1. Real Sequences – An Interview Question

(i) *Let* $n \geq 1$ *be a fixed natural number and let* $0 < x_1 < \cdots < x_N < 2n + 1$ *be real numbers such that* $|kx_i - x_j| \geq 1$ *for all natural numbers* i, j *and* k *with* $1 \leq i < j \leq N$. *Then* $N \leq n$.
(ii) *Let* $n \geq 1$ *be a fixed natural number and let* $0 < x_1 < \cdots < x_N < (3n+1)/2$ *be real numbers such that* $|kx_i - x_j| \geq 1$ *for all natural numbers* i, j *and* k *with* $1 \leq i < j \leq N$ *and* $k \geq 1$ *odd. Then* $N \leq n$.

Proof. (i) Set $x = x_N$. For every i, $1 \leq i \leq N$, let $k_i \geq 0$ be the unique integer such that $x/2 < 2^{k_i} x_i \leq x$. Then $2^{k_i} x_i \geq x/2 + 1/2$ since otherwise $|2^{k_i+1} x_i - x_N| < 1$. Also, $|2^{k_i} x_i - 2^{k_j} x_j| \geq 1$ for all i, j with $1 \leq i < j \leq N$, since $|2^{k_i} x_i - 2^{k_j} x_j| < 1$ would imply

$$|2^{k_i - k_j} x_i - x_j| = 2^{-k_j} |2^{k_i} x_i - 2^{k_j} x_j| < 2^{-k_j} \leq 1.$$

Hence

$$x/2 + 1/2 \leq 2^{k_i} x_i \leq x$$

for every i and $|2^{k_i} x_i - 2^{k_j} x_j| \leq 1$ if $i \neq j$. Consequently,

$$x - (x/2 + 1/2) \geq N - 1, \quad \text{i.e.} \quad 2n + 1 > x_N = x \geq 2N - 1,$$

so $N \leq n$, as claimed.

(ii) We shall copy the proof of the first part verbatim: the only change is that we replace 2 by 3. Thus, set $x = x_N$ so that $2x < 3n + 1$, and for every i, $1 \leq i \leq N$, let $k_i \geq 0$ be the unique integer such that $x/3 < 3^{k_i} x_i \leq x$. Then $3^{k_i} x_i \geq x/3 + 1/3$ as otherwise $|3^{k_i+1} x_i - x_N| < 1$. Also, $|3^{k_i} x_i - 3^{k_j} x_j| \geq 1$ for all i, j with $1 \leq i < j \leq N$, since otherwise

$$|3^{k_i - k_j} x_i - x_j| = 3^{-k_j} |3^{k_i} x_i - 3^{k_j} x_j| < 3^{-k_j} \leq 1.$$

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Hence

$$x/3 + 1/3 \leq 3^{k_i} x_i \leq x$$

for every i and $|3^{k_i} x_i - 3^{k_j} x_j| \leq 1$ if $i \neq j$. Consequently,

$$x - (x/3 + 1/3) \geq N - 1, \quad \text{i.e.} \quad 3n + 1 > 2x_N = 2x \geq 3N - 2,$$

so $N \leq n$, as claimed. □

Notes. The results above are sharp. For example, if in (i) we weaken the strict inequality $x_N < 2n + 1$ to $x_N \leq 2n + 1$ then N can be as large as $n + 1$. Indeed, the $n + 1$ integers $n + 1 < n + 2 < \cdots < 2n + 1$ are such that none is at distance less than 1 from a multiple of another.

The alert reader must have realized that part (ii) holds in greater generality. We postulated that the multiplier k was *odd*, but what we used was that it was at least 3. Clearly, the proof above (given twice, with tiny changes) applies to whatever we take instead of the bounds 2 and 3 above.

This problem is an extension of a basic 'Erdős Problem for Epsilons', namely Problem 2(i) in CTM, a problem Erdős invented and asked from clever students in their early teens. It would have been an ideal question when interviewing candidates for admission to Trinity College, but I had stopped interviewing years before I thought of this problem.

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