GROUPS WITH AN AUTOMORPHISM SQUARING MANY ELEMENTS

Dedicated to the memory of Hanna Neumann

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(Received 29 March 1972)

Communicated by M. F. Newman

1. Introduction

A universal power automorphism (Cooper [1]) of a group is an automorphism mapping every element x to a power x^n for some fixed integer n. It is long known that a group admitting such an automorphism with $n = -1$, 2 or 3 must be Abelian. Miller [5] showed that for every other non-zero integral value of n there exist non-Abelian groups admitting a non-trivial universal power automorphism $x \rightarrow x^n$.

The problem remained of how large a proportion of elements of a non-Abelian group can be mapped to their nth power by some automorphism, when $n = -1$, 2 or 3. For the case $n = -1$, it is known that if G is a finite non-Abelian group and if p is the smallest prime dividing its order, then not more than $\frac{3}{4}|G|$ or $|G|/p$ of its elements can be inverted by an automorphism according as $p = 2$ or p is an odd prime. MacHale and the present author ([3] and [4]) classified all finite non-Abelian groups G that admit an automorphism inverting exactly $\frac{3}{4}|G|$ ($|G|/p$) elements when $p = 2$ (p is odd). In a sense such groups are "almost" Abelian and it is no surprise to find that they either have an Abelian subgroup of index p or they are nilpotent groups of small nilpotency class.

In this paper we turn to the problem of classifying finite non-Abelian groups G that admit an automorphism squaring as large a proportion of group elements as possible. We prove (Theorem 3.5) that if p is the smallest prime dividing $|G|$ then not more than $|G|/p$ *of the elements of* G *can be squared by an automorphism.* (This includes the case $p = 2$.) We then classify all finite non-Abelian groups G that have an automorphism squaring exactly $|G|/p$ elements. The classification is given in Theorem 4.1 for fixed-point-free automorphisms and Theorems 4.5, 4.10 and 4.11 for automorphisms with a non-trivial fixed group, which turns out to be of order p.

It must be expected that there is an overlap between groups having an automorphism inverting many elements and groups having an automorphism squaring many elements, for in groups of exponent 3 the concepts coincide. Indeed, the groups of Theorems 4.1 and 4.10 are precisely the groups of [4, Theorem 4.9]; the groups of Theorem 4.5 resemble those of [4, Theorem 5.1], and the groups of Theorem 4.11 are those of [4, Theorem 4.13] with $p = 3$.

The proofs in [4] rested heavily on the fact that the automorphisms concerned had order 2. This is not necessarily the case here, and new methods of proof had to be found to settle the present question.

2. Notation

$\mathscr{G}_p$	The set of all groups with order divisible by the prime p but by no smaller prime.
α	An automorphism of a group G.
T_α	$\{g \in G \mid g\alpha = g^2\}$.
A_α	A subgroup of G maximal in T_α.
F_α	Subgroup of all elements fixed by α.
$C_A(g)$	Centralizer in subgroup A of element $g \in G$.
$\langle S, x_1, \cdots, x_r \rangle$	Group generated by the elements $x_1, \cdots, x_r$ of G and the elements of the subset $S \subseteq G$.
$\lvert S \rvert$	Number of elements in set S.
$S \setminus T$	Set of elements in S but not in T.

3. Preliminary results

Let G be a non-Abelian group in $\mathscr{G}_p$ and let α be an automorphism of G. The set T_α and subgroup A_α (as defined in section 2) clearly contain elements of odd order only. Moreover, A_α is Abelian.

We aim to prove that $|T_\alpha| \leqq |G|/p$. For the proof we require four lemmas.

LEMMA 3.1. *Let $f \in F_\alpha$. Then $T_\alpha f^r \cap T_\alpha f^s$ is empty whenever $f^{r-s} \neq 1$.*

PROOF. Suppose $t_1 f^r = t_2 f^s$ where t_1, $t_2 \in T_\alpha$. Applying α we obtain $t_1^2 f^r = t_2^2 f^s$. Hence $t_1 = t_2$ and $f^{r-s} = 1$.

LEMMA 3.2. (i) *Let $g \in G \setminus A_\alpha$, and suppose that the set $A_\alpha g \cap T_\alpha$ is not empty. Then*

$$|A_\alpha g \cap T_\alpha| = |C_{A_\alpha}(g)| \leqq \frac{1}{p} |A_\alpha|.$$

(ii) *Suppose $g \in G \setminus A_\alpha$ and $g^{-1} A_\alpha g = A_\alpha$. Then $A_\alpha g \cap T_\alpha$ is empty.*

(iii) *Suppose $|T_\alpha| > |G|/p$. Then A_α is self-normalizing.*

PROOF. For convenience of notation put $A = A_\alpha$.

(i) Suppose $t \in Ag \cap T_\alpha$. Then $t = a_1 g$ for some $a_1 \in A$, and $Ag = At$.

Now, given $a \in A$, $(at)\alpha = (at)^2$ if and only if a and t commute. So

$$Ag \cap T_\alpha = (C_A(t))t,$$

and clearly $C_A(t) = C_A(g)$.

Finally, $C_A(g)$ is a proper subgroup of A, for otherwise $\langle C_A(g), t\rangle$ is a subgroup in T_α containing A properly, and this contradicts the definition of A.

(ii) We first show that if t in T_α normalizes A then it centralizes A. For consider a in A such that $t^{-1}at \in A$. Then

$$t^{-2}a^2t^2 = (t^{-1}at)\alpha = t^{-1}a^2t,$$

from which it follows that t commutes with a^2 and also with a, since A has odd order.

Now suppose that g in $G \setminus A$ normalizes A, and that $a_1 g \in T_\alpha$ for some a_1 in A. Then $a_1 g$ centralizes A and hence $C_A(g) = A$. This contradicts (i) and completes the proof of (ii).

(iii) Suppose g in $G \setminus A$ normalizes A. Then also g^r $(r = 1, 2, \cdots, p-1)$ normalizes A, and $g^r \notin A$. By (ii), the set $A \cup Ag \cup \cdots \cup Ag^{p-1}$ has $1/p$ of its elements in T_α. It now follows from (i) that $|T_\alpha|$ cannot exceed $|G|/p$.

LEMMA 3.3. *If α is fixed-point-free (f.p.f.) then for every t in T_α, $g \in C_G(t)$ iff $g^{-1}(g\alpha) \in C_G(t)$.*

PROOF. $[g, t] = 1 \Rightarrow [g\alpha, t^2] = 1 \Rightarrow [g\alpha, t] = 1$, since t has odd order. Hence $g \in C_G(t) \Rightarrow g^{-1}(g\alpha) \in C_G(t)$. Since α is f.p.f. the correspondence $g \leftrightarrow g^{-1}(g\alpha)$ is one-one and the lemma follows.

LEMMA 3.4. *If α is f.p.f. then T_α contains at most one element from any conjugacy class of G.*

PROOF. Suppose $gtg^{-1} \in T_\alpha$, where $t \in T_\alpha$. Then

$$gt^2g^{-1} = (gtg^{-1})\alpha = (g\alpha)t^2(g\alpha)^{-1}.$$

So $g^{-1}(g\alpha) \in C_G(t^2) = C_G(t)$. By Lemma 3.3, $g \in C_G(t)$ and so $gtg^{-1} = t$.

We can now prove the main result of this section, that the non-Abelian group G in $\mathscr{G}_p$ has $|T_\alpha| \leqq |G|/p$. This is an immediate consequence of the following theorem.

THEOREM 3.5. *Let G in $\mathscr{G}_p$ have an automorphism α such that $|T_\alpha| > |G/p|$. Then G is Abelian, and $T_\alpha = G$.*

PROOF. The theorem is clearly true for groups of prime order. Let G be a minimal counterexample; so G is a non-Abelian group as specified in the theorem, for some prime p. Then we may suppose that α is fixed-point-free; for consider a coset decomposition of G relative to the subgroup A_α. If $f\alpha = f$, $f \neq 1$, then by

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Lemma 3.1 none of the cosets $A_\alpha f^i$ $(i = 1, \cdots, p-1)$ contains an element of T_α; all other cosets except A_α have not more than $1/p$ of their elements in T_α. Thus the existence of a non-trivial element fixed under α implies that $p|T_\alpha| \leqq |G|$.

We next show that G must have non-trivial centre Z. If $Z = 1$ then each conjugacy class other than $\{1\}$ contains at least p elements. By Lemma 3.4, G has at least $|T_\alpha|$ conjugacy classes, and so $|G| \geqq (|T_\alpha| - 1)p + 1$. But $|T_\alpha| > |G|/p$, and so $|T_\alpha| \geqq 1 + |G|/p$, the last inequality because both $|T_\alpha|$ and $|G|/p$ are integers. Hence $|G| \geqq |G| + 1$, and we have a contradiction.

It remains to consider a group G with non-trivial centre Z and such that α is f.p.f. Put $Z^* = Z \cap T_\alpha$. This is a subgroup of Z. If Z^* is a proper subgroup of Z then T_α contains at most $1/p$ of the central elements and, by Lemma 3.4, at most $1/p$ of the remaining elements; thus $p|T_\alpha| \leqq |G|$. Next, suppose that $Z^* = Z$. By Lemma 3.2 (iii) G has a self-normalizing proper subgroup A_α and so G is not nilpotent. Therefore we may suppose that G/Z is non-Abelian. Now α induces an automorphism $\bar{\alpha}$ in G/Z, such that $(Zg)\bar{\alpha} = Z(g\alpha)$. Furthermore, G/Z belongs to $\mathscr{G}_q$, for some prime $q \geqq p$, and since $|G/Z| < |G|$, we may assume that $q|T_{\bar{\alpha}}| \leqq |G/Z|$. But $|T_{\bar{\alpha}}| \leqq |T_\alpha|\,|Z|$, and hence $p|T_\alpha| \leqq |G|$. This contradiction completes the proof.

4. Groups with $|T_\alpha| = |G|/p$

Having established that an automorphism of a non-Abelian group in $\mathscr{G}_p$ cannot send more than $1/p$ of the elements to their squares, we shall now study the situation at the upper bound. So let G be a non-Abelian group in $\mathscr{G}_p$ having an automorphism α such that $p|T_\alpha| = |G|$. The analysis depends on whether α is f.p.f. or not, and we treat the two cases separately.

4A. The case α *f.p.f.*

In this case T_α cannot be a subgroup. For suppose $T_\alpha = A$, a subgroup of index p in G. Then A is Abelian and, because p is the smallest prime divisor of G, A is normal in G. Now G is assumed non-Abelian, so there exist $x \notin A$, $a \in A$ such that $a \neq x^{-1}ax$. This contradicts Lemma 3.4.

Next we show that not all of the centre Z of G can belong to T_α. We shall assume that $Z \subseteq T_\alpha$ and obtain a contradiction. Our argument falls into two cases, depending on whether (i) every conjugacy class of G contains an element in T_α, or (ii) some conjugacy class has no element in T_α.

Case (i): by Lemma 3.4, G has $|T_\alpha|$ $(= |G|/p)$ conjugacy classes. We need a result of Joseph [2, Proposition 2.6 (iv)] which states that if a group G in $\mathscr{G}_p$ has k conjugacy classes then the order of the commutator subgroup G' satisfies the inequality

$$\frac{k}{|G|} \leqq \frac{1}{p^2}\left(1 + \frac{p^2 - 1}{|G'|}\right).$$

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The proof runs as follows: consider the absolutely irreducible characters of G. There are $(G: G')$ linear characters and $k-(G: G')$ characters whose degrees $m_1, \cdots, m_i, \cdots$ exceed 1 and divide $|G|$. Thus $m_i \geqq p$. Now $|G| = (G: G') + \Sigma m_i^2$, and hence $|G| \geqq (G: G') + (k-(G: G'))p^2$, which gives the above inequality.

From Joseph's lemma we obtain $|G'| \leqq p+1$, when $pk = |G|$. If p is odd then $|G'| = p$ and all non-central conjugacy classes contain p elements. This contradicts G having $|G|/p$ conjugacy classes. If $p = 2$ then $|G'| = 2$ or 3. Because α is assumed fixed-point-free, the only possibility is that G' is generated by an element c of order 3 and that $c\alpha = c^2$. Therefore $G' \subseteq T_\alpha$. By Lemma 3.4, c and c^2 are not conjugate and so $G' \subseteq Z$. Hence G is the direct product of its unique Sylow 3-group and a central subgroup of even order. But our assumption that $Z \subseteq T_\alpha$ implies that Z has odd order, and we have a contradiction.

Case (*ii*): Suppose the element d in G is not conjugate to an element in T_α. It follows easily that the $(p-1)\,|Z|$ elements zd^r, $z \in Z$, $r = 1, \cdots, p-1$, fall into conjugacy classes with no elements in T_α. In order to satisfy Lemma 3.4 and the condition $p|T_\alpha| = |G|$, all the remaining conjugacy classes have p elements with exactly one member in T_α. Clearly α permutes these latter classes and so we may assume that for some $z_1 \in Z$ and integer q not divisible by p,

$$d\alpha = z_1 d^q.$$

Moreover, every conjugate of d has the form zd^r for some $z \in Z$.

Now d is not central, and since the elements of T_α generate G, there exists $t \in T_\alpha$, $z \in Z$ and integer k such that

$$d \neq t^{-1}dt = zd^k.$$

Applying the automorphism α, we obtain

$$t^{-2}z_1 d^q t^2 = z^2 z_1^{kqk} p = z^{2-q} z_1^k t^{-1} d^q t.$$

Hence

$$d^q t d^{-q} \in Zt \subset T_\alpha.$$

By Lemma 3.4, d^q commutes with t, and hence d commutes with t, a contradiction. Thus we have proved that $Z \nsubseteq T_\alpha$.

The only other case that can arise with α a f.p.f. automorphism such that $p|T_\alpha| = |G|$ is as follows: the subgroup $Z^* = Z \cap T_\alpha$ has index p in Z and every conjugacy class containing non-central elements has exactly p elements, and exactly one of these lies in T_α. Choose z in $Z \setminus Z^*$. It follows easily that the sets $T_\alpha z^i$ $(i = 0, 1, \cdots, p-1)$ are pairwise disjoint and, since $|T_\alpha| = |G|/p$, that $G = T_\alpha Z$. Hence $(xZ)\alpha = x^2 Z$ for all $x \in G$, and G/Z is Abelian. Now let t_1, t_2 be non-commuting elements of T_α and put $c = [t_1, t_2]$. Since G is nilpotent of class 2, $c \in Z$. Moreover,

$$c\alpha = [t_1, t_2]\alpha = [t_1^2, t_2^2] = c^4.$$

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It follows that $c \notin Z^*$, for $c\alpha = c^2$ gives $c\alpha = 1$, which is false. Thus we have proved that the centre of G is the direct product $Z = Z^* \times G'$, where G' is generated by an element c of order p.

Now G' is characteristic, and so $c\alpha = c^r$ for some integer r. Since α is assumed f.p.f. we require $r \neq 1$, and since $c \notin T_\alpha$, the cases $p = 2$ or 3 are ruled out.

Finally we note that $t^p \in Z \cap T_\alpha = Z^*$ for all $t \in T_\alpha$, and, since p is odd, that $(t_i t_j)^p = t_i^p t_j^p \in Z^*$, for all t_i, $t_j \in T_\alpha$. Thus G^p is a subgroup of Z^*.

We can now state the structure theorem.

THEOREM 4.1. *A necessary and sufficient condition that a non-Abelian group $G \in \mathscr{G}_p$ have a f.p.f. automorphism α such that $p|T_\alpha| = |G|$ is that*

(i) *G be nilpotent of class 2 with $|G'| = p$;*

(ii) *$G^p \cap G' = 1$; and*

(iii) *$p \geqq 5$.*

PROOF. The necessity of the condition has been established. Conversely, suppose that G satisfied (i)–(iii). Let Z denote the centre of G. Then G/Z is an elementary Abelian p-group, and Z is expressible as a direct product $Z = Z^* \times G'$, where Z^* is a subgroup containing G^p. A simple commutator calculation shows that G/Z can be generated by elements $Za_1, \cdots, Za_k, Zx_1, \cdots, Zx_k$ such that

$$[x_i, x_j] = [a_i, a_j] = 1 \text{ for } i, j = 1, \cdots, k,$$

$$[a_i, x_j] = 1 \qquad (i \neq j),$$

$$[a_i, x_i] = c \qquad (i = 1, \cdots, k),$$

where c generates G'. Put $A = \langle a_1, \cdots, a_k, Z^* \rangle$. Then every element of G is uniquely expressible in the form $g = ac^s x_1^{q_1} \cdots x_k^{q_k}$, where $a \in A$, $0 \leqq s < p$, and $0 \leqq q_i < p$ $(i = 1, \cdots, k)$. The map α such that

$$g\alpha = (ac^s x_1^{q_1} \cdots x_k^{q_k})\alpha = a^2 c^{4s} x_1^{2q_1} \cdots x_k^{2q_k}$$

defines an automorphism of G. Moreover, $p|T_\alpha| = |G|$, for, given any a in A and integers $q_1, \cdots, q_k$, there is precisely one s, $0 \leqq s < p$, such that $g\alpha = g^2$. This is true for all odd primes p. However, α is f.p.f. if and only if $p \geqq 5$.

4B. The case α not *f.p.f.*

For the remainder of this section let G be a non-Abelian group in $\mathscr{G}_p$ having an automorphism α with non-trivial fixed group F_α and such that $p|T_\alpha| = |G|$. It follows from Lemma 3.1 that F_α has order p. Suppose F_α is generated by f. We note the decomposition of G into disjoint sets

$$G = T_\alpha \cup T_\alpha f \cup \cdots \cup T_\alpha f^{p-1}. \qquad (4.2)$$

We require the following lemmas.

LEMMA 4.3. *The conjugacy class containing f has no element in T_α.*

PROOF. If not, then for some $t \in T_\alpha$ and integer r,

$$(tf^r)f^2(tf^r)^{-1} = \{(tf^r)f(tf^r)^{-1}\}\alpha = t^2ft^{-2}.$$

Hence $f^2 = tft^{-1}$, and a further application of α yields $f = tft^{-1}$ and the contradiction $f^2 = f$.

LEMMA 4.4. *Let $G_1 = AF_\alpha$ be a subgroup of G such that* (i) *A is a subgroup of index p in G_1, and* (ii) *A admits the automorphism α. Then $A \subseteq T_\alpha$ (and so A is Abelian).*

PROOF. Suppose $a \in A$. It follows from (4.2) that exactly one of the elements af^i $(0 \leqq i < p)$ belongs to T_α. Suppose then that $af^j \in T_\alpha$. Then $(af^j)^2 = (a\alpha)f^j$, that is, $f^j = a^{-1}(a\alpha)\, a^{-1} \in A$. But clearly $A \cap F_\alpha = 1$, and so it follows that $j = 0$, and the proof is complete.

The case where T_α is a subgroup is easily disposed of.

THEOREM 4.5. *A group $G \in \mathscr{G}_p$ has an automorphism α such that T_α is a subgroup of index p in G if and only if G has an odd order Abelian subgroup A of index p and an element $f \notin A$ of order p.*

PROOF. If T_α is a subgroup it must clearly have odd order. Moreover, in this case α is not f.p.f. and so the generator f of F_α is an element of order p as required. Conversely, given A and f as stated, the map $(af^r)\,\alpha = a^2f^r$, $a \in A$, $r = 0, \cdots, p-1$, defines an automorphism of G with $T_\alpha = A$.

Next we consider the case where T_α is not a subgroup (and F_α has order p).

LEMMA 4.6. *Let $A(=A_\alpha)$ be a subgroup of G which is maximal in T_α and suppose $A \neq T_\alpha$. Then there exists a coset decomposition*

$$G = A \cup Af \cup \cdots \cup Af^{p-1} \cup Ag_1 \cup \cdots \cup Ag_n,$$

such that

(i) *$Af^j \cap T_\alpha$ is empty $(j = 1, \cdots, p-1)$;*

(ii) $|Ag_i \cap T_\alpha| = |C_A(g_i)| = |A|/p$ $(i = 1, \cdots, n)$.

PROOF. Result (i) is a consequence of (4.2). Clearly exactly $1/p$ of the elements of $A \cup Af \cup \cdots \cup Af^{p-1}$ belong to T_α. By Lemma 3.2 every other coset must have exactly $1/p$ of its elements in T_α, for otherwise the condition $p|T_\alpha| = |G|$ is violated. This proves (ii).

LEMMA 4.7. *The conjugacy class containing $t \in T_\alpha$ either has exactly one element in T_α, when $[t,f] = 1$, or has exactly p elements in T_α, when $[t,f] \neq 1$. These elements are $f^{-r}tf^r$ $(r = 0, 1, \cdots, p-1)$.*

PROOF. $g^{-1}tg \in T_\alpha \Rightarrow (g\alpha)^{-1}t^2(g\alpha) = g^{-1}t^2g$

$\Rightarrow (g\alpha)g^{-1}$ commutes with t^2, and with t.

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But $g = t_1 f^r$ for some $t_1 \in T_\alpha$ and some integer r. It follows that t_1 commutes with t, and that $g^{-1}tg = f^{-r}tf^r$.

LEMMA 4.8. *Suppose T_α is not a subgroup. Suppose $t \in T_\alpha \backslash Z$, and denote by $\{t\}$ the conjugacy class containing t. Then either $|\{t\}| = q$, a prime $\geqq p$, and $[t,f] = 1$, or $|\{t\}| \geqq p^2$.*

PROOF. Deny the lemma. Put $A = C_G(t)$. Then A has index q in G but $[t,f] \neq 1$, that is, $f \notin A$. So $G = \langle A,f \rangle$. Now $at = ta \Leftrightarrow (a\alpha)t^2 = t^2(a\alpha) \Leftrightarrow (a\alpha)t = t(a\alpha)$, and so A admits the automorphism α.

We show that A is a subgroup in T_α. This follows from (4.2) if we can prove that $A \cap T_\alpha f$ is empty for every generator f of F_α. Suppose that $t_1 f \in A$ for some $t_1 \in T_\alpha$. Then also $(t_1 f)\alpha = t_1^2 f \in A$, whence $t_1 \in A$ and $f \in A$, a contradiction.

It is thus clear that A is an Abelian subgroup of index q in G, and that $a\alpha = a^2$ for all $a \in A$. Moreover, since T_α is not a subgroup of G, we must have $q > p$. By Lemma 4.6 there exists $t^* \in T_\alpha \backslash A$ such that $|C_A(t^*)| = |A|/p$. Since $G = \langle A, t^* \rangle$ it follows that $C_A(t^*) = Z$, the centre of G. So G/Z is a non-Abelian group of order pq. It follows that the subgroups A and $\langle Z,f \rangle$ of order $p|Z|$ are conjugate in G. But this contradicts Lemma 4.3, and the proof is complete.

As an immediate consequence of Lemmas 4.7 and 4.8 we have

COROLLARY 4.9. *If T_α is not a subgroup then at most $1/p$ of the elements of a non-central conjugacy class belong to T_α.*

We now consider the structure of G according as its centre Z is a subgroup in T_α or not. Put $Z^* = Z \cap T_\alpha$ and suppose Z^* is a proper subgroup of Z. In order to satisfy $p|T_\alpha| = |G|$ we must have $(Z : Z^*) = p$ and every non-central conjugacy class must have exactly $1/p$ of its elements in T_α. By Lemma 4.3, $f \in Z$ and $Z = \langle Z^*, f \rangle$. By (4.2) $G = T_\alpha F_\alpha$ and hence G/F_α is squared by α and so is Abelian. Thus $G' \subseteq F_\alpha$, and since $|F_\alpha| = p$, it follows that $G' = F_\alpha$. Now $\langle T_\alpha \rangle = G$, and so there exist $t_1, t_2 \in T_\alpha$ such that $[t_1, t_2] = f$. Applying α we obtain

$$f = f\alpha = [t_1^2, t_2^2] = [t_1, t_2]^4 = f^4,$$

so that $p = 3$.

Finally it follows as in Section 4A that G^p is a subgroup of Z^*. The following structure theorem is now clear.

THEOREM 4.10. *A necessary and sufficient condition that a non-Abelian group $G \in \mathscr{G}_p$ have an automorphism α with non-trivial fixed group such that T_α is not a subgroup and does not contain the centre of G, and $p|T_\alpha| = |G|$, is given by conditions (i) and (ii) of Theorem 4.1 and condition (iii)': $p = 3$.*

Of course sufficiency is proved as in the proof of Theorem 4.1.

It remains to consider groups for which the centre Z is contained in T_α. In this case it follows from Lemma 4.3 that for integer r, $1 \leqq r \leqq p-1$, and $z \in Z$, the conjugacy class containing zf^r has no elements in T_α. In order to satisfy $p|T_\alpha| = |G|$ it is necessary that every such conjugacy class lies entirely in $Zf \cup Zf^2 \cup \cdots \cup Zf^{p-1}$ and that exactly $1/p$ of the elements of every other non-central conjugacy class belong to T_α. The first of these two conditions implies that given any $g \in G$, there exists $z \in Z$ and integer s, such that

$$gfg^{-1} = zf^s.$$

Putting $g = tf^i$ (according to (4.2)) and applying α, one finds that $s = 1$. Thus $Zgf = Zfg$, and so Zf lies in the centre of G/Z.

However, f is not central in G, and so, by a remark above and Lemma 4.7, there exists an element $t \in T_\alpha$ whose conjugacy class has p^2 elements, exactly p of which lie in T_α (these are the elements $f^{-r}tf^r$).

Put $A = C_G(t)$. Then $(G: A) = p^2$. It can be shown, as in the proof of Lemma 4.8, that A admits the automorphism α. Put $G_1 = \langle A, f\rangle$. Since A contains Z, $|G_1| = p|A|$ and $G_1 = AF_\alpha$. By Lemma 4.4, A is an Abelian subgroup contained in T_α. Indeed, since T_α is assumed not to be a subgroup, A is a subgroup maximal in T_α. Now G_1 has index p in G, so $G_1 \lhd G$. There exists $t^* \in T_\alpha$ such that $G = \langle G_1, t^*\rangle = \langle A, f, t^*\rangle$. By Lemma 4.6 both $C_A(t^*)$ and $C_A(ft^*)$ have index p in A. We show that they are equal. For let $a \in C_A(ft^*)$. Then $t^*at^{*-1} = f^{-1}af$, and applying α, we find that $a \in C_A(t^*)$. Thus $C_A(ft^*) \subseteq C_A(t^*)$ and equality follows because both subgroups have index p in A. It also follows that $C_A(t^*) = C_A(f)$, and this subgroup therefore is the centre Z of G.

Thus we have proved that G/Z has order p^3. This group is non-Abelian. To see this, put $A = \langle Z, a\rangle$. We show that $[t^*, a] \notin Z$. Firstly, $[t^*, a] \neq 1$, for otherwise $\langle A, t^*\rangle \subseteq T_\alpha$, and this contradicts the maximality of A in T_α. Secondly, if there exists $z \neq 1$ such that $1 \neq [t^*, a] = z \in Z$, an application of α gives $z^2 = 1$, a contradiction, since T_α contains no elements of even order.

Next we show that G/Z has exponent p. Clearly $a^p \in Z$; and $t^{*p} \in Z$ also, since otherwise $\langle Z, t^*\rangle$ has index p in G and is contained in T_α, which by hypothesis is not a subgroup of G.

Finally, consider the commutator $[t^*, a]$. Since Zf generates the centre of G/Z we have

$$[t^*, a] \in Zf^r$$

for some integer r, $1 \leqq r < p$.

Applying α we obtain

$$[t^*, a]^3 \in Z,$$

which leads to the contradiction $[t^*, a] \in Z$ unless $p = 3$. We can now state our final structure theorem.

https://doi.org/10.1017/S1446788700013926 Published online by Cambridge University Press

THEOREM 4.11. *A necessary and sufficient condition that a non-Abelian group $G \in \mathscr{G}_p$ have an automorphism α with non-trivial fixed group such that T_α is not a subgroup but contains the centre of G, and $p|T_\alpha| = |G|$, is that $p = 3$ and G/Z be the non-Abelian group of order* 27 *and exponent* 3.

PROOF. The necessity has been proved. The converse is established as in [4], Theorem 4.13.

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