^{-2} - \sigma_r^{-2}| Y_r^2 e^{-\frac{|Y_r|^2}{2\sigma_r}}.$$

Hence, by (4.11),

$$\|R_4\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} \sigma_s^{-1+\frac{1}{m}} (1 - \sigma_s^2 \sigma_r^{-2}) \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} |s - v|^{-1+(-1+\frac{1}{m})H} |t - s|.$$

To estimate R_5, observe

$$\sigma_s |R_6| \lesssim e^{-\frac{|Y_s - a|^2}{8\sigma_s^2}} \sigma_s^{-1} |Y_{s,r}| + \mathbb{1}_{\{|Y_s - a| \leq 2|Y_{s,r}|\}}.$$

By (4.12),

$$\|e^{-\frac{|Y_s - a|^2}{8\sigma_s^2}} \sigma_s^{-2} |Y_{s,r}|\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} \sigma_s^{-2+\frac{1}{m}} |s - v|^{H-1} |t - s| \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} |s - v|^{(-1+\frac{1}{m})H-1} |t - s|.$$

To estimate $\mathbb{P}(|Y_s - a| \leq 2|Y_{s,r}|)$, consider the decomposition

$$Y_{s,r} = \mathbb{E}[Y_s^2]^{-1} \mathbb{E}[Y_s Y_{s,r}] Y_s + (Y_{s,r} - \mathbb{E}[Y_s^2]^{-1} \mathbb{E}[Y_s Y_{s,r}] Y_s).$$

If $\frac{t-s}{s-v}$ is sufficiently small, then $|\mathbb{E}[Y_s^2]^{-1} \mathbb{E}[Y_s Y_{s,r}]| \leq \frac{1}{4}$. Therefore,

$$\mathbb{P}(|Y_s - a| \leq 2|Y_{s,r}|) \leq \mathbb{P}(\chi_s |Y - \chi_s^{-1} a| \lesssim_H |s - v|^{H-1} |t - s||X|) \lesssim \chi_s^{-1} e^{-\frac{ca^2}{2\chi_s^2}} |s - v|^{H-1} |t - s|.$$

This gives an estimate of R_5. Thus, we conclude

$$\mathbb{E}[L_t - L_s | \mathcal{F}_v] = \frac{e^{-\frac{|Y_s - a|^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s} |t - s| + R_6,$$

where

$$\|R_6\|_{L_m(\mathbb{P})} \lesssim \chi_s^{-\frac{1}{m}} e^{-\frac{ca^2}{2\chi_s^2}} \left(\frac{t-s}{s-v}\right)^{\frac{1}{m}+(1-\frac{1}{m})H} |t-s|^{1-H+\frac{H}{m}}.$$

This completes the proof of the lemma. □

Now we can complete the proof of Theorem 4.1. The above lemmas show

$$\|\mathfrak{c}_{H,\gamma}(L_t(a) - L_t(a)) - A_{s,t}(a)\|_{L_m(\mathbb{P})} \begin{cases} \lesssim_T |t-s|^{1-H}, & \text{for all } H \in (0,1), \\ \lesssim_T s^{-\frac{H}{m}} |t-s|^{1-H+\frac{H}{m}}, & \text{if } H > \frac{1}{2}, \end{cases}$$

and if $\frac{t-s}{s-v}$ is sufficiently small, then

$$\|\mathbb{E}[\mathfrak{c}_{H,\gamma}(L_t(a) - L_s(a)) - A_{s,t}(a)|\mathcal{F}_v]\|_{L_m(\mathbb{P})}$$
$$\begin{cases} \lesssim_T e^{-c(\mathbb{E}[B(0)^2]+T^{2H})^{-1}a^2} \left(\frac{t-s}{s-v}\right)^{\min\{2H,\frac{1}{m}+H\}-\frac{H}{m}} |t-s|^{1-H}, & H < \frac{1}{2}, \\ \lesssim_T s^{-\frac{H}{m}} e^{-c(\mathbb{E}[B(0)^2]+T^{2H})^{-1}a^2} \left(\frac{t-s}{s-v}\right)^{\min\{1,\frac{1}{m}+H\}-\frac{H}{m}} |t-s|^{1-H+\frac{H}{m}}, & H > \frac{1}{2}. \end{cases}$$

Noting that the exponents satisfy the assumption of Theorem 1.1 or Corollary 2.7, Remark 1.4 implies

$$\|\mathfrak{c}_{H,\gamma}L_T(a) - \sum_{[s,t]\in\pi} A_{s,t}(a)\|_{L_m(\mathbb{P})} \lesssim_T e^{-c(\mathbb{E}[B(0)^2]+T^{2H})^{-1}a^2} |\pi|^\varepsilon. \tag{4.17}$$

Hence, we complete the proof of Theorem 4.1. □

Corollary 4.8. *We have*

$$\lim_{n\to\infty} \left(\frac{T}{n}\right)^{1-H} \#\{k \in \{1,\dots,n\} \mid B_{\frac{(k-1)T}{n}} < a < B_{\frac{kT}{n}}\} = \sqrt{\frac{c_H}{2\pi}} L_T(a),$$

where the convergence is in $L_m(\mathbb{P})$ with m satisfying (4.1).

Proof. The claim is a special case of Theorem 4.1 with $\gamma = 0$. When $m = 2$, it is proved in [4, Theorem 5]. □

For applications to pathwise stochastic calculus, a representation of the local time as in (b) above is more useful. In [9, Theorem 3.2], a pathwise Itô-Tanaka formula is derived under the assumption that

$$\tilde{L}_T^\pi(a) := \sum_{[s,t]\in\pi:B_s<a<B_t} |B_t - a|^{\frac{1}{H}-1} \tag{4.18}$$

converges weakly in $L_m(\mathbb{R})$ for some $m > 1$. But as already suggested by [9, Lemma 3.5], this weak convergence in $L_m(\mathbb{R})$ follows from our convergence result in Theorem 4.1:

Corollary 4.9. *Let $B \in C([0,T],\mathbb{R})$, and for any partition π of $[0,T]$, let $\tilde{L}_T^\pi(a)$ be defined as in (4.18). Assume that $m > 1$ and that (π_n) is a sequence of partitions of $[0,T]$, such that $\lim_{n\to\infty} \sup_{[s,t]\in\pi_n} |B_t - B_s| = 0$ and for a limit $L_T \in L_m(\mathbb{R})$:*

$$\lim_{n\to\infty} \int_\mathbb{R} |\sum_{[s,t]\in\pi_n:B_s<a<B_t} |B_t - B_s|^{\frac{1}{H}-1} - \mathfrak{c}_{H,\frac{1}{H}-1} L_T(a)|^m \, \mathrm{d}a = 0. \tag{4.19}$$

Then

$$\lim_{n\to\infty} \tilde{L}_T^{\pi_n}(\cdot) = H\mathfrak{c}_{H,\frac{1}{H}-1} L_T(\cdot) \quad \text{weakly in } L_m(\mathbb{R}).$$

Remark 4.10. If B is a sample path of the fractional Brownian motion with Hurst index $H \in (0, 1)$, then by Theorem 4.1, the convergence (4.19) holds in probability for any sequence of partitions $(\pi_n)_{n \in \mathbb{N}}$, provided that m satisfies (4.1). Therefore, we can find a subsequence so that the convergence along the subsequence holds almost surely. In fact, by (4.17), we even control the convergence rate in terms of the mesh size of the partition, and this easily gives us specific sequences of partitions along which the convergence holds almost surely and not only in probability. For example, if π_n is the nth dyadic partition of $[0, T]$, the estimate (4.17) gives

$$\|\mathfrak{c}_{H,\gamma} L_T(a) - \sum_{[s,t] \in \pi_n} A_{s,t}(a)\|_{L_m(\mathbb{P})} \lesssim_T e^{-c(\mathbb{E}[B(0)^2] + T^{2H})^{-1} a^2} 2^{-n\varepsilon}.$$

Since the right-hand side is summable with respect to n, the Borel–Cantelli lemma implies the almost sure convergence. Along any such sequence of partitions, we therefore obtain the almost sure weak convergence of $\tilde{L}_T^{\pi_n}$ in $L_m(\mathbb{R})$.

Proof. Set

$$\tilde{A}_{s,t}(a) := (|B_t - a|^{\frac{1}{H} - 1} - H|B_t - B_s|^{\frac{1}{H} - 1}) \mathbb{1}_{\{B_s < a < B_t\}}.$$

It suffices to show that $\sum_{[s,t] \in \pi_n} \tilde{A}_{s,t}(\cdot)$ converge weakly to 0 in $L_m(\mathbb{R})$. Since

$$|\tilde{A}_{s,t}(a)| \leq \min\{H, 1 - H\} |B_t - a|^{\frac{1}{H} - 1} \mathbb{1}_{\{B_s < a < B_t\}}$$

and since $\sum_{[s,t] \in \pi_n} |B_t - a|^{\frac{1}{H} - 1} \mathbb{1}_{\{B_s < a < B_t\}}$ is bounded in $L_m(\mathbb{P})$ by assumption, it suffices to show that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \left(\sum_{[s,t] \in \pi_n} \tilde{A}_{s,t}(a) \right) g(a) \, da = 0$$

for every compactly supported continuous function g. Since for $B_s < B_t$ we have

$$B_{s,t}^{-1} \int_{B_s}^{B_t} |B_t - a|^{\frac{1}{H} - 1} \, da = H|B_{s,t}|^{\frac{1}{H} - 1},$$

we obtain

$$\int_{B_s}^{B_t} \tilde{A}_{s,t}(a) g(a) \, da = \int_{B_s}^{B_t} |B_t - a|^{\frac{1}{H} - 1} (g(a) - B_{s,t}^{-1} \int_{B_s}^{B_t} g(x) \, dx) \, da.$$

Therefore,
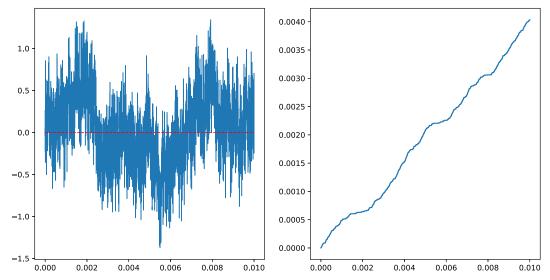
$$\int_{\mathbb{R}} \left(\sum_{[s,t] \in \pi_n} \tilde{A}_{s,t}(a) \right) g(a) \, da = \sum_{[s,t] \in \pi_n} \int_{B_s}^{B_t} |B_t - a|^{\frac{1}{H} - 1} (g(a) - B_{s,t}^{-1} \int_{B_s}^{B_t} g(x) \, dx) \, da$$

$$\leq \int_{\mathbb{R}} \tilde{L}_T^{\pi_n}(a) \, da \times \sup_{|x-y| \leq \sup_{[s,t] \in \pi} |B_t - B_s|} |g(x) - g(y)|$$

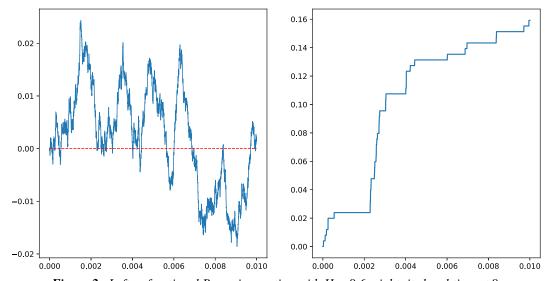which converges to 0. $\square$

Remark 4.11. As noted in [33], we can use Theorem 4.1 to simulate the local time of a fractional Brownian motion (see Figures 1 ($H = 0.1$) and 2 ($H = 0.6$)).[1]

---

[1]Fractional Brownian motions are simulated by the Python package **fbm**: https://pypi.org/project/fbm/.

Figure 1. *Left: a fractional Brownian motion with $H = 0.1$, right: its local time at 0.*

Figure 2. *Left: a fractional Brownian motion with $H = 0.6$, right: its local time at 0.*

5. Regularization by noise for diffusion coefficients

Let $y \in C^{\alpha}([0, T], \mathbb{R}^d)$ with $\alpha \in (\frac{1}{2}, 1)$. We consider a Young differential equation

$$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}y_t, \quad x_0 = x. \tag{5.1}$$

We suppose that the drift coefficient b belongs to $C_b^1(\mathbb{R}^d, \mathbb{R}^d)$, where $C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ is the space of continuously differentiable bounded functions between $\mathbb{R}^d$ with bounded derivatives. If the diffusion coefficient σ belongs to $C_b^1(\mathbb{R}^d, \mathcal{M}_d)$, where $\mathcal{M}_d$ is the space of $d \times d$ matrices, then we can prove the existence of a solution to (5.1). However, to prove the uniqueness of solutions, the coefficient σ needs to be more regular. The following result is well-known (e.g. [29]), but we give a proof for the sake of later discussion.

Proposition 5.1. *Let $b \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C^{1+\delta}(\mathbb{R}^d, \mathcal{M}_d)$ with $\delta > \frac{1-\alpha}{\alpha}$. Then the Young differential equation (5.1) has a unique solution.*

Proof. The argument is very similar to that of [24, Theorem 6.2]. Let $x^{(i)}$ ($i = 1, 2$) be two solutions to (5.1). Then,

$$x_t^{(1)} - x_t^{(2)} = \int_0^t \{b(x_s^{(1)}) - b(x_s^{(2)})\}\, ds + \int_0^t \{\sigma(x_s^{(1)}) - \sigma(x_s^{(2)})\}\, dy_s = \int_0^t \{x_s^{(1)} - x_s^{(2)}\}\, dv_s,$$

where

$$v_t := \int_0^t \int_0^1 \nabla b(\theta x_s^{(1)} + (1 - \theta)x_s^{(2)})\, d\theta\, ds + \int_0^t \int_0^1 \nabla \sigma(\theta x_s^{(1)} + (1 - \theta)x_s^{(2)})\, d\theta\, dy_s.$$

Note that the second term is well-defined as a Young integral since

$$s \mapsto \nabla \sigma(\theta x_s^{(1)} + (1 - \theta)x_s^{(2)})$$

is $\delta\alpha$-Hölder continuous and $\delta\alpha + \alpha > 1$ by our assumption of δ. Therefore, $x^{(1)} - x^{(2)}$ is a solution of the Young differential equation

$$dz_t = z_t\, dv_t, \quad z_0 = 0.$$

The uniqueness of this linear Young differential equation is known. Hence, $x^{(1)} - x^{(2)} = 0$. □

Proposition 5.1 is sharp in the sense that for any $\alpha \in (1, 2)$ and any $\delta \in (0, \frac{1-\alpha}{\alpha})$, we can find $\sigma \in C^\gamma(\mathbb{R}^2, \mathcal{M}_2)$ and $y \in C^\alpha([0, T], \mathbb{R}^2)$, such that the Young differential equation

$$dx_t = \sigma(x_t)\, dy_t$$

has more than one solution (see Davie [11, Section 5]). However, if the driver y is random, we could hope to obtain the uniqueness of solutions in a probabilistic sense even when the regularity of σ does not satisfy the assumption of Proposition 5.1. For instance, if the driver y is a Brownian motion and the integral is understood in Itô's sense, the condition $\sigma \in C_b^1$ is sufficient to prove pathwise uniqueness.

The goal of this section is to prove the following.

Theorem 5.2. *Suppose that B is an $(\mathcal{F}_t)$-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ in the sense of Definition 3.1. Let $b \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C_b^1(\mathbb{R}^d, \mathcal{M}_d)$. Assume one of the following.*

1. *We have $b \equiv 0$ and $\sigma \in C^{1+\delta}(\mathbb{R}^d, \mathcal{M}_d)$ with*

$$\delta > \frac{(1 - H)(2 - H)}{H(3 - H)}. \tag{5.2}$$

2. *For all $x \in \mathbb{R}^d$, the matrix $\sigma(x)$ is symmetric and satisfies*

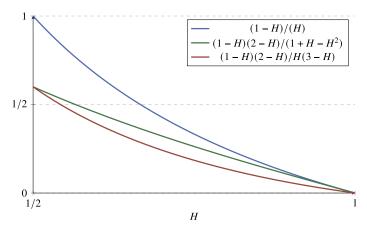$$y \cdot \sigma(x)y > 0, \quad \forall y \in \mathbb{R}^d,$$

   and $\sigma \in C^{1+\delta}(\mathbb{R}^d, \mathcal{M}_d)$ with δ satisfying (5.2).
3. *We have $\sigma \in C^{1+\delta}(\mathbb{R}^d; \mathcal{M}_d)$ with*

$$\delta > \frac{(1 - H)(2 - H)}{1 + H - H^2}. \tag{5.3}$$

The graphs of (5.2) and (5.3) can be found in Figure 3.

Then, for every $x \in \mathbb{R}^d$, there exists a unique, up to modifications, process $(X_t)_{t \in [0,\infty)}$ with the following properties.

Figure 3. *Some graphs of H from Theorem 5.2.*

○ *The process (X_t) is $(\mathcal{F}_t)$-adapted and is α-Hölder continuous for every $\alpha < H$.*
○ *The process (X_t) solves the Young differential equation*

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x. \tag{5.4}$$

Furthermore, in that case the process $(X_t)_{t \in [0, \infty)}$ is a strong solution, that is, it is adapted to the natural filtration generated by the Brownian motion W appearing in the Mandelbrot–Van Ness representation (3.1).

Remark 5.3. In the case 2, we assume the positive-definiteness of σ to ensure that for every $x, y \in \mathbb{R}^d$ and $\theta \in [0, 1]$, the matrix $\theta\sigma(x) + (1 - \theta)\sigma(y)$ is invertible.

Remark 5.4. Since the seminal work [8] of Catellier and Gubinelli, many works have appeared to establish weak or strong existence or uniqueness to the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}B_t$$

for an irregular drift b and a fractional Brownian motion B. In contrast, there are much fewer works that attempt to optimize the regularity of the diffusion coefficient σ. The work [20] by Hinz et al., where $b \equiv 0$, considers certain existence and uniqueness for σ that is merely of bounded variation, at the cost of additional restrictive assumptions (variability and [20, Assumption 3.15]). It seems that Theorem 5.2 is the first result to improve the regularity of σ without any additional assumption (except σ being invertible for the case 2). However, we believe that our assumption of δ is not optimal (see Remark 5.9).

The proof of Proposition 5.1 suggests that the pathwise uniqueness holds if, for any two $(\mathcal{F}_t)$-adapted solutions $X^{(1)}$ and $X^{(2)}$ to (5.4), we can construct the integral

$$\int_0^t \int_0^1 \nabla\sigma(\theta X_s^{(1)} + (1 - \theta)X_s^{(2)})\,\mathrm{d}\theta\,\mathrm{d}B_s. \tag{5.5}$$

If $\theta X_s^{(1)} + (1 - \theta)X_s^{(2)}$ is replaced by B_s, then the integral is constructed in Proposition 3.3. The difficulty here is that $X^{(i)}$ is not Gaussian and the Wiener chaos decomposition crucially used in the proof of Proposition 3.3 cannot be applied. Yet, the process $X^{(i)}$ is *locally controlled by the Gaussian process B* (whose precise meaning will be clarified later), and by taking advantage of this fact, we can still make sense of the integral (5.5).

As a technical ingredient, we need a variant of Theorem 1.1.

Lemma 5.5. *Let $(A_{s,t})_{0 \leq s < t \leq T}$ be a family of two-parameter random variables, and let $(\mathcal{F}_t)_{t \in [0,T]}$ be a filtration, such that $A_{s,t}$ is $\mathcal{F}_t$-measurable for every $0 \leq s \leq t \leq T$. Suppose that for some $m \geq 2$, $\Gamma_1, \Gamma_2, \Gamma_3 \in [0, \infty)$ and $\alpha, \gamma, \beta_1, \beta_2, \beta_3 \in [0, \infty)$, we have for every $0 \leq v < s < u < t \leq T$*

$$\|\mathbb{E}[\delta A_{s,u,t} | \mathcal{F}_v]\|_{L_m(\mathbb{P})} \leq \Gamma_1 |s-v|^{-\alpha} |t-s|^{\beta_1} + \Gamma_2 |t-v|^{\gamma} |t-s|^{\beta_2}, \quad \text{if } t-s \leq s-v,$$

$$\|\delta A_{s,u,t}\|_{L_m(\mathbb{P})} \leq \Gamma_3 |t-s|^{\beta_3}.$$

Suppose that

$$\min\{\beta_1, 2\beta_3\} > 1, \quad \gamma + \beta_2 > 1, \quad 1 + \alpha - \beta_1 < \alpha \min\left\{\frac{\gamma + \beta_2 - 1}{\gamma}, 2\beta_3 - 1\right\}. \tag{5.6}$$

Finally, suppose that there exists a stochastic process $(\mathcal{A}_t)_{t \in [0,T]}$, such that

$$\mathcal{A}_t = \lim_{|\pi| \to 0; \pi \text{ is a partition of } [0,t]} \sum_{[u,v] \in \pi} A_{u,v},$$

where the convergence is in $L_m(\mathbb{P})$. Then, we have

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L_m(\mathbb{P})} \lesssim_{\alpha, \gamma, \beta_1, \beta_2, \beta_3} \Gamma_1 |t-s|^{\beta_1 - \alpha} + \Gamma_2 |t-s|^{\gamma + \beta_2} + \kappa_{m,d} \Gamma_3 |t-s|^{\beta_3}.$$

Remark 5.6. It should be possible to formulate Lemma 5.5 at the generality of Theorem 1.1. However, such generality is irrelevant to prove Theorem 5.2, and we do not pursue the generality to simplify the presentation.

Proof. Here, we consider dyadic partitions. Fix $s < t$, and set

$$A_{s,t}^n := \sum_{k=0}^{2^n - 1} A_{s + \frac{k}{2^n}(t-s), s + \frac{k+1}{2^n}(t-s)}.$$

Since $\mathcal{A}_{s,t} = \lim_{n \to \infty} A_{s,t}^n$, it suffices to show

$$\|A_{s,t}^n - A_{s,t}^{n+1}\|_{L_m(\mathbb{P})} \lesssim 2^{-n\delta} (\Gamma_1 |t-s|^{\beta_1 - \alpha} + \Gamma_2 |t-s|^{\gamma + \beta_2} + \kappa_{m,d} \Gamma_3 |t-s|^{\beta_3})$$

for some $\delta > 0$ and all sufficiently large n. As in the proof of Theorem 1.1, we decompose

$$A_{s,t}^n - A_{s,t}^{n+1} = \sum_{l=0}^{L} \sum_{j: l+jL \leq 2^n - 1} \{\delta A_{l+jL}^n - \mathbb{E}[\delta A_{l+jL}^n | \mathcal{F}_{l+(j-1)L+1}^n]\} + \sum_{l=0}^{L} \sum_{j: l+jL \leq 2^n - 1} \mathbb{E}[\delta A_{l+jL}^n | \mathcal{F}_{l+(j-1)L+1}^n].$$

By the BDG inequality,

$$\left\| \sum_{j: l+jL \leq 2^n - 1} \{\delta A_{l+jL}^n - \mathbb{E}[\delta A_{l+jL}^n | \mathcal{F}_{l+(j-1)L+1}^n]\} \right\|_{L_m(\mathbb{P})}$$

$$\lesssim \kappa_{m,d} \left(\sum_{j: l+jL \leq 2^n - 1} \|\delta A_{l+jL}^n\|_{L_m(\mathbb{P})}^2 \right)^{\frac{1}{2}} \leq \kappa_{m,d} \Gamma_3 \left(\sum_{j: l+jL \leq 2^n - 1} (2^{-n} |t-s|)^{2\beta_3} \right)^{\frac{1}{2}}.$$

Thus,

$$\left\| \sum_{l=0}^{L} \sum_{j: l+jL \leq 2^n - 1} \{\delta A_{l+jL}^n - \mathbb{E}[\delta A_{l+jL}^n | \mathcal{F}_{l+(j-1)L+1}^n]\} \right\|_{L_m(\mathbb{P})} \lesssim \kappa_{m,d} \Gamma_3 L^{\frac{1}{2}} 2^{-n(\beta_3 - \frac{1}{2})} |t-s|^{\beta_3}.$$

Furthermore,

$$\Big\| \sum_{l=0}^{L} \sum_{j:l+jL\leq 2^n-1} \mathbb{E}[\delta A_{l+jL}^n | \mathcal{F}_{l+(j-1)L+1}^n] \Big\|_{L_m(\mathbb{P})}$$

$$\leq \sum_{l=0}^{L} \sum_{j:l+jL\leq 2^n-1} \|\mathbb{E}[\delta A_{l+jL}^n | \mathcal{F}_{l+(j-1)L+1}^n]\|_{L_m(\mathbb{P})}$$

$$\leq \Gamma_1 \sum_{l=0}^{L} \sum_{j:l+jL\leq 2^n-1} \left(\frac{L-1}{2^n}|t-s|\right)^{-\alpha} (2^{-n}|t-s|)^{\beta_1}$$

$$+ \Gamma_2 \sum_{l=0}^{L} \sum_{j:l+jL\leq 2^n-1} \left(\frac{L}{2^n}|t-s|\right)^{\gamma} (2^{-n}|t-s|)^{\beta_2}$$

$$\lesssim \Gamma_1 2^{-n(\beta_1-\alpha-1)} L^{-\alpha}|t-s|^{\beta_1-\alpha} + \Gamma_2 2^{-n(\gamma+\beta_2-1)} L^{\gamma}|t-s|^{\gamma+\beta_2}.$$

Therefore,

$$\|A_{s,t}^n - A_{s,t}^{n+1}\|_{L_m(\mathbb{P})} \lesssim \Gamma_1 2^{-n(\beta_1-\alpha-1)} L^{-\alpha}|t-s|^{\beta_1-\alpha}$$

$$+ \Gamma_2 2^{-n(\gamma+\beta_2-1)} L^{\gamma}|t-s|^{\gamma+\beta_2} + \kappa_{m,d}\Gamma_3 L^{\frac{1}{2}} 2^{-n(\beta_3-\frac{1}{2})}|t-s|^{\beta_3}.$$

We choose $L = \lfloor 2^{n\varepsilon} \rfloor$ with $\varepsilon \in (0,1)$ so that

$$\min\left\{\beta_1 - \alpha - 1 + \alpha\varepsilon, \gamma + \beta_2 - 1 - \gamma\varepsilon, \beta_3 - \frac{1+\varepsilon}{2}\right\} > 0.$$

Namely,

$$\frac{1+\alpha-\beta_1}{\alpha} < \varepsilon < \min\left\{\frac{\gamma+\beta_2-1}{\gamma}, 2\beta_3 - 1\right\}.$$

Such an ε exists exactly under our assumption (5.6). □

We mentioned that a solution to (5.4) is controlled by B. Here comes a more precise statement. We fix $\alpha \in (\frac{1}{2}, H)$ and let X be a solution to (5.4). We have the estimates

$$\left|\int_s^t b(X_r)\,\mathrm{d}r - b(X_s)(t-s)\right| \leq \|b\|_{C_b^1}\|X\|_{C^\alpha}(t-s)^{1+\alpha},$$

$$\left|\int_s^t \sigma(X_r)\,\mathrm{d}B_r - \sigma(X_s)B_{s,t}\right| \lesssim_\alpha \|\sigma\|_{C_b^1}\|X\|_{C^\alpha}\|B\|_{C^\alpha}|t-s|^{2\alpha}. \tag{5.7}$$

Furthermore, the a priori estimate of the Young differential equation ([14, Proposition 8.1]) gives

$$\|X\|_{C^\alpha([0,T])} \lesssim_{T,\alpha} |x| + (\|b\|_{C_b^1} + \|\sigma\|_{C_b^1}\|B\|_{C^\alpha})(1 + \|b\|_{C_b^1} + \|\sigma\|_{C_b^1}\|B\|_{C^\alpha}). \tag{5.8}$$

Therefore, we have

$$X_t = X_s + \sigma(X_s)B_{s,t} + b(X_s)(t-s) + R_{s,t},$$

where

$$|R_{s,t}| \lesssim_{T,\alpha} (|x| + \|b\|_{C_b^1} + \|\sigma\|_{C_b^1}\|B\|_{C^\alpha})(1 + \|b\|_{C_b^1} + \|\sigma\|_{C_b^1}\|B\|_{C^\alpha})^2|t-s|^{2\alpha}. \tag{5.9}$$

This motivates the following definition. Recall that B is an $(\mathcal{F}_t)$-fractional Brownian motion in the sense of Definition 3.1.

Definition 5.7. Let Z be a random path in $C^\alpha([0,T], \mathbb{R}^d)$. For $\beta \in (\alpha, \infty)$, we write $Z \in \mathcal{D}(\alpha, \beta)$, if for every $s < t$, we have

$$Z_t = z^{(1)}(s) + z^{(2)}(s)B_t + z^{(3)}(s)(t-s) + R_{s,t},$$

where

○ the random variables $z^{(1)}(s), z^{(3)}(s) \in \mathbb{R}^d$, and $z^{(2)}(s) \in \mathcal{M}_d$ are $\mathcal{F}_s$-measurable and
○ there exists a (random) constant $C \in [0, \infty)$, such that for all $s < t$

$$|R_{s,t}| \le C|t-s|^\beta.$$

We set

$$\|Z\|_{\mathcal{D}(\alpha,\beta)} := \|Z\|_{C^\alpha([0,T])} + \|z^{(2)}\|_{L_\infty([0,T])} + \|z^{(3)}\|_{L_\infty([0,T])} + \sup_{0 \le s < t \le T} \frac{|R_{s,t}|}{(t-s)^\beta}.$$

Furthermore, we set

$$\mathcal{D}_0(\alpha, \beta) := \{Z \in \mathcal{D}(\alpha, \beta) \mid z^{(3)} \equiv 0\},$$
$$\mathcal{D}_1(\alpha, \beta) := \{Z \in \mathcal{D}(\alpha, \beta) \mid z^{(2)}(s) \text{ is invertible for all } s \in [0, T)\},$$

and $\|Z\|_{\mathcal{D}_1(\alpha,\beta)} := \|Z\|_{\mathcal{D}_1(\alpha,\beta)} + \sup_{s \in [0,T]} |(z^{(2)}(s))^{-1} z^{(3)}(s)|$.

Proposition 5.8. *Let $f \in C_b^1(\mathbb{R}^d; \mathbb{R}^d)$ and $Z \in \mathcal{D}_1(\alpha, \beta)$. If*

$$\max\left\{\frac{1-H}{\beta}, \frac{2-3H+H^2}{\beta+H-H^2}, \frac{3-\sqrt{3}}{2H} - 1\right\} < \delta < 1, \tag{5.10}$$

and if α is sufficiently close to H, then for every $m \in [2, \infty)$,

$$\left\| \int_s^t f(Z_r)\, \mathrm{d}B_r - \frac{f(Z_s) + f(Z_t)}{2} \cdot B_{s,t} \right\|_{L_m(\mathbb{P})}$$
$$\lesssim_{T,\alpha,\delta,m} \|f\|_{C^\delta} \|\|Z\|_{\mathcal{D}_1(\alpha,\beta)}^\delta\|_{L_{3m}(\mathbb{P})} (1 + \|\|Z\|_{\mathcal{D}_1(\alpha,\beta)}\|_{L_{3m}(\mathbb{P})}) |t-s|^{H+\alpha\delta}.$$

If $Z \in \mathcal{D}_0(\alpha, \beta)$, a similar estimate holds with $\|Z\|_{\mathcal{D}_1(\alpha,\beta)}$ replaced by $\|Z\|_{\mathcal{D}(\alpha,\beta)}$.

Proof. Our tool is Lemma 5.5. Since arguments are similar, we only prove the claim for $Z \in \mathcal{D}_1(\alpha, \beta)$. We set

$$A_{s,t} := \frac{f(Z_s) + f(Z_t)}{2} \cdot B_{s,t}.$$

We have

$$\delta A_{s,u,t} = -\frac{f(Z_u) - f(Z_s)}{2} B_{u,t} + \frac{f(Z_t) - f(Z_u)}{2} B_{s,u}.$$

Hence

$$\|\delta A_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\delta} \|\|Z\|_{C^\alpha}^\delta\|_{L_{2m}(\mathbb{P})} |t-s|^{H+\alpha\delta}. \tag{5.11}$$

Let $v < s$ with $t - s \le s - v$. As $Z \in \mathcal{D}_1(\alpha, \beta)$, we write

$$Z_r = z^{(1)}(v) + z^{(2)}(v)B_r + z^{(3)}(v)(r-v) + R_{v,r}, \quad r \in [s, t].$$

Then, if we write $\hat{Z}_r := z^{(1)}(v) + z^{(2)}(v)B_r + z^{(3)}(v)(r - v)$,

$$|f(Z_r) - f(\hat{Z}_r)| \leq \|f\|_{C^\delta}|R_{v,r}|^\delta \leq \|f\|_{C^\delta}\|Z\|_{\mathcal{D}(\alpha,\beta)}^\delta|v - r|^{\delta\beta}.$$

Hence, if we write $\hat{A}_{s,t} := \frac{f(\hat{Z}_s) + f(\hat{Z}_t)}{2} \cdot B_{s,t}$, then

$$\|\delta A_{s,u,t} - \delta\hat{A}_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\delta,T} \|f\|_{C^\delta}\|\|Z\|_{\mathcal{D}(\alpha,\beta)}^\delta\|_{L_{2m}(\mathbb{P})}|t - v|^{\delta\beta}|t - s|^H. \tag{5.12}$$

Next, we will estimate $\|\mathbb{E}[\delta\hat{A}_{s,u,t}|\mathcal{F}_v]\|_{L_m(\mathbb{P})}$. The rest of the calculation resembles the proof of Proposition 3.5. We write $Y_r := \mathbb{E}[B_r|\mathcal{F}_v]$ and $\tilde{B}_r := B_r - Y_r$ as before. We can decompose

$$\hat{Z}_r = \hat{Y}_r + z^{(2)}(v)\tilde{B}_r, \quad \text{with } \hat{Y}_r := z^{(1)}(v) + z^{(2)}(v)Y_r + z^{(3)}(v)(r - v),$$

where $\hat{Y}_r$ is $\mathcal{F}_v$-measurable and $\tilde{B}_r$ is independent of $\mathcal{F}_v$. We set

$$\hat{a}_0(r) := \mathbb{E}[f(\hat{Y}_r + z^{(2)}(v)\tilde{B}_r)|\mathcal{F}_v] = \mathbb{E}[f(z^{(2)}(v)\{(z^{(2)}(v))^{-1}\hat{Y}_r + \tilde{B}_r\})|\mathcal{F}_v],$$
$$\hat{a}_i^j(r) := 2H(s - v)^{-2H}\mathbb{E}[f^j(\hat{Y}_r + z^{(2)}(v)\tilde{B}_r)\tilde{B}_r^i|\mathcal{F}_v].$$

Then, as in the proof of Proposition 3.5, we have the decomposition

$$\mathbb{E}[\delta\hat{A}_{s,u,t}|\mathcal{F}_v] = \hat{D}_{s,u,t}^0 + \sum_{i=1}^d \hat{D}_{s,u,t}^i,$$

where, as in (3.8) and (3.11),

$$\hat{D}_{s,u,t}^0 := (\hat{a}_0(t) - \hat{a}_0(u)) \cdot Y_{s,u} + (\hat{a}_0(s) - \hat{a}_0(u)) \cdot Y_{u,t},$$
$$D_{s,u,t}^i := \mathbb{E}[(\hat{a}_i^i(s)\tilde{B}_s^i + \hat{a}_i^i(t)\tilde{B}_t^i)\tilde{B}_{s,t}^i|\mathcal{F}_v]$$
$$\qquad - \mathbb{E}[(\hat{a}_i^i(s)\tilde{B}_s^i + \hat{a}_i^i(u)\tilde{B}_u^i)\tilde{B}_{s,u}^i|\mathcal{F}_v] - \mathbb{E}[(\hat{a}_i^i(u)\tilde{B}_u^i + \hat{a}_i^i(t)\tilde{B}_t^i)\tilde{B}_{u,t}^i|\mathcal{F}_v]$$
$$\qquad = (\hat{a}_i^i(t) - \hat{a}_i^i(u))\mathbb{E}[\tilde{B}_t^i\tilde{B}_{s,t}^i] + (\hat{a}_i^i(s) - \hat{a}_i^i(u))\mathbb{E}[\tilde{B}_s^i\tilde{B}_{s,t}^i]$$
$$\qquad - (\hat{a}_i^i(s) - \hat{a}_i^i(u))\mathbb{E}[\tilde{B}_s^i\tilde{B}_{s,u}^i] - (\hat{a}_i^i(t) - \hat{a}_i^i(u))\mathbb{E}[\tilde{B}_t^i\tilde{B}_{u,t}^i].$$

The map $\mathbb{R}^d \ni x \mapsto f(z^{(2)}(v)x) \in \mathbb{R}^d$ belongs to $C^\delta(\mathbb{R}^d)$ with its norm bounded by

$$|z^{(2)}(v)|^\delta\|f\|_{C^\delta(\mathbb{R}^d)}.$$

Therefore, by repeating the argument used to obtain (3.9), we obtain

$$|\hat{D}_{s,u,t}^0| \leq |z^{(2)}(v)|^\delta\|f\|_{C^\delta(\mathbb{R}^d)}\Big\{(s - v)^{\delta H-1}(t - s)(|Y_{s,u}| + |Y_{u,t}|)$$
$$+ (s - v)^{(\delta-1)H}\Big(|(z^{(2)}(v))^{-1}\hat{Y}_{s,u}||Y_{u,t}| + |z^{(2)}(v)^{-1}\hat{Y}_{u,t}||Y_{s,u}|\Big)\Big\}.$$

Referring to (3.4), we have

$$\|(z^{(2)}(v))^{-1}\hat{Y}_{s,u}\|_{L_m(\mathbb{P})} \lesssim (s - v)^{H-1}(t - s) + \|z^{(2)}(v)^{-1}z^{(3)}(v)\|_{L_m(\mathbb{P})}(t - s).$$

Therefore,

$$\|\hat{D}_{s,u,t}^0\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\delta}\||z^{(2)}(v)|^\delta\|_{L_{3m}(\mathbb{P})}$$
$$\times \{(s - v)^{(\delta+1)H-2}(t - s)^2 + \|z^{(2)}(v)^{-1}z^{(3)}(v)\|_{L_{3m}(\mathbb{P})}(s - v)^{\delta H-1}(t - s)^2\}.$$

Similarly as before, we have

$$|a_i^i(t) - a_i^i(s)| \lesssim |z^{(2)}(v)|^\delta \|f\|_{C^\delta(\mathbb{R}^d)} (s-v)^{(\delta-2)H} (|(z^{(2)}(v))^{-1}\hat{Y}_{s,u}| + (s-v)^{H-1}(t-s)).$$

As $H > \frac{1}{2}$, by Lemma 3.2, we have

$$|\mathbb{E}[\tilde{B}_s\tilde{B}_{s,t}]| \lesssim (s-v)^{2H-1}(t-s).$$

Therefore,

$$\|\hat{D}_{s,u,t}^i\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\delta}\||z^{(2)}(v)|^\delta\|_{L_{3m}(\mathbb{P})}$$
$$\times \{(s-v)^{(\delta+1)H-2}(t-s)^2 + \|z^{(2)}(v)^{-1}z^{(3)}(v)\|_{L_{3m}(\mathbb{P})}(s-v)^{\delta H-1}(t-s)^2\}.$$

Combining our estimates, we have

$$\|\mathbb{E}[\delta\hat{A}_{s,u,t}|\mathcal{F}_v]\|_{L_m(\mathbb{P})} \lesssim_T \|f\|_{C^\delta}\||z^{(2)}(v)|^\delta\|_{L_{3m}(\mathbb{P})}$$
$$\times (1 + \|z^{(2)}(v)^{-1}z^{(3)}(v)\|_{L_{3m}(\mathbb{P})})(s-v)^{(\delta+1)H-2}(t-s)^2. \tag{5.13}$$

Hence, combining (5.11), (5.12), and (5.13)

$$\|\delta A_{s,u,t}\|_{L_m(\mathbb{P})} \lesssim \|f\|_{C^\delta}\|\|Z\|_{C^\alpha}^\delta\|_{L_{2m}(\mathbb{P})}|t-s|^{H+\alpha\delta},$$

and

$$\|\mathbb{E}[\delta A_{s,u,t}|\mathcal{F}_v]\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\delta,T} \|f\|_{C^\delta}\||z^{(2)}(v)|^\delta\|_{L_{3m}(\mathbb{P})}$$
$$\times (1 + \|z^{(2)}(v)^{-1}z^{(3)}(v)\|_{L_{3m}(\mathbb{P})})(s-v)^{(\delta+1)H-2}(t-s)^2$$
$$+ \|f\|_{C^\delta}\|\|Z\|_{\mathcal{D}(\alpha,\beta)}^\delta\|_{L_{2m}(\mathbb{P})}|t-v|^{\delta\beta}|t-s|^H.$$

To apply Lemma 5.5, we need

$$\delta\beta + H > 1, \quad \frac{1-(1+\delta)H}{2-(1+\delta)H} < \min\left\{\frac{\delta\beta + H - 1}{\delta\beta}, 2(H+\alpha\delta) - 1\right\},$$

which, if α is sufficiently close to H, are fulfilled under (5.10). □

Proof of Theorem 5.2. We first prove the pathwise uniqueness. We suppose that the assumption in the case 2 mentioned in Theorem 5.2 holds, and the other cases will be discussed later. Let $X^{(1)}$ and $X^{(2)}$ be $(\mathcal{F}_t)$-adapted solutions. Our strategy is similar to Proposition 5.1, but, here, we must construct the integral (5.5) stochastically. For each $k \in \mathbb{N}$, we set

$$\lambda_k := \inf_{x:|x|\le k} \inf_{y:|y|=1} y \cdot \sigma(x)y,$$

we let $\sigma^k \in C^{1+\delta}(\mathbb{R}^d; \mathcal{M}_d)$ be such that $\sigma^k = \sigma$ in $\{x \mid |x| \le k\}$ and

$$\inf_{x\in\mathbb{R}^d} \inf_{y:|y|=1} y \cdot \sigma^k(x)y \ge \frac{\lambda_k}{2},$$

and we set

$$X_t^{(i),k} := x + \int_0^t b(X_r^{(i)})\,dr + \int_0^t \sigma^k(X_r^{(i)})\,dB_r, \quad i = 1, 2.$$

If we write

$$\Omega_k := \{\omega \in \Omega \mid \sup_{t \in [0,T]} \max\{|X_t^{(1)}(\omega)|, |X_t^{(2)}(\omega)|\} \le k\},$$

then in the event Ω_k, we have $X_t^{(i)} = X_t^{(i),k}$, $t \in [0,T]$.

Let $\{\sigma^{k,n}\}_{n=1}^{\infty}$ be a smooth approximation of σ^k. In general, we can only guarantee the convergence in $C^{1+\delta'}(\mathbb{R}^d, \mathcal{M}_d)$ for any $\delta' < \delta$, which is still sufficient to make the following argument work. To simplify the notation, we assume that we can take $\delta' = \delta$.

We have

$$\int_0^t \sigma^k(X_r^{(i)}) \, \mathrm{d}B_r = \lim_{n \to \infty} \int_0^t \sigma^{k,n}(X_r^{(i)}) \, \mathrm{d}B_r$$

and in Ω_k

$$\int_0^t \{\sigma^{k,n}(X_r^{(1)}) - \sigma^{k,n}(X_r^{(2)})\} \, \mathrm{d}B_r = \int_0^t \{X_r^{(1),k} - X_r^{(2),k}\} \, \mathrm{d}V_r^{k,n},$$

where

$$V_t^{k,n} := \int_0^t \int_0^1 \nabla \sigma^{k,n}(\theta X_r^{(1),k} + (1-\theta) X_r^{(2),k}) \, \mathrm{d}\theta \, \mathrm{d}B_r.$$

For a fixed $\theta \in (0,1)$, we set

$$Z_t^{\theta,k} := \theta X_t^{(1),k} + (1-\theta) X_t^{(2),k}.$$

By the a priori estimate (5.9), for $\alpha \in (\frac{1}{2}, H)$, we have

$$\begin{aligned}
Z_t^{\theta,k} - Z_s^{\theta,k} &= \{\theta \sigma^k(X_s^{(1),k}) + (1-\theta)\sigma^k(X_s^{(1),k})\} B_{s,t} \\
&\quad + \{\theta b(X_s^{(1),k}) + (1-\theta) b(X_s^{(1),k})\}(t-s) + R_{s,t}
\end{aligned}$$

with

$$|R_{s,t}| \lesssim_{T,\alpha} (1 + |x| + \|b\|_{C_b^1} + \|\sigma\|_{C_b^1} \|B\|_{C^\alpha})^3 |t-s|^{2\alpha}.$$

Note that we have

$$\inf_{y:|y|=1} y \cdot \{\theta \sigma^k(X_s^{(1),k}) + (1-\theta)\sigma^k(X_s^{(1),k})\} y \ge \frac{\lambda_k}{2},$$

and hence

$$|\{\theta \sigma^k(X_s^{(1),k}) + (1-\theta)\sigma^k(X_s^{(1),k})\}^{-1}| \lesssim \lambda_k^{-1}.$$

Therefore, we have $Z^{\theta,k} \in \mathcal{D}_1(\alpha, 2\alpha)$ with

$$\|Z^{\theta,k}\|_{\mathcal{D}_1(\alpha,2\alpha)} \lesssim_{T,\alpha} (1 + |x| + \|b\|_{C_b^1} + \|\sigma\|_{C_b^1} \|B\|_{C^\alpha})^3 + (1 + \lambda_k^{-1})(\|b\|_{L_\infty} + \|\sigma\|_{L_\infty}).$$

Since

$$\max\left\{\frac{1-H}{2\alpha}, \frac{2 - 3H + H^2}{2\alpha + H - H^2}, \frac{3 - \sqrt{3}}{2H} - 1\right\} = \frac{2 - 3H + H^2}{2\alpha + H - H^2} < \delta$$

if α is sufficiently close to H, by Proposition 5.8,

$$\|V_{s,t}^{k,n_1} - V_{s,t}^{k,n_2}\|_{L_m(\mathbb{P})} \lesssim_{T,\alpha,\delta,b,\sigma,k,m} \|\sigma^{k,n_1} - \sigma^{k,n_2}\|_{C^{1+\delta}(\mathbb{R}^d,\mathcal{M}_d)}|t-s|^H.$$

By Kolmogorov's continuity theorem, the sequence $(V^{k,n})_{n\in\mathbb{N}}$ converges to some V^k in $C^\alpha([0,T],\mathbb{R}^d)$.

Therefore, we conclude that almost surely in Ω_k, the path $z = X^{(1)} - X^{(2)}$ solves the linear Young equation

$$z_t = \int_0^t z_r \, dU_r^k, \quad U_t^k := V_t^k + \int_0^t \int_0^1 \nabla b(\theta X_r^{(1),k} + (1-\theta)X_r^{(2),k}) \, d\theta \, dr,$$

and hence $X^{(1)} = X^{(2)}$. Since $\mathbb{P}(\Omega_k) \to 1$, we conclude $X^{(1)} = X^{(2)}$ almost surely. Thus, we completed the proof of the uniqueness under the case 2. The other cases can be handled similarly. Indeed, under the case 1, we have $X^{(i)} \in \mathcal{D}_0(\alpha, 2\alpha)$, and under the case 3, we have $X^{(i)} \in \mathcal{D}_0(\alpha, 1)$.

Now, it remains to prove the existence of a strong solution. However, in view of the Yamada-Watanabe theorem (Proposition B.2), it suffices to show the existence of a weak solution, which will be proved in Lemma B.3 based on a standard compactness argument. □

Remark 5.9. We believe that our assumption in Theorem 5.2 is not optimal. One possible approach to relax the assumption is to consider a higher order approximation in (5.7). Yet, we believe that this will not lead to an optimal assumption, as long as we apply Lemma 5.5. Thus, finding an optimal regularity of σ for the pathwise uniqueness and the strong existence remains an interesting open question that is likely to require a new idea.

A. Proofs of technical results

Proofs of Lemmas 2.3 and 2.4

Proof of Lemma 2.3 without (1.12). Let us first recall our previous strategy under (1.12). We used Lemma 2.1 to write

$$A_{t_0,t_N} - \sum_{i=1}^{N} A_{t_{i-1},t_i} = \sum_{n\in\mathbb{N}_0} \sum_{i=0}^{2^n-1} R_i^n. \tag{A.1}$$

Then, we decomposed

$$\sum_{i=0}^{2^n-1} R_i^n = \sum_{l=0}^{L-1} \sum_{j=0}^{2^n/L} \left(R_{Lj+l}^n - \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \mathbb{1}_{\{j\geq1\}} \right)$$
$$+ \sum_{l=0}^{L-1} \sum_{j=1}^{2^n/L} \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n], \tag{A.2}$$

where $\mathcal{F}_k^n := \mathcal{F}_{t_0 + \frac{k}{2^n}(t_N-t_0)}$. We estimated the first term of (A.2) by the BDG inequality and (1.8):

$$\|\sum_{l=0}^{L-1} \sum_{j=0}^{2^n/L} \left(R_{Lj+l}^n - \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n] \mathbb{1}_{\{j\geq1\}} \right)\|_{L_m(\mathbb{P})} \lesssim \kappa_{m,d} \Gamma_2 L^{\frac{1}{2}} 2^{-n(\beta_2-\frac{1}{2})}|t_N-t_0|^{\beta_2}. \tag{A.3}$$

In the proof under (1.12), we estimated the second term of (A.2) by the triangle inequality and (1.7):

$$\|\sum_{l=0}^{L-1} \sum_{j=1}^{2^n/L} \mathbb{E}[R_{Lj+l}^n | \mathcal{F}_{L(j-1)+l+1}^n]\|_{L_m(\mathbb{P})} \lesssim_\alpha \Gamma_1 L^{-\alpha} 2^{-(\beta_1-\alpha-1)n}|t_N-t_0|^{\beta_1-\alpha}. \tag{A.4}$$

Then, we chose L so that both (A.3) and (A.4) are summable with respect to n, for which to be possible, we had to assume (1.12).

In order to remove the assumption (1.12), let us think again why we did the decomposition (A.2). This is because we do not want to apply the simplest estimate, namely, the triangle inequality, since the condition (1.7) implies that $(A_{s,t})_{[s,t]\in\pi}$ are not so correlated. This point of view teaches us that, to estimate

$$\sum_{l=0}^{L-1}\sum_{j=1}^{2^n/L}\mathbb{E}[R_{Lj+l}^n|\mathcal{F}_{L(j-1)+l+1}^n],$$

we should not simply apply the triangle inequality. That is, we should again apply the decomposition as in (A.2).

To carry out our new strategy, set

$$S_j^{(1),l} := R_{Lj+l}^n, \qquad \mathcal{G}_j^{(1),l} := \mathcal{F}_{L(j-1)+l+1}^n, \qquad j\in\mathbb{N}.$$

For this new strategy, we can set $L := \max\{2, \lceil M \rceil\}$. In particular, L does not depend on n. We use the convention $\mathbb{E}[X|\mathcal{G}_j^{(1),l}] = 0$ for $j \leq 0$. Then,

$$\sum_{l=0}^{L-1}\sum_{j=1}^{2^n/L}\mathbb{E}[R_{Lj+l}^n|\mathcal{F}_{L(j-1)+l+1}^n] = \sum_{l=0}^{L-1}\sum_{j=1}^{L^{-1}2^n}\mathbb{E}[S_j^{(1),l}|\mathcal{G}_j^{(1),l}] = \sum_{l_1=0}^{L-1}\sum_{l_2=0}^{L-1}\sum_{j=0}^{L^{-2}2^n}\mathbb{E}[S_{jL+l_2}^{(1),l_1}|\mathcal{G}_{jL+l_2}^{(1),l_1}]. \quad \text{(A.5)}$$

By setting

$$S_j^{(2),l_1,l_2} := S_{jL+l_2}^{(1),l_1}, \qquad \mathcal{G}_j^{(2),l_1,l_2} := \mathcal{G}_{(j-1)L+l_2}^{(1),l_1},$$

the quantity (A.5) equals to

$$\sum_{l_1=0}^{L}\sum_{l_2=0}^{L}\sum_{j=0}^{L^{-2}2^n}\left(\mathbb{E}[S_j^{(2),l_1,l_2}|\mathcal{G}_{j+1}^{(2),l_1,l_2}] - \mathbb{E}[S_j^{(2),l_1,l_2}|\mathcal{G}_j^{(2),l_1,l_2}]\right) + \sum_{l_1=0}^{L}\sum_{l_2=0}^{L}\sum_{j=0}^{L^{-2}2^n}\mathbb{E}[S_j^{(2),l_1,l_2}|\mathcal{G}_j^{(2),l_1,l_2}].$$

The $L_m(\mathbb{P})$-norm of the first term can be estimated by the BDG inequality: it is bounded by

$$2\kappa_{m,d}\sum_{l_1,l_2\leq L}\left(\sum_{j\leq L^{-2}2^n}\|\mathbb{E}[S_j^{(2),l_1,l_2}|\mathcal{G}_{j+1}^{(2),l_1,l_2}]\|_{L_m(\mathbb{P})}^2\right)^{\frac{1}{2}}. \quad \text{(A.6)}$$

By (1.7), we have

$$\|\mathbb{E}[S_j^{(2),l_1,l_2}|\mathcal{G}_{j+1}^{(2),l_1,l_2}]\|_{L_m(\mathbb{P})} \leq \Gamma_1(L2^{-n}|t_N - t_0|)^{-\alpha}(2^{-n}|t_N - t_0|)^{\beta_1}.$$

Therefore, the quantity (A.6) is bounded by

$$2\kappa_{m,d}\Gamma_1 L^{1-\alpha}2^{-n(\beta_1-\alpha-\frac{1}{2})}|t_N - t_0|^{\beta_1-\alpha}.$$

As the reader may realize, we will repeat the same argument for

$$\sum_{l_1=0}^{L}\sum_{l_2=0}^{L}\sum_{j=1}^{L^{-2}2^n}\mathbb{E}[S_j^{(2),l_1,l_2}|\mathcal{G}_{j-1}^{(2),l_1,l_2}]$$

and continue. To state more precisely, set inductively,

$$S_j^{(k),l_1,...,l_k} := S_{Lj+l_k}^{(k-1),l_1,...,l_{k-1}}, \quad \mathcal{G}_j^{(k),l_1,...,l_k} := \mathcal{G}_{L(j-1)+l_k}^{(k-1),l_1,...,l_{k-1}}, \quad j \in [1, L^{-k}2^n] \cap \mathbb{N}. \tag{A.7}$$

We claim that, if $L^k \le 2^n$, we have

$$\| \sum_{i=0}^{2^n-1} R_i^n \|_{L_m(\mathbb{P})} \le 2\kappa_{m,d} \Gamma_2 L^{\frac{1}{2}} 2^{-n(\beta_2 - \frac{1}{2})} |t_N - t_0|^{\beta_2}$$

$$+ 2\kappa_{m,d} \Gamma_1 \Big(\sum_{j=1}^{k-1} L^{\frac{j}{2}-(j-1)\alpha} \Big) L^{\frac{1}{2}} 2^{-n(\beta_1-\alpha-\frac{1}{2})} |t_N - t_0|^{\beta_1 - \alpha}$$

$$+ \| \sum_{l_1,...,l_k \le L} \sum_{j \le L^{-k}2^n} \mathbb{E}[S_j^{(k),l_1,...,l_k} | \mathcal{G}_j^{(k),l_1,...,l_k}] \|_{L_m(\mathbb{P})}.$$

The proof of the claim is based on induction. The case $k = 1$ and $k = 2$ is obtained. Suppose that the claim is correct for $k \ge 2$, and consider the case $k + 1$. Again, decompose

$$\sum_{l_1,...,l_k \le L} \sum_{j \le L^{-k}2^n} \mathbb{E}[S_j^{(k),l_1,...,l_k} | \mathcal{G}_j^{(k),l_1,...,l_k}]$$

$$= \sum_{l_1,...,l_k,l_{k+1} \le L} \sum_{j \le L^{-(k+1)}2^n} \big(\mathbb{E}[S_j^{(k+1),l_1,...,l_k} | \mathcal{G}_{j+1}^{(k+1),l_1,...,l_k,l_{k+1}}] - \mathbb{E}[S_j^{(k+1),l_1,...,l_k} | \mathcal{G}_j^{(k+1),l_1,...,l_k,l_{k+1}}] \big)$$

$$+ \sum_{l_1,...,l_k,l_{k+1} \le L} \sum_{j \le L^{-(k+1)}2^n} \mathbb{E}[S_j^{(k+1),l_1,...,l_k} | \mathcal{G}_j^{(k+1),l_1,...,l_k,l_{k+1}}].$$

To prove the claim, it suffices to estimate the first sum in the right-hand side. By the BDG inequality, its $L_m(\mathbb{P})$-norm is bounded by

$$2\kappa_{m,d} \sum_{l_1,...,l_k,l_{k+1} \le L} \Big(\sum_{j \le L^{-(k+1)}2^n} \| \mathbb{E}[S_j^{(k+1),l_1,...,l_k,l_{k+1}} | \mathcal{G}_{j+1}^{(k+1),l_1,...,l_k,l_{k+1}}] \|_{L_m(\mathbb{P})}^2 \Big)^{1/2}. \tag{A.8}$$

By (1.7),

$$\| \mathbb{E}[S_j^{(k+1),l_1,...,l_k,l_{k+1}} | \mathcal{G}_{j+1}^{(k+1),l_1,...,l_k,l_{k+1}}] \|_{L_m(\mathbb{P})} \le \Gamma_1 (L^k 2^{-n} |t_N - t_0|)^{-\alpha} (2^{-n} |t_N - t_0|)^{\beta_1}. \tag{A.9}$$

Therefore, the quantity (A.8) is bounded by

$$2\kappa_{m,d} \Gamma_1 L^{\frac{1}{2}} L^{(\frac{1}{2}-\alpha)k} 2^{-n(\beta_1-\alpha-\frac{1}{2})} |t_N - t_0|^{(\beta_1-\alpha)}$$

and the claim follows.

Now let us estimate

$$\| \sum_{l_1,...,l_k \le L} \sum_{j \le L^{-k}2^n} \mathbb{E}[S_j^{(k),l_1,...,l_k} | \mathcal{G}_j^{(k),l_1,...,l_k}] \|_{L_m(\mathbb{P})} \tag{A.10}$$

by the triangle inequality:

$$\| \sum_{l_1,...,l_k \le L} \sum_{j \le L^{-k}2^n} \mathbb{E}[S_j^{(k),l_1,...,l_k} | \mathcal{G}_j^{(k),l_1,...,l_k}] \|_{L_m(\mathbb{P})}$$

$$\le \sum_{l_1,...,l_k \le L} \sum_{j \le L^{-k}2^n} \| \mathbb{E}[S_j^{(k),l_1,...,l_k} | \mathcal{G}_j^{(k),l_1,...,l_k}] \|_{L_m(\mathbb{P})}.$$

By (1.7) (or essentially the estimate (A.9)),

$$\|\mathbb{E}[S_j^{(k),l_1,\ldots,l_k}|\mathcal{G}_j^{(k),l_1,\ldots,l_k}]\|_{L_m(\mathbb{P})} \le \Gamma_1(L^k 2^{-n}|t_N - t_0|)^{-\alpha}(2^{-n}|t_N - t_0|)^{\beta_1},$$

and hence, the quantity (A.10) is bounded by

$$\Gamma_1 L^{-\alpha k} 2^{-n(\beta_1 - \alpha - 1)}|t_N - t_0|^{\beta_1 - \alpha}.$$

In conclusion, we obtained for $L^k \le 2^n$,

$$\|\sum_{i=0}^{2^n-1} R_i^n\|_{L_m(\mathbb{P})} \lesssim \kappa_{m,d}\Gamma_2 L^{\frac{1}{2}} 2^{-n(\beta_2 - \frac{1}{2})}|t_N - t_0|^{\beta_2}$$
$$+ \kappa_{m,d}\Gamma_1 f_k 2^{-n(\beta_1 - \alpha - \frac{1}{2})}|t_N - t_0|^{\beta_1 - \alpha} + \Gamma_1 L^{-\alpha k} 2^{-n(\beta_1 - \alpha - 1)}|t_N - t_0|^{\beta_1 - \alpha}, \qquad (A.11)$$

where

$$f_k = \begin{cases} L^{\frac{k}{2} - \alpha(k-1)}\mathbb{1}_{\{k \ge 2\}}, & \text{if } \alpha < \frac{1}{2}, \\ (k-1)L^{\frac{1}{2}}, & \text{if } \alpha \ge \frac{1}{2}. \end{cases} \qquad (A.12)$$

We recall that $L = \max\{2, \lceil M \rceil\}$. To respect $L^k \le 2^n$, we set $k := \lfloor \frac{n \log 2}{\log L} \rfloor$. We then have

$$f_k 2^{-n(\beta_1 - \alpha - \frac{1}{2})} \lesssim_M \begin{cases} 2^{-n(\beta_1 - 1)}, & \text{if } \alpha < \frac{1}{2}, \\ n 2^{-n(\beta_1 - \alpha - \frac{1}{2})}, & \text{if } \alpha \ge \frac{1}{2}, \end{cases}$$

and

$$L^{-\alpha k} 2^{-n(\beta_1 - \alpha - 1)} \lesssim_{\alpha, M} 2^{-n(\beta_1 - 1)}.$$

Therefore, we note that the right-hand side of (A.11) is summable with respect to n and

$$\|A_{t_0, t_N} - \sum_{i=1}^{N} A_{t_{i-1}, t_i}\|_{L_m(\mathbb{P})} \lesssim_{\alpha, \beta_1, \beta_2, M} \kappa_{m,d}\Gamma_2|t_N - t_0|^{\beta_2} + \kappa_{m,d}\Gamma_1|t_N - t_0|^{\beta_1 - \alpha_1}. \qquad \square$$

Proof of Lemma 2.4. The proof is similar to Lemma 2.3. Write

$$\pi' =: \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$$

and

$$\{[s,t] \in \pi \mid t_j \le s < t \le t_{j+1}\} =: \{t_j = t_0^j < t_1^j < \cdots < t_{N_j - 1}^j < t_{N_j}^j = t_{j+1}\}.$$

By (2.6), we have $N \le 3|\pi'|^{-1}T$. We fix a parameter L, which will be chosen later, and set

$$Z_j^{(1),l} := A_{t_{jL+l}, t_{jL+l+1}} - \sum_{k=1}^{N_{jL+l}} A_{t_{k-1}^{jL+l}, t_k^{jL+l}}, \quad \mathcal{H}_j^{(1),l} := \mathcal{F}_{t_{(j-1)L+l+1}}.$$

Inductively, we set

$$Z_j^{(k),l_1,\ldots,l_k} := Z_{jL+l_k}^{(k-1),l_1,\ldots,l_{k-1}}, \quad \mathcal{H}_j^{(k),l_1,\ldots,l_k} := \mathcal{H}_{(j-1)L+l_k}^{(k-1),l_1,\ldots,l_{k-1}}.$$

As in Lemma 2.3, for each $k \in \mathbb{N}$, we consider the decomposition

$$A_T^{\pi'} - A_T^{\pi} = A + B,$$

where

$$A := \sum_{p=1}^{k} \sum_{l_1,\ldots,l_p \leq L} \sum_{j \leq NL^{-p}} \left\{ \mathbb{E}[Z_j^{(p),l_1,\ldots,l_p} | \mathcal{H}_{j+1}^{(p),l_1,\ldots,l_p}] - \mathbb{E}[Z_j^{(p),l_1,\ldots,l_p} | \mathcal{H}_j^{(p),l_1,\ldots,l_p}] \right\}$$

$$B := \sum_{l_1,\ldots,l_k \leq L} \sum_{j \leq NL^{-k}} \mathbb{E}[Z_j^{(k),l_1,\ldots,l_k} | \mathcal{H}_j^{(k),l_1,\ldots,l_k}].$$

For this decomposition, we must have $L^k \leq N$. By the BDG inequality and the Cauchy-Schwarz inequality,

$$\|A\|_{L_m(\mathbb{P})} \lesssim \kappa_{m,d} \sum_{p=1}^{k} L^{\frac{p}{2}} \left(\sum_{l_1,\ldots,l_p \leq L} \sum_{j \leq NL^{-p}} \|\mathbb{E}[Z_j^{(p),l_1,\ldots,l_p} | \mathcal{H}_{j+1}^{(p),l_1,\ldots,l_p}]\|_{L_m(\mathbb{P})}^2 \right)^{\frac{1}{2}}.$$

By Lemma 2.3,

$$\|Z_j^{(1),l}\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2} \Gamma_1 |t_{jL+l+1} - t_{jL+l}|^{\beta_1-\alpha} + \kappa_{m,d}\Gamma_2 |t_{jL+l+1} - t_{jL+l}|^{\beta_2}.$$

For $p \geq 2$, by Lemma 2.2 and (2.6),

$$\|\mathbb{E}[Z_j^{(p),l_1,\ldots,l_p} | \mathcal{H}_{j+1}^{(p),l_1,\ldots,l_p}]\|_{L_m(\mathbb{P})} \lesssim_{\beta_1} \Gamma_1 L^{-(p-1)\alpha} |\pi'|^{-\alpha} |\pi'|^{\beta_1}.$$

Therefore, we obtain

$$\|A\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,m,d,T} L^{\frac{1}{2}} (\Gamma_1 |\pi'|^{\beta_1-\alpha-\frac{1}{2}} + \Gamma_2 |\pi'|^{\beta_2-\frac{1}{2}}) + \Gamma_1 f_k |\pi'|^{\beta_1-\alpha-\frac{1}{2}}, \tag{A.13}$$

where f_k is defined by (A.12).

We move to estimate B. By Lemma 2.2 and (2.6),

$$\|\mathbb{E}[Z_j^{(k),l_1,\ldots,l_k} | \mathcal{H}_j^{(k),l_1,\ldots,l_k}]\|_{L_m(\mathbb{P})} \lesssim_{\beta_1} \Gamma_1 L^{-\alpha k} |\pi'|^{\beta_1-\alpha}.$$

Therefore,

$$\|B\|_{L_m(\mathbb{P})} \lesssim_{\beta_1,T} \Gamma_1 L^{-\alpha k} |\pi'|^{\beta_1-\alpha-1}. \tag{A.14}$$

Combining (A.13) and (A.14), we obtain

$$\|A_T^{\pi'} - A_T^{\pi}\|_{L_m(\mathbb{P})}$$
$$\lesssim_{\alpha,\beta_1,\beta_2,m,d,T} L^{\frac{1}{2}} (\Gamma_1 |\pi'|^{\beta_1-\alpha-\frac{1}{2}} + \Gamma_2 |\pi'|^{\beta_2-\frac{1}{2}}) + \Gamma_1 f_k |\pi'|^{\beta_1-\alpha-\frac{1}{2}} + \Gamma_1 L^{-\alpha k} |\pi'|^{\beta_1-\alpha-1}.$$

As in the proof of Lemma 2.3, we set $L := \max\{2, \lceil M \rceil\}$ and $k := \lfloor \frac{\log N}{\log L} \rfloor$. We then obtain the claimed estimate. □

Proof of Corollary 2.7

The argument is similar to that of Theorem 1.1. Therefore, we only prove an analogue of Lemma 2.3.

Analogue of Lemma 2.3. Given a partition

$$0 \leq t_0 < t_1 < \cdots < t_{N-1} < t_N \leq T,$$

we have

$$\|A_{t_0,t_N} - \sum_{i=1}^{N} A_{t_{i-1},t_i}\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,M} \kappa_{m,d}\Gamma_1 t_0^{-\gamma_1}|t_N - t_0|^{\beta_1-\alpha} + \kappa_{m,d}\Gamma_2 t_0^{-\gamma_2}|t_N - t_0|^{\beta_2}, \quad \text{(A.15)}$$

$$\|A_{t_0,t_N} - \sum_{i=1}^{N} A_{t_{i-1},t_i}\|_{L_m(\mathbb{P})}$$
$$\lesssim_{\alpha,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,M} \kappa_{m,d}(\Gamma_1|t_N - t_0|^{\beta_1-\alpha_1-\gamma_1} + \Gamma_2|t_N - t_0|^{\beta_2-\gamma_2} + \Gamma_3|t_N - t_0|^{\beta_3}), \quad \text{(A.16)}$$

where $t_0 > 0$ is assumed for (A.15). In fact, the proof of (A.15) is the same as that of Lemma 2.3, since we can simply replace Γ_1 and Γ_2 by $t_0^{-\gamma_1}\Gamma_1$ and $t_0^{-\gamma_2}\Gamma_2$. Therefore, we focus on proving (A.16).

Yet, the proof of (A.16) is similar to that of Lemma 2.3. Recalling the notation therein, namely, (A.1) and (A.7), we have

$$A_{t_0,t_N} - \sum_{i=1}^{N} A_{t_{i-1},t_i} = \sum_{n=0}^{\infty} \sum_{i=0}^{2^n} R_i^n,$$

$$\sum_{i=0}^{2^n} R_i^n = \sum_{p=1}^{k} \sum_{l_1,\ldots,l_p \leq L} \sum_{j \leq 2^n L^{-p}} \left\{ \mathbb{E}[S_j^{(p),l_1,\ldots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\ldots,l_p}] - \mathbb{E}[S_j^{(p),l_1,\ldots,l_p}|\mathcal{G}_j^{(p),l_1,\ldots,l_p}] \right\}$$
$$+ \sum_{l_1,\ldots,l_k \leq L} \sum_{j \leq 2^n L^{-k}} \mathbb{E}[S_j^{(k),l_1,\ldots,l_k}|\mathcal{G}_j^{(k),l_1,\ldots,l_k}]. \quad \text{(A.17)}$$

We fix a large n. To estimate the first term of (A.17), we apply the BDG inequality to obtain

$$\| \sum_{j \leq 2^n L^{-p}} \left\{ \mathbb{E}[S_j^{(p),l_1,\ldots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\ldots,l_p}] - \mathbb{E}[S_j^{(p),l_1,\ldots,l_p}|\mathcal{G}_j^{(p),l_1,\ldots,l_p}] \right\}\|_{L_m(\mathbb{P})}$$
$$\lesssim \kappa_{m,d} \left(\sum_{j \leq 2^n L^{-p}} \|\mathbb{E}[S_j^{(p),l_1,\ldots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\ldots,l_p}]\|_{L_m(\mathbb{P})}^2 \right)^{\frac{1}{2}}.$$

For $p = 1$, since $S_j^{(1),l_1} = R_{jL+l_1}^n$, by the Cauchy-Schwarz inequality,

$$\sum_{l_1 \leq L} \left(\sum_{j \leq 2^n L^{-1}} \|S_j^{(1),l_1}\|_{L_m(\mathbb{P})}^2 \right)^{\frac{1}{2}} \leq L^{\frac{1}{2}} \left(\sum_{i=0}^{2^n} \|R_i^n\|_{L_m(\mathbb{P})}^2 \right)^{\frac{1}{2}}.$$

For $i \geq 1$, by (2.11), we have

$$\|R_i^n\|_{L_m(\mathbb{P})} \leq 2\Gamma_2(t_0 + 2^{-n}i|t_N - t_0|)^{-\gamma_2}(2^{-n}|t_N - t_0|)^{\beta_2}$$

and by (2.12)

$$\|R_0^n\|_{L_m(\mathbb{P})} \leq 2\Gamma_3(2^{-n}|t_N - t_0|)^{\beta_3}.$$

Therefore,

$$\left(\sum_{i=0}^{2^n} \|R_i^n\|_{L_m(\mathbb{P})}^2 \right)^{\frac{1}{2}} \lesssim \Gamma_3 2^{-n\beta_3}|t_N - t_0|^{\beta_3} + \Gamma_2 2^{-n\beta_2}|t_N - t_0|^{\beta_2} \left(\sum_{i=1}^{2^n} (t_0 + 2^{-n}i|t_N - t_0|)^{-2\gamma_2} \right)^{\frac{1}{2}}.$$

We observe

$$\sum_{i=1}^{2^n}(t_0 + 2^{-n}i|t_N - t_0|)^{-2\gamma_2} \le 2^n|t_N - t_0|^{-1}\int_0^{|t_N - t_0|} s^{-2\gamma_2}\,\mathrm{d}s = \frac{2^n|t_N - t_0|^{-2\gamma_2}}{1 - 2\gamma_2}, \qquad (A.18)$$

where the condition $\gamma_2 < \frac{1}{2}$ is used. We conclude

$$\sum_{l_1 \le L}\Big(\sum_{j \le 2^n L^{-1}}\|S_j^{(1),l_1}\|_{L_m(\mathbb{P})}^2\Big)^{\frac{1}{2}} \lesssim_{\gamma_2} L^{\frac{1}{2}}(\Gamma_3 2^{-n\beta_3}|t_N - t_0|^{\beta_3} + \Gamma_2 2^{-n(\beta_2 - \frac{1}{2})}|t_N - t_0|^{\beta_2 - \gamma_2}).$$

For $2 \le p \le k$, the argument is similar but now we use (2.10). By the Cauchy-Schwarz inequality,

$$\sum_{l_1,\dots,l_p \le L}\Big(\sum_{j \le 2^n L^{-p}}\|\mathbb{E}[S_j^{(p),l_1,\dots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\dots,l_p}]\|_{L_m(\mathbb{P})}^2\Big)^{\frac{1}{2}}$$

$$\le L^{\frac{p}{2}}\Big(\sum_{l_1,\dots,l_p \le L}\sum_{j \le 2^n L^{-p}}\|\mathbb{E}[S_j^{(p),l_1,\dots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\dots,l_p}]\|_{L_m(\mathbb{P})}^2\Big)^{\frac{1}{2}}.$$

We note that for each index $l_1,\dots,l_p$ and j in the sum, there exists a unique $i = i(l_1,\dots,l_p;j)$, such that

$$S_j^{(p),l_1,\dots,l_p} = R_i^n.$$

As $p \ge 2$, we know $i \ge L$. By (2.10) (as in the estimate (A.9)),

$$\|\mathbb{E}[S_j^{(p),l_1,\dots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\dots,l_p}]\|_{L_m(\mathbb{P})}$$
$$\le 2\Gamma_1(L^{p-1}2^{-n}|t_N - t_0|)^{-\alpha}(t_0 + 2^{-n}i(l_1,\dots,l_p;j)|t_N - t_0|)^{-\gamma_1}(2^{-n}|t_N - t_0|)^{\beta_1}.$$

Therefore,

$$\Big(\sum_{l_1,\dots,l_p \le L}\sum_{j \le 2^n L^{-p}}\|\mathbb{E}[S_j^{(p),l_1,\dots,l_p}|\mathcal{G}_{j+1}^{(p),l_1,\dots,l_p}]\|_{L_m(\mathbb{P})}^2\Big)^{\frac{1}{2}}$$

$$\lesssim \Gamma_1(L^{p-1}2^{-n}|t_N - t_0|)^{-\alpha}(2^{-n}|t_N - t_0|)^{\beta_1}\Big(\sum_{i=1}^{2^n}(t_0 + 2^{-n}i|t_N - t_0|)^{-2\gamma_1}\Big)^{\frac{1}{2}}$$

$$\lesssim_{\gamma_1} \Gamma_1 L^{-\alpha(p-1)}2^{-n(\beta_1 - \alpha - \frac{1}{2})}|t_N - t_0|^{\beta_1 - \alpha - \gamma_1},$$

where to obtain the second inequality, we applied the estimate (A.18).

Now we consider the estimate of the second term of (A.17). By the triangle inequality,

$$\|\sum_{l_1,\dots,l_k \le L}\sum_{j \le 2^n L^{-k}}\mathbb{E}[S_j^{(k),l_1,\dots,l_k}|\mathcal{G}_j^{(k),l_1,\dots,l_k}]\|_{L_m(\mathbb{P})}$$

$$\le \sum_{l_1,\dots,l_k \le L}\sum_{j \le 2^n L^{-k}}\|\mathbb{E}[S_j^{(k),l_1,\dots,l_k}|\mathcal{G}_j^{(k),l_1,\dots,l_k}]\|_{L_m(\mathbb{P})}.$$

But the estimate of the right-hand side was just discussed. In fact, we have

$$\sum_{l_1,\dots,l_k \le L}\sum_{j \le 2^n L^{-k}}\|\mathbb{E}[S_j^{(k),l_1,\dots,l_k}|\mathcal{G}_j^{(k),l_1,\dots,l_k}]\|_{L_m(\mathbb{P})} \lesssim_{\gamma_1} \Gamma_1 L^{-\alpha k}2^{-n(\beta_1 - \alpha - 1)}|t_N - t_0|^{\beta_1 - \alpha - \gamma_1}.$$

Hence, we obtain the estimate

$$\Big\| \sum_{i=0}^{2^n} R_i^n \Big\|_{L_m(\mathbb{P})} \lesssim_{\gamma_1,\gamma_2} \kappa_{m,d} L^{\frac{1}{2}} (\Gamma_3 2^{-n\beta_3} |t_N - t_0|^{\beta_3} + \Gamma_2 2^{-n(\beta_2-\frac{1}{2})} |t_N - t_0|^{\beta_2-\gamma_2})$$

$$+ \kappa_{m,d} \Gamma_1 \sum_{p=2}^{k} L^{\frac{p}{2}-\alpha(p-1)} 2^{-n(\beta_1-\alpha-\frac{1}{2})} |t_N - t_0|^{\beta_1-\alpha-\gamma_1}$$

$$+ \Gamma_1 L^{-\alpha k} 2^{-n(\beta_1-\alpha-1)} |t_N - t_0|^{\beta_1-\alpha-\gamma_1}.$$

By choosing L and k exactly as in the proof of Lemma 2.3, we conclude that there exists an $\varepsilon = \varepsilon(\alpha, \beta_1, \beta_2, \beta_3) > 0$, such that for all large n

$$\Big\| \sum_{i=0}^{2^n} R_i^n \Big\|_{L_m(\mathbb{P})} \lesssim_{\alpha,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2} \kappa_{m,d} 2^{-n\varepsilon} (\Gamma_1 |t_N - t_0|^{\beta_1-\alpha_1-\gamma_1} + \Gamma_2 |t_N - t_0|^{\beta_2-\gamma_2} + \Gamma_3 |t_N - t_0|^{\beta_3}),$$

from which we obtain (A.16).

Proof of Lemma 3.2

Let $d = 1$. For $u > v$, we set

$$B_u^{(1)} := \int_{-\infty}^{v} K(u,r) \, dW_r, \quad B_u^{(2)} := \int_{v}^{u} K(u,r) \, dW_r$$

so that $B_u - B(0) = B_u^{(1)} + B_u^{(2)}$ and $B^{(1)}$ and $B^{(2)}$ are independent. Then, we have

$$\mathbb{E}[B_s^{(2)} B_t^{(2)}] = \int_{v}^{s} K(s,r) K(t,r) \, dr,$$

and by (3.2), we have

$$\frac{c_H}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) = \mathbb{E}[B_s^{(1)} B_t^{(1)}] + \mathbb{E}[B_s^{(2)} B_t^{(2)}],$$

and thus, we will estimate $\mathbb{E}[B_s^{(1)} B_t^{(1)}]$. We have

$$\mathbb{E}[B_s^{(1)} B_t^{(1)}] = \int_{0}^{\infty} \big[(s+r)^{H-1/2} - r^{H-1/2}\big]\big[(t+r)^{H-1/2} - r^{H-1/2}\big] \, dr$$

$$+ \int_{0}^{v} (s-r)^{H-1/2}(t-r)^{H-1/2} \, dr. \tag{A.19}$$

By [38, Theorem 33], the first term of (A.19) equals to

$$(c_H - (2H)^{-1})s^{2H} + \int_{0}^{\infty} \big[(s+r)^{H-1/2} - r^{H-1/2}\big]\big[(t+r)^{H-1/2} - (s+r)^{H-1/2}\big] \, dr. \tag{A.20}$$

Since

$$(t+r)^{H-1/2} - (s+r)^{H-1/2} = (H-1/2)(s+r)^{H-3/2}(t-s) + O((s+r)^{H-5/2}(t-s)^2),$$

the second term of (A.20) equals to

$$s^{2H-1}(t-s)(H-1/2)\int_0^\infty \left[(1+r)^{H-1/2} - r^{H-1/2}\right](1+r)^{H-3/2}\,\mathrm{d}r + O(s^{2H-2}(t-s)^2).$$

By [38, Theorem 33],

$$(H-1/2)\int_0^\infty \left[(1+r)^{H-1/2} - r^{H-1/2}\right](1+r)^{H-3/2}\,\mathrm{d}r = -\frac{1}{2} + Hc_H.$$

Similarly, the second term of (A.19) equals to

$$\frac{1}{2H}(s^{2H} - (s-v)^{2H}) + \frac{t-s}{2}(s^{2H-1} - (s-v)^{2H-1}) + O((s-v)^{2H-2}(t-s)^2).$$

Therefore, $\mathbb{E}[B_s^{(1)} B_t^{(1)}]$ equals to

$$c_H s^{2H} + Hc_H s^{2H-1}(t-s) - \frac{1}{2H}(v-s)^{2H} - \frac{1}{2}(s-v)^{2H-1}(t-s) + O((s-v)^{2H-2}(t-s)^2).$$

Since

$$\frac{c_H}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) - c_H s^{2H} + Hc_H s^{2H-1}(t-s) - \frac{1}{2H}(v-s)^{2H}$$
$$= -\frac{c_H}{2}(t-s)^{2H} + O((s-v)^{2H-2}(t-s)^2),$$

the proof is complete.

B. Yamada-Watanabe theorem for fractional SDEs

We consider a Young differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x, \tag{B.1}$$

where $b \in L_\infty(\mathbb{R}^d, \mathbb{R}^d)$ and B is an $(\mathcal{F}_t)_{t\in\mathbb{R}}$ fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. We fix $\alpha \in (\frac{1}{2}, H)$, and we assume that $\sigma \in C^{\frac{1-\alpha}{\alpha}}(\mathbb{R}^d; \mathcal{M}_d)$ so that the integral

$$\int_s^t \sigma(X_r)\,\mathrm{d}B_r$$

is interpreted as a Young integral.

Definition B.1. We say that a quintuple $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P}, B, X)$ is a *weak solution* to (B.1) if (B, X) are random variables defined on the filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{R})$, if B is an $(\mathcal{F}_t)$-fractional Brownian motion, if $X \in C^\alpha([0,T])$ is adapted to $(\mathcal{F}_t)$, and if X solves the Young differential equation (B.1). Given a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ and an $(\mathcal{F}_t)$-fractional Brownian motion B, we say that a $C^\alpha([0,T])$-valued random variable X defined on $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ is a *strong solution* if it solves (B.1) and if it is adapted to the natural filtration generated by B. We say that the *pathwise uniqueness* holds for (B.1) if, for any two adapted $C^\alpha([0,T])$-valued random process X and Y defined on a common filtered probability space that solve (B.1) driven by a common $(\mathcal{F}_t)$-Brownian motion, we have $X = Y$ almost surely.

We will prove a Yamada-Watanabe type theorem for (B.1) based on Kurtz [25]. To this end, we recall that an $(\mathcal{F}_t)$-fractional Brownian motion has the representation (3.1), and we view (B.1) as an equation of X and the Brownian motion W.

Proposition B.2. *Suppose that a weak solution to (B.1) exists and that the pathwise uniqueness holds for (B.1). Then, there exists a strong solution to (B.1).*

Proof. We would like to apply [25, Theorem 3.14]. For this purpose, we need a setup. We follow the notation in [25]. We fix $\beta > 0$ that is less than but sufficiently close to $\frac{1}{2}$. As before, we set $K_H(t,r) := (t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}$, and we set $S_1 := C^\alpha([0,T])$ and define S_2 as a subspace of

$$\{w \in C^\beta(\mathbb{R}) \mid \lim_{r\to-\infty} |w_r|(-r)^{H-\frac{3}{2}} = 0, \ \int_{-\infty}^{-1} |w(r)|(-r)^{H-\frac{3}{2}} \, dr < \infty\}$$

that is Polish and the Brownian motion lives in S_2. We note that for $w \in S_2$, the improper integral

$$\int_{-\infty}^t K_H(t,r) \, dw_r = \lim_{M\to\infty} \int_{-M}^t K_H(t,r) \, dw_r$$

is well-defined. For $t \in [0,T]$, we denote by $(\mathcal{B}_t^{S_1})_{t\in[0,T]}$ and $(\mathcal{B}_t^{S_2})_{t\in[0,T]}$ the filtration generated by the coordinate maps in S_1 and S_2, respectively. We set

$$\mathcal{C} := \{(\mathcal{B}_t^{S_1}, \mathcal{B}_t^{S_2}) \mid t \in [0,T]\}$$

as our compatibility structure in the sense of [25, Definition 3.4]. We denote by $\mathcal{S}_{\Gamma,\mathcal{C},W}$ the set of probability measures μ on $S_1 \times S_2$, such that

○ we have

$$\mu(\{(x,y) \in S_1 \times S_2 \mid x_t = x + \int_0^t b(x_r) \, dr + \int_0^t \sigma(x_r) \, dIy_r \ \text{for all } t \in [0,T]\}) = 1,$$

where $(Iy)_t := \int_{-\infty}^t K_H(t,r) \, dy_r$;
○ μ is $\mathcal{C}$-compatible in the sense of [25, Definition 3.6];
○ $\mu(S_1 \times \cdot)$ has the law of the Brownian motion.

By [25, Lemma 3.8], $\mathcal{S}_{\Gamma,\mathcal{C},W}$ is convex. In view of [25, Lemma 3.2], the existence of weak solutions implies $\mathcal{S}_{\Gamma,\mathcal{C},W} \neq \varnothing$.

Therefore, to apply [25, Theorem 3.14], it remains to prove the pointwise uniqueness in the sense of [25, Definition 3.12]. Suppose that (X_1, X_2, W) are defined on a common probability space, that the laws of (X_1, Y) and (X_2, Y) belong to $\mathcal{S}_{\Gamma,\mathcal{C},W}$, and that (X_1, X_2) are jointly compatible with W in the sense of [25, Definition 3.12]. But then, if we denote by $(\mathcal{F}_t)$ the filtration generated by (X_1, X_2, W), by [25, Lemma 3.2], the joint compatibility implies that W is an $(\mathcal{F}_t)$-Brownian motion, and therefore the pathwise uniqueness implies $X_1 = X_2$ almost surely.

Hence, by [25, Theorem 3.14], there exists a measurable map $F : S_2 \to S_1$, such that for a Brownian motion W, the law of $(F(W), W)$ belongs to $\mathcal{S}_{\Gamma,\mathcal{C},W}$. Then, [25, Lemma 3.11] implies that $F(W)$ is a strong solution. □

Lemma B.3. *Let $b \in C_b^1(\mathbb{R}^d)$ and $\sigma \in C_b^1(\mathbb{R}^d)$. Then, there exists a weak solution to (B.1).*

Proof. Let $(\sigma^n)_{n\in\mathbb{N}}$ be a smooth approximation to σ, and let X^n be the solution to

$$X_t^n = x + \int_0^t b(X_r^n) \, dr + \int_0^t \sigma^n(X_r^n) \, dB_r.$$

Let W be the Brownian motion, such that $B_t = \int_{-\infty}^t K_H(t,r) \, dW_r$. Let ε be greater than but sufficiently

close to 0, and let S be a subspace of

$$\{w \in C^{\frac{1}{2}-\varepsilon}(\mathbb{R}) \mid \lim_{r \to -\infty} |w_r|(-r)^{H-\frac{3}{2}} = 0, \int_{-\infty}^{-1} |w(r)|(-r)^{H-\frac{3}{2}} \, dr < \infty\}$$

that is Polish and where the Brownian motion lives. By the a priori estimate (5.8), we see that a sequence of the laws of (X^n, W) is tight in $C^{H-\varepsilon}([0,T]) \times S$. Thus, replacing it with a subsequence, we suppose that the sequence (X^n, W) converges to some limit $(\tilde{X}, \tilde{W})$ in law.

To see that $(\tilde{X}, \tilde{W})$ solves (5.4), we write $Iw_t := \int_{-\infty}^{t} K_H(t, r) \, dw_r$, and for $\delta > 0$, we set

$$A_\delta := \{(y, w) \in C^{H-\varepsilon}([0,T]) \times S \mid \sup_{t \in [0,T]} |y_t - x - \int_0^t b(y_r) \, dr - \int_0^t \sigma(y_r) \, d(Iw)_r| > \delta\}.$$

Then, we have

$$\mathbb{P}((\tilde{X}, \tilde{W}) \in A_\delta) \le \liminf_{n \to \infty} \mathbb{P}((X^n, W) \in A_\delta)$$

$$\le \liminf_{n \to \infty} \mathbb{P}(\sup_{t \in [0,T]} |\int_0^t \{\sigma - \sigma^n\}(X_r^n) \, dB_r| > \delta).$$

However, by the estimate of Young's integral,

$$|\int_0^t \{\sigma - \sigma^n\}(X_r^n) \, dB_r| \lesssim_{H,\varepsilon} \|\sigma - \sigma^n\|_{C_b^1} \|X^n\|_{C_b^1} \|B\|_{C_b^1}.$$

Thus, combined with the a priori estimate (5.8), we observe

$$\liminf_{n \to \infty} \mathbb{P}(\sup_{t \in [0,T]} |\int_0^t \{\sigma - \sigma^n\}(X_r^n) \, dB_r| > \delta) = 0,$$

and hence, $\mathbb{P}((\tilde{X}, \tilde{W}) \in A_\delta) = 0$. Since δ is arbitrary, this implies

$$\mathbb{P}(\tilde{X}_t = x + \int_0^t b(\tilde{X}_r) \, dr + \int_0^t \sigma(\tilde{X}_r) \, d(I\tilde{W})_r \; \forall t \in [0,T]) = 1.$$

Finally, since $W_t - W_s$ is independent of the σ-algebra generated by $(X_r^n)_{r \le s}$ and $(W_r)_{r \le s}$, we know that $\tilde{W}_t - \tilde{W}_s$ is independent of

$$\tilde{\mathcal{F}}_s := \sigma(\tilde{X}_r, \tilde{W}_r | r \le s),$$

or equivalently, $\tilde{W}$ is an $(\tilde{\mathcal{F}}_t)$-Brownian motion. □

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