ON CONVEX UNIVALENT FUNCTIONS

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In what follows, we suppose that $f(z) = \sum_0^\infty a_n z^n$ is regular for $|z| < 1$. Let

$$s_n(z) = \sum_0^n a_k z^k, \qquad \sigma_n(z) = \frac{1}{n+1}\sum_0^n s_k(z),$$

$$k(r, \theta) = \frac{1 - r^2}{2(1 - 2r\cos\theta + r^2)},$$

$$k_n(r, \theta) = \frac{1 - r^2 - 2r^{n+1}\{\cos(n+1)\theta - r\cos n\theta\}}{2(1 - 2r\cos\theta + r^2)},$$

and

$$K_n(r, \theta) = \frac{1}{2n\pi}\int_0^{2\pi} \frac{\sin^2 \frac{1}{2}n(\theta - \phi)}{\sin^2 \frac{1}{2}(\theta - \phi)} k(r, \phi)\, d\phi, \qquad 0 \leqq r < 1.$$

Then (see, for example, [**6**, pp. 235–236]), for $0 \leqq r < \rho < 1$, we have:

(1)
$$s_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)}) k_n\left(\frac{r}{\rho}, \phi\right) d\phi,$$
$$\sigma_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)}) K_{n+1}\left(\frac{r}{\rho}, \phi\right) d\phi.$$

The following results are well known.

THEOREM A. *If* $|f(z)| < M$ *for* $|z| < 1$, *then* $|\sigma_n(z)| < M$ *for all* n *and* $|z| < 1$. *Conversely, if* $|\sigma_n(z)| < M$ *for all* n *and* $|z| < 1$, *then* $|f(z)| < M$.

THEOREM B. *If* $|f(z)| < M$ *for* $|z| < 1$, *then* $|s_n(z)| < M$ *for all* n *and* $|z| < \frac{1}{2}$. *The number* $\frac{1}{2}$ *is best possible.*

The proof of Theorem A (see, for example, [**6**, pp. 235–236]) depends on the facts that $K_n(r, \theta) > 0$ for $r < 1$ and

(2)
$$\frac{1}{\pi}\int_0^{2\pi} K_n\left(\frac{r}{\rho}, \phi\right) d\phi = 1,$$

(3)
$$\lim_{n\to\infty} \sigma_n(z) = f(z);$$

the proof of Theorem B (see, for example, [**6**, pp. 235–236]) follows in a similar way from the properties

$$k_n(r, \theta) > 0 \ (r < \tfrac{1}{2}) \quad \text{and} \quad \frac{1}{\pi}\int_0^{2\pi} k_n\left(\frac{r}{\rho}, \phi\right) d\phi = 1,$$

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of $k_n(r, \theta)$. The function $f(z) = (z - a)/(az - 1)$, where $0 < a < 1$, satisfies $|f(z)| < 1$, and the fact that $s_1(-\frac{1}{2}a^{-1}) = (a^2 + 1)/(2a) > 1$ (with a arbitrarily near to 1) shows that the number $\frac{1}{2}$ in Theorem B is best possible.

Theorems A and B are very special cases of the following more general results.

THEOREM 1. (i) *Suppose that the values taken by $f(z)$ for $|z| < 1$ lie in a convex domain D. Then the values taken by $\sigma_n(z)$ also lie in D for all n and $|z| < 1$.*

(ii) *Conversely, if the values taken by $\sigma_n(z)$ lie in a convex domain D for all n and $|z| < 1$, then the values taken by $f(z)$ lie in D for $|z| < 1$.*

Proof. (i) By (1) and (2) we have

$$\sigma_n(re^{i\theta}) = \int_0^{2\pi} f(\rho e^{i(\theta-\phi)}) K_{n+1}\left(\frac{r}{\rho}, \phi\right) d\phi \Big/ \int_0^{2\pi} K_{n+1}\left(\frac{r}{\rho}, \phi\right) d\phi.$$

For fixed $re^{i\theta}$, the right-hand side is the centre of mass of a positive linear mass distribution of density $K_{n+1}(r/\rho, \phi)$ along the curve $w = f(\rho e^{i(\theta-\phi)})$ described as ϕ varies from 0 to 2π. Since D is convex, this centre of mass lies in D.

(ii) The converse follows from (3).

THEOREM 2. *Under the conditions of Theorem* 1 (i), *the values taken by $s_n(z)$ lie in D for all n and $|z| < \frac{1}{2}$.*

Proof. This is exactly as in the proof of Theorem 1 (i), but with $K_n(r/\rho, \phi)$ replaced by $k_n(r/\rho, \phi)$.

Alternatively, we may reduce the proof of Theorem 1 as well as the proof of Theorem 2 to the special (and classical) case when D is the half-plane $Rw > 0$, using the fact that a convex domain is the intersection of half-planes.

Suppose that $g(z)$ and $h(z)$ are regular for $|z| < 1$, $h(z)$ is univalent, and $g(z)$ is subordinate to $h(z)$. We shall then write

$$g(z) \prec h(z).$$

In the case when the function of Theorem 2 is univalent and of the normalized form $f(z) = z + \sum_2^\infty a_n z^n$, and maps $|z| < 1$ onto a convex domain D in the w-plane, the conclusion of the theorem for $n = 2$ is the familiar fact (see, for example, [**2**, p. 13]) that D contains the disc $|w| < \frac{1}{2}$, i.e., that

$$\frac{1}{2}z \prec f(z). \tag{4}$$

The purpose of the remainder of this note is to show the solution of the problem of determining *necessary and sufficient conditions on complex numbers* λ, μ *under which, for all convex* $f(z) = z + \sum_2^\infty a_n z^n$, $\lambda z + \mu a_2 z^2$ *is convex and*

$$\frac{1}{2}z \prec \lambda z + \mu a_2 z^2 \prec f(z), \tag{5}$$

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a stronger form of (4). A necessary condition for the left-hand relation in (5) is that $|\lambda| \geqq \frac{1}{2}$ (see, for example, [**4**, p. 228]), and it is easily seen (by considering $\lambda e^{i\theta}z + \mu e^{2i\theta}a_2z^2$) that we may suppose with no effective loss of generality that λ is real and positive. The solution of the problem is then as follows.

THEOREM 3. (i) *If, for all convex* $f(z) = z + \sum_2^\infty a_n z^n$, $\lambda z + \mu a_2 z^2$ *is convex and*

$$\tfrac{1}{2}z \prec \lambda z + \mu a_2 z^2 \prec f(z),$$

then $\lambda = \mu + \frac{1}{2}$, $\mu \leqq \frac{1}{6}$.

(ii) *Conversely, for all convex* $f(z) = z + \sum_2^\infty a_n z^n$ *and* $\mu \leqq \frac{1}{6}$, $(\mu + \frac{1}{2})z + \mu a_2 z^2$ *is convex and*

$$\tfrac{1}{2}z \prec (\mu + \tfrac{1}{2})z + \mu a_2 z^2 \prec \tfrac{2}{3}z + \tfrac{1}{6}a_2 z^2 \prec f(z).$$

The theorem is the combination of a number of lemmas proved below.

LEMMA 1. *The function* $z + cz^2$ *is convex if and only if* $|c| \leqq \frac{1}{4}$.

Lemma 1 is equivalent to the fact that $z + cz^2$ is starlike if and only if $|c| \leqq \frac{1}{2}$ (see, for example, [**1**]).

LEMMA 2. *If for all convex* $f(z) = z + \sum_2^\infty a_n z^n$, $\lambda z + \mu a_2 z^2$ *is convex and*

$$\tfrac{1}{2}z \prec \lambda z + \mu a_2 z^2 \prec f(z),$$

then μ *is real and non-negative and*

$$\lambda = \mu + \tfrac{1}{2}, \qquad \mu \leqq \tfrac{1}{6}.$$

Proof. If $\lambda z + \mu a_2 z^2$ is convex for all convex $f(z)$, then, with

$$f(z) = z/(1 - z) = z + z^2 + \ldots,$$

by Lemma 1 we must have

$$|\mu| \leqq \tfrac{1}{4}\lambda. \tag{6}$$

The minimum value of $|\lambda z + \mu z^2|$ on $|z| = 1$ is then $\lambda - |\mu|$; hence $\frac{1}{2}z \prec \lambda z + \mu z^2$ implies

$$\lambda - |\mu| \geqq \tfrac{1}{2}. \tag{7}$$

With the same $f(z)$, if $\lambda z + \mu z^2 \prec f(z)$, then, for real x, $-1 < x < 1$, we have $\lambda x + R\mu x^2 > -\frac{1}{2}$, and allowing $x \to -1$,

$$\lambda \leqq R\mu + \tfrac{1}{2}. \tag{8}$$

Combination of (6), (7), and (8) yields the conclusions stated. We suppose from now on that μ is real and non-negative.

LEMMA 3. *Suppose that* b_0, b_1, *and* b_2 *are complex numbers*, $b_2 \neq 0$, *and let* $P(z) = b_0 + b_1 z + b_2 z^2$.

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(i) *If* $|b_0| < |b_2|$ *and*

(9) $$|b_0\bar{b}_1 - \bar{b}_2 b_1| \leqq |b_2|^2 - |b_0|^2,$$

then the zeros of $P(z)$ *lie on* $|z| \leqq 1$.

(ii) *If the zeros of* $P(z)$ *lie on* $|z| \leqq 1$, *then* $|b_0| \leqq |b_2|$ *and* (9) *holds.*

A proof of this is given in [**3**].

LEMMA 4. *For all convex* $f(z) = z + \sum_2^\infty a_n z^n$ *and all* $\mu \leqq \frac{1}{6}$, $(\mu + \frac{1}{2})z + \mu a_2 z^2$ *is convex and*

$$\tfrac{1}{2}z \prec (\mu + \tfrac{1}{2})z + \mu a_2 z^2 \prec \tfrac{2}{3}z + \tfrac{1}{6}a_2 z^2.$$

Proof. By Lemma 1, since $|a_2| \leqq 1$ (see, for example, [**2**, p. 12]) and $\mu \leqq \frac{1}{6}$, $(\mu + \frac{1}{2})z + \mu a_2 z^2$ is convex. Furthermore, in $|z| = 1$ we have

$$|(\mu + \tfrac{1}{2})z + \mu a_2 z^2| \geqq \mu + \tfrac{1}{2} - \mu|a_2| \geqq \tfrac{1}{2};$$

hence

$$\tfrac{1}{2}z \prec (\mu + \tfrac{1}{2})z + \mu a_2 z^2.$$

It is now sufficient to show that, for each real α and $\mu < \frac{1}{6}$, the polynomial

(10) $$\tfrac{2}{3}z + \tfrac{1}{6}a_2 z^2 - (\mu + \tfrac{1}{2})e^{i\alpha} - \mu a_2 e^{2i\alpha}$$

has a zero on $|z| \leqq 1$. We shall show, in fact, that except when $|a_2| = 1$ and α takes a certain value, it has a zero in $|z| < 1$. Suppose that for some α it has no zero in $|z| < 1$. Then the polynomial

$$[(\mu + \tfrac{1}{2})e^{i\alpha} + \mu a_2 e^{2i\alpha}]z^2 - \tfrac{2}{3}z - \tfrac{1}{6}a_2$$

has both zeros on $|z| \leqq 1$; hence by Lemma 3,

$$|(1/9)a_2 + (2/3)[(\mu + (1/2))e^{-i\alpha} + \mu\bar{a}_2 e^{-2i\alpha}]| \leqq |(\mu + (1/2)) + \mu a_2 e^{i\alpha}|^2 - (1/36)|a_2|^2.$$

Writing $a_2 = \rho e^{i\phi}$, $\alpha + \phi = \Psi$, this is equivalent to

(11) $$|6\mu + 3 + 6\mu\rho e^{i\Psi}|^2 - 4|\rho e^{i\Psi} + 3(2\mu + 1) + 6\mu\rho e^{-i\Psi}| \geqq \rho^2.$$

But, since $\mu < \frac{1}{6}$, we have

(12) $$\begin{aligned} |\rho e^{i\Psi} + 3(2\mu + 1) + 6\mu\rho e^{-i\Psi}| &= |6\mu\rho e^{i\Psi} + 3(2\mu + 1) + 6\mu\rho e^{-i\Psi} \\ &\qquad + \rho(1 - 6\mu)e^{i\Psi}| \\ &\geqq 3(2\mu + 1) + 12\mu\rho\cos\Psi - \rho(1 - 6\mu), \end{aligned}$$

and (11), (12) yield

$$-(1 - 36\mu^2)\rho^2 - 12\mu\rho(1 - 6\mu)\cos\Psi - 3(1 + 2\mu)(1 - 6\mu) + 4\rho(1 - 6\mu) \geqq 0.$$

Again since $\mu < \frac{1}{6}$, we may divide this inequality by $1 - 6\mu$, and we obtain

$$4\rho(1 - 3\mu\cos\Psi) \geqq (1 + 6\mu)\rho^2 + 3(1 + 2\mu).$$

This implies that

$$4\rho(1 + 3\mu) \geqq (1 + 6\mu)\rho^2 + 3(1 + 2\mu),$$

or

$$(\rho - 1)\left(\rho - 1 - \frac{2}{1 + 6\mu}\right) \leqq 0.$$

Since $\rho \leqq 1$, this is a contradiction (and shows that (10) has a zero in $|z| < 1$) unless $\rho = 1$, $\Psi = \pi$. But (10) then has a zero $-e^{-i\phi}$, and this completes the proof.

LEMMA 5. *For all convex* $f(z) = z + \sum_2^\infty a_n z^n$, *we have* $\frac{2}{3}z + \frac{1}{6}a_2 z^2 \prec f(z)$.

This is the particular case $n = 2$ of the relation

$$V_n(z, f) \prec f(z)$$

proved by Pólya and Schoenberg [**5**], where $V_n(z, f)$ is the de la Vallée Poussin mean defined by

$$V_n(z, f) = \binom{2n}{n}^{-1} \sum_1^n \binom{2n}{n + k} a_k z^k.$$

Theorem 3 now follows on combining Lemmas 2, 4, and 5.

In conclusion we remark that we have also a proof of Lemma 6 which is exactly on the lines of the proof of the relation $\frac{3}{8}z + \frac{1}{16}a_2 z^2 \prec f(z)$ for starlike $f(z)$ given in [**3**]. This consists of showing first that, for any $w(z) = w_1 z + w_2 z^2 + \ldots$ regular for $|z| < 1$ and satisfying $|w(z)| < 1$, we have

$$\frac{2}{3} w_1 z + \frac{1}{6}(w_2 + w_1^2) z^2 \prec \frac{z}{1 - z},$$

and then using the fact that any convex domain may be expressed as the intersection of half-planes.

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