Can. J. Math., Vol. XXVIII, No. 5, 1976, pp. 897–904

AVERAGING OPERATORS AND $C(X)$-SPACES WITH THE SEPARABLE PROJECTION PROPERTY

JOHN WARREN BAKER AND JOHN WOLFE

1. Introduction. The Banach space of bounded continuous real or complex-valued functions on a topological space X is denoted $C(X)$. An *averaging operator* for an onto continuous function $\phi : X \to Y$ is a bounded linear projection of $C(X)$ onto the subspace $\{f \in C(X) : f$ is constant on each set $\phi^{-1}(y)$ for $y \in Y\}$. The *projection constant* $p(\phi)$ for an onto continuous map ϕ is the lower bound for the norms of all averaging operators for ϕ ($p(\phi) = \infty$ if there is no averaging operator for ϕ). The *derived set*, X^1, of a topological space X is the set of non-isolated points of X. We define $X^0 = X$ and for each ordinal number α, the αth derived set X^α of X is defined inductively by $X^\alpha = (X^{\alpha-1})^1$ if α is a successor ordinal or by $X^\alpha = \bigcap_{\beta<\alpha} X^\beta$ if α is not a successor ordinal.

Let Y be a compact metric space. The present paper develops a technique for constructing compact metric spaces X and onto continuous maps $\phi : X \to Y$ such that $p(\phi)$ is large. The main result is the following theorem.

THEOREM 1. *Suppose that Y is a compact metric space and $Y^n \neq \phi$ for each integer $n \geqq 0$. Then there is a compact metric space X and an onto continuous map $\phi : X \to Y$ for which there is no averaging operator.*

This result resolves a problem raised by Pełczyński [**12**, Problem 30] whose solution is contained in the following corollary.

COROLLARY 2. *Suppose Y is a compact metric space. Then the following are equivalent.*

(1) *$Y^n = \phi$ for some integer n.*

(2) *Every continuous map $\phi : X \to Y$ of a compact metric space X onto Y has an averaging operator.*

(3) *Every continuous map $\phi : C \to Y$ from the Cantor set C onto Y has an averaging operator.*

These three conditions are equivalent to $C(Y)$ having the separable projection property, i.e. if $C(Y)$ is embedded as a subspace of the separable Banach space Z, then there is a projection of Z onto the image of $C(Y)$. Other equivalent conditions are contained in Amir [**1**], Baker ([**3**, Theorem 2.11]) and Pełczyński ([**12**, Theorem 9.13]).

Theorem 1 and Corollary 2 are proved at the end of Section 2 in which the

Received August 22, 1973 and in revised form, January 9, 1976. The work of the second author was done under NSF grant 35347X.

https://doi.org/10.4153/CJM-1976-087-4 Published online by Cambridge University Press

method is developed for constructing maps with large projection constants. Section 3 contains additional applications, remarks and open problems.

2. Construction of maps having no averaging operators. The next definition and the following two lemmas provide the criterion used to show that certain maps have no averaging operators.

Definition 3. Let $\phi : X \to Y$ be a continuous map. For a sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers, we inductively define a sequence of subsets $\sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ of Y for $k \geqq 0$. Let $\sigma_\phi^{(0)} = \phi(X)$. For $k > 0$, $y \in \sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ if and only if there are n_k open disjoint sets $W_1, W_2, \ldots, W_{n_k}$ in Y such that $y \in \bigcap_{i=1}^{n_k} \text{Cl}\ [W_i \cap \sigma_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})]$ and, for $i \neq j$, $\text{Cl}\ [\phi^{-1}(W_i)] \cap \text{Cl}\ [\phi^{-1}(W_j)] = \emptyset$. We denote the special case $\sigma_\phi^{(k)}(2, 2, \ldots, 2)$ by $\Delta^k(\phi)$.

The next lemma is a reformulation of Corollary 5.4 of Ditor [**8**]. Related results are found in Corollary 1.4 of Baker [**3**] and Theorem 1 of Amir [**2**].

LEMMA 4. *Let $\phi : X \to Y$ be an onto continuous map where X and Y are compact Hausdorff spaces and suppose that $\Delta^n(\phi) \neq \emptyset$ for some $n \geqq 0$. Then $p(\phi) \geqq n + 1$.*

Remarks. In the proof of this lemma we actually show that if $\sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k) \neq \emptyset$, then $p(\phi) \geqq 2k + 1 - \sum_{i=1}^{k} 2/n_i$.

We establish this lemma by showing the essential equivalence between the sets $\sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ of Definition 3 and the sets $\Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ introduced by Ditor [**8**] to calculate lower bounds for $p(\phi)$. In defining the sets $\Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$, Ditor used the finite topology defined on the family of closed subsets of a compact Hausdorff space. Definition 3 provides an alternative definition of the sets $\Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ which does not use the finite topology.

Proof of Lemma 4. The definition of the sets $\Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ and a discussion of the finite topology can be found in Ditor [**8**]. Lemma 4 follows immediately from Corollary 5.4 of Ditor [**8**] and the following equivalence: if X and Y are compact Hausdorff spaces and $\phi : X \to Y$ is an onto continuous map, then $y \in \sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ if and only if $\phi^{-1}(y) \in \Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$.

Since this is obvious for $k = 0$, we inductively suppose it holds for $k - 1 \geqq 0$. Suppose $y_0 \in Y$ and $\phi^{-1}(y_0) \in \Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$. Then there exist closed pairwise disjoint subsets $A_1, A_2, \ldots, A_{n_k}$ of $\phi^{-1}(y_0)$ which are in the closure of $\Delta_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})$ in the finite topology on the closed subsets of X. For each i choose an open set U_i in X such that $A_i \subset U_i$ and $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$. Define the sets W_i needed in Definition 3 by $W_i = \{y \in Y : \phi^{-1}(y) \subset U_i\}$. The sets W_i are open since ϕ is a closed map and $W_i = Y \backslash \phi(X \backslash U_i)$. For $i \neq j$, $\overline{\phi^{-1}(W_i)} \cap \overline{\phi^{-1}(W_j)} \subset \bar{U}_i \cap \bar{U}_j = \emptyset$. It remains to show that $y_0 \in \text{Cl}\ [W_i \cap \sigma_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})]$. If G is a neighborhood of y_0, then $\phi^{-1}(G) \cap U_i$ is a neighborhood of A_i. Since A_i is in the closure of $\Delta_\phi^{(k-1)}$

https://doi.org/10.4153/CJM-1976-087-4 Published online by Cambridge University Press

$(n_1, n_2. \ldots, n_{k-1})$ in the finite topology, there is a point $y \in Y$ with $\phi^{-1}(y) \in \Delta_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})$ and $\phi^{-1}(y) \subset \phi^{-1}(G) \cap U_i$. Thus $y \in \sigma_\phi^{(k-1)}$ (by inductive hypothesis) and $y \in G \cap W_i$. Thus $y_0 \in \mathrm{Cl}\,[W_i \cap \sigma_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})]$ and so $y_0 \in \sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k)$. Conversely, suppose $y_0 \in \sigma_\phi^{(k)}(n_1, n_2, \ldots, n_k)$. Then there exist disjoint open sets $W_1, W_2, \ldots, W_{n_k}$ in Y such that $y_0 \in \mathrm{Cl}\,[W_i \cap \sigma_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})]$ for each i and $\overline{\phi^{-1}(W_i)} \cap \overline{\phi^{-1}(W_j)} = \emptyset$ for $i \neq j$. Fix i and let $\{y_\alpha\}$ be a net in $W_i \cap \sigma_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})$ converging to y_0. Since X is compact, the finite topology is a compact Hausdorff topology (Theorem 4.2, Michael [**11**]). Consequently the net $\{\phi^{-1}(y_\alpha)\}$ has a convergent subnet which we also denote by $\{\phi^{-1}(y_\alpha)\}$. Let A_i be the limit of $\{\phi^{-1}(y_\alpha)\}$. Since $\{y_\alpha\}$ converges to y_0, $A_i \subset \phi^{-1}(y_0)$. Since $y_\alpha \in \sigma_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})$, by the inductive hypothesis $\phi^{-1}(y_\alpha) \in \Delta_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})$ and so A_i is in the closure in the finite topology of the collection $\Delta_\phi^{(k-1)}(n_1, n_2, \ldots, n_{k-1})$. Since $y_\alpha \in W_i$, $A_i \subset \phi^{-1}(W_i)$ and so the sets A_i are disjoint. Thus $\phi^{-1}(y_0) \in \Delta_\phi^{(k)}(n_1, n_2, \ldots, n_k)$ and the proof is complete.

The next lemma follows easily from Definition 3.

LEMMA 5. *Let $\phi : X \to Y$ be an onto continuous map and let V be an open subset of Y. Suppose that $\phi^{-1}(V) \subset X_0$, where X_0 is closed in X and $\phi(X_0) = \bar{V}$. If $\phi_r = \phi|X_0 : X_0 \to \bar{V}$, then $\Delta^n(\phi_r) \cap V \subset \Delta^n(\phi)$.*

For two topological spaces X_1 and X_2, the *disjoint topological union*, which is denoted $X_1 \oplus X_2$, is defined on the set $X_1 \times \{1\} \cup X_2 \times \{2\} = \{(x, i) : x \in X_i\}$. The topology on $X_1 \oplus X_2$ is determined by the condition that, for $i = 1$ or 2, $X_i \times \{i\}$ is a closed and open subset of $X_1 \oplus X_2$ which is homeomorphic to X_i. If Y is a topological space and $\phi_i : X_i \to Y$ is a continuous map for $i = 1, 2$, then the continuous map $\phi_1 \oplus \phi_2 : X_1 \oplus X_2 \to Y$ is defined by $(\phi_1 \oplus \phi_2)(x, i) = \phi_i(x)$. If $\{X_n\}_{n=1}^\infty$ is a sequence of topological spaces and if $\{\phi_n : X_n \to Y\}_{n=1}^\infty$ is a sequence of continuous maps, then $\oplus_{n=1}^\infty X_n$ and the continuous map $\oplus_{n=1}^\infty \phi_n : \oplus_{n=1}^\infty X_n \to Y$ are defined in the obvious way.

The next two lemmas contain straightforward topological results about metric spaces.

LEMMA 6. *Suppose Y is a metric space and $y \in Y^{n+1}$ for some $n \geqq 0$. If U is a neighborhood of y then there are disjoint open subsets U_1 and U_2 of U with $\{y\} = \bar{U}_1 \cap \bar{U}_2 = \overline{(U_1 \cap Y^n)} \cap \overline{(U_2 \cap Y^n)}$.*

Proof. Suppose $y \in Y^{n+1}$ and $y \in U$. Then there is a sequence of distinct points in $Y^n \cap U$ which converges to y. Choose a sequence of open balls contained in U which are centered on the points of this sequence whose closures are disjoint and whose diameters go to zero. Let U_1 be the union of the even balls and let U_2 be the union of the odd balls. Then $y = \bar{U}_1 \cap \bar{U}_2 = \overline{(U_1 \cap Y^n)} \cap \overline{(U_2 \cap Y^n)}$.

LEMMA 7. *Let Y be a separable metric space. For some $n \geqq 1$ let B be a closed set and U an open set in Y such that $B \cap U = \emptyset$ and $B \subset \overline{U \cap Y^n}$. Then there*

https://doi.org/10.4153/CJM-1976-087-4 Published online by Cambridge University Press

are disjoint open subsets U_1 *and* U_2 *of* U *such that if* $B_1 = \bar{U}_1 \cap \bar{U}_2$, *then*

$$B_1 = \overline{(U_1 \cap Y^{n-1})} \cap \overline{(U_2 \cap Y^{n-1})} \quad \textit{and} \quad B \subset \overline{B_1 \cap U}.$$

Proof. Let $\{y_i\}$ be a countable dense subset of B. For each $m \geqq 1$ choose m points $\{x_1^m, \ldots, x_m^m\}$ in $U \cap Y^n$ such that $d(y_i, x_i^m) < 1/m$ for $1 \leqq i \leqq m$. Let S be the union of all the sets $\{x_1^m, \ldots, x_m^m\}$. Then S is discrete in its relative topology, $S \subset U \cap Y^n$ and $\bar{S} = B \cup S$. Let $\{x_i\}$ be an indexing of S. For each $i \geqq 1$ let $V_i = \{y \in Y : d(x_i, y) < 1/i$ and $d(x_i, y) < \frac{1}{3} \inf d(x_i, x_j) : i \neq j\}\}$. By Lemma 6, for each $i \geqq 1$ we can choose disjoint open subsets $U_1{}^i$ and $U_2{}^i$ of $V_i \cap U$ such that $\{x_i\} = \overline{U_1{}^i} \cap \overline{U_2{}^i} = \overline{(U_1{}^i \cap Y^{n-1})} \cap \overline{(U_2{}^i \cap Y^{n-1})}$. Now define the desired sets by $U_1 = \cup_{i=1}^{\infty} U_1{}^i$ and $U_2 = \cup_{i=1}^{\infty} U_2{}^i$ and let $B_1 = \bar{U}_1 \cap \bar{U}_2$. Since $\lim_{i \to \infty}$ (diam $U_j{}^i) = 0$ for $j = 1, 2$, it follows that $B_1 = \overline{\{x_i\}} = \overline{(U_1 \cap Y^{n-1})} \cap \overline{(U_2 \cap Y^{n-1})}$. Thus $B_1 \cap U = \{x_i\}$ and so $B \subset B \cup \{x_i\} = \overline{\{x_i\}} = \overline{B_1 \cap U}$ and the lemma is proved.

Theorem 1 will follow from the next lemma.

LEMMA 8. *Let* U_1 *and* U_2 *be disjoint open subsets of a compact metric space* Y. *For some* $n \geqq 0$ *suppose* $\bar{U}_1 \cap \bar{U}_2 = \overline{(U_1 \cap Y^n)} \cap \overline{(U_2 \cap Y^n)}$. *Then there is a compact metric space* X *and an onto continuous map* $\phi : X \to Y$ *with* $\bar{U}_1 \cap \bar{U}_2 \subset \Delta^{n+1}(\phi)$.

Proof. We prove the lemma first for the case $n = 0$ and then proceed by induction. For $n = 0$, $Y^n = Y^0 = Y$. Let $B = \bar{U}_1 \cap \bar{U}_2$ and let V_1 and V_2 be disjoint open sets with $U_i \subset V_i$ and $\bar{V}_1 \cup \bar{V}_2 = Y$. Let $X = \bar{V}_1 \oplus \bar{V}_2$ and $\phi_i : \bar{V}_i \to Y$ be the inclusion map. Then $\phi = \phi_1 \oplus \phi_2$ is the desired map since $B = \bar{U}_1 \cap \bar{U}_2 \subset \bar{V}_1 \cap \bar{V}_2 = \Delta^1(\phi)$. Inductively suppose the lemma is true for $n - 1$. Suppose U_1 and U_2 are disjoint open sets and $B = \bar{U}_1 \cap \bar{U}_2 = \overline{(U_1 \cap Y^n)} \cap \overline{(U_2 \cap Y^n)}$. By Lemma 7 choose for each i disjoint open subsets $U_{i,1}$ and $U_{i,2}$ of U_i such that, if $B_i = \bar{U}_{i,1} \cap \bar{U}_{i,2}$, then $B_i = \overline{(U_{i,1} \cap Y^{n-1})} \cap \overline{(U_{i,2} \cap Y^{n-1})}$ and $B \subset \overline{U_i \cap B_i}$. Let V_1 and V_2 be disjoint open sets with $U_i \subset V_i$ and $\bar{V}_1 \cup \bar{V}_2 = Y$. Since the lemma holds for $n - 1$, there are compact metric spaces X_i and onto continuous maps $\phi_i : X_i \to \bar{V}_i$ such that $B_i \subset \Delta^n(\phi_i)$. Let $X = X_1 \oplus X_2$ and let $\phi = \phi_1 \oplus \phi_2$. By Lemma 5, $B_i \cap U_i \subset \Delta^n(\phi)$. Thus $B \subset \overline{(U_1 \cap B_1)} \cap \overline{(U_2 \cap B_2)} \subset \overline{(U_1 \cap \Delta^n(\phi))} \cap \overline{(U_2 \cap \Delta^n(\phi))}$ and clearly $\overline{\phi^{-1}(U_1)} \cap \overline{\phi^{-1}(U_2)} = \emptyset$ and so $B \subset \Delta^{n+1}(\phi)$ by Definition 3.

Proof of Theorem 1. Since Y^n is a decreasing sequence of closed sets, choose $y \in \cap_{n=1}^{\infty} Y^n$. For each $n \geqq 1$ choose a $y_n \in Y^n$ with $d(y_n, y) < 1/n$ (d is the metric on Y). We may suppose that the points of the sequence $\{y_n\}$ are distinct. Choose a sequence U_n of open sets such that $y_n \in U_n$, the sets $\bar{U}_n$ are disjoint, and $U_n \subset \{x \in Y : d(x, y_n) < 1/n\}$. Let $U_0 = Y \backslash \cup_{n=1}^{\infty} \bar{U}_n$. By Lemmas 6 and 8, for each $n \geqq 1$, there is a compact metric space X_n and an onto continuous map $\phi_n : X_n \to \bar{U}_n$ such that $y_n \in \Delta^n(\phi_n)$. Let $X_0 = \bar{U}_0$ and let $\phi_0 : X_0 \to \bar{U}_0$ be the identity map. Let $X = (\oplus_{n=0}^{\infty} X_n) \cup \{\infty\}$ be the one-point compactification of $\oplus_{n=0}^{\infty} X_n$ and define the onto map $\phi : X \to Y$ by $\phi|X_n = \phi_n$ and $\phi(\infty) = y$. Then ϕ is continuous since the diameters of the sets U_n go to zero. By Lemma 5, $y_n \in \Delta^n(\phi)$ for each n. Thus by Lemma 4, $p(\phi) \geqq n + 1$ for every n; i.e. $p(\phi) = \infty$.

COROLLARY 9. *Suppose that Y is a compact metric space and $Y^n \neq \emptyset$ for each $n \geqq 0$. Then there is a continuous map $\phi : C \to Y$ of the Cantor set C onto Y which has no averaging operator.*

Proof. By Theorem 1, choose a compact metric space X and an onto continuous map $\phi_1 : X \to Y$ which has no averaging operator. Let $\psi_2 : C \to X$ be any onto continuous map. Then $\phi_1\psi_2 : C \to Y$ has no averaging operator since if u is an averaging operator for $\phi_1\psi_2$, then $u\psi_2^0$ is an averaging operator for ϕ_1.

Proof of Corollary 2. For (1) $\Rightarrow$ (2) see Amir [**1**] or Baker [**3**, Theorem 2.11]. (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (1) is Corollary 9.

3. Applications, remarks and open problems. Suppose X and Y are compact metric spaces. The next theorem summarizes and extends known results on the following questions: When is $C(Y)$ a subalgebra (with the same unit) of $C(X)$? When is $C(Y)$ a complemented subalgebra of $C(X)$? When is $C(Y)$ an uncomplemented subalgebra of $C(X)$? From the theory of commutative Banach algebras it follows that $C(Y)$ is a subalgebra of $C(X)$ if and only if there is an onto continuous map $\phi : X \to Y$ (Semadeni [**15**, 7.7.1 and 7.7.2]). Also $C(Y)$ is a complemented or uncomplemented subalgebra if and only if the corresponding onto continuous map $\phi : X \to Y$ has or does not have an averaging operator.

THEOREM 10. *Suppose that X and Y are compact metric spaces at least one of which is totally disconnected. Consider the following statements:*

(a) *There is an onto continuous map $\phi : X \to Y$.*

(b) *There is an onto continuous map $\phi : X \to Y$ which has an averaging operator.*

(c) *There is an onto continuous map $\phi : X \to Y$ which does not have an averaging operator.*

(d) *The cardinality of X^α is greater than or equal to the cardinality of Y^α for each ordinal number α.*

Then

(1) (a) *and* (b) *are equivalent.*

(2) *If $Y^n \neq \emptyset$ for all integers $n \geqq 0$, then* (a), (b) *and* (c) *are equivalent.*

(3) *If X is totally disconnected, then* (a), (b) *and* (d) *are equivalent.*

Proof. Suppose that ϕ_1 and ϕ_2 are two onto continuous maps such that the composition $\phi_2\phi_1$ makes sense. We use the following elementary facts: (i) if there are averaging operators for ϕ_1 and ϕ_2 then there is an averaging operator for $\phi_2\phi_1$ (Pełczyński [**12**, Proposition 4.3] or Ditor [**7**, Lemma 2.3 (ii)]), (ii) if there is no averaging operator for ϕ_2 then there is no averaging operator for

$\phi_2\phi_1$ (Ditor [**7**, Lemma 2.3 (i)]). Also (iii) retraction maps (a map $r : X \to Y$ where $Y \subset X$ and $r(y) = y$ for $y \in Y$) have averaging operators (Pełczyński [**12**, Proposition 3.4]). In proving that (a) implies (b) and (c) we consider two cases.

Case 1. Suppose Y is an uncountable space and (a) holds, i.e. there is an onto continuous map $\phi : X \to Y$. Then there is an uncountable totally disconnected compact metric space Z and an onto continuous map $\psi : X \to Z$ (if X is totally disconnected let $Z = X$ and let ψ be the identity; if Y is totally disconnected let $Z = Y$ and let $\psi = \phi$.). There is a subset C of X which is homeomorphic to the Cantor set such that $\psi|C$ is a homeomorphism (Kuratowski [**9**, page 444]). Also there is a retraction $r_0 : Z \to \psi(C)$ since there is a retraction map of a totally disconnected compact metric space onto any closed subset (Kuratowski [**9**, Cor. 2, § 26, II]; Semadeni [**15**, Th. 8.3.4]). Then $r = \psi^{-1}r_0\psi : X \to C$ is a retraction. By Theorem 1 of Ditor there is an onto continuous map $\theta : C \to X$ which has an averaging operator. Thus $\theta r : X \to Y$ has an averaging operator by (i) and (iii) and so (a) implies (b). If $Y^n \neq \emptyset$ for each $n \geqq 1$, then Corollary 9 implies there is an onto continuous map $\theta_1 : C \to Y$ which has no averaging operator. Thus $\theta_1 r : X \to Y$ has no averaging operator (by (ii)) and (a) implies (c).

Case 2. Suppose Y is a countable space and $\phi : X \to Y$ is an onto continuous map. By Proposition 2(II) of Pełczyński [**13**], there is a subset Y_0 of X with Y_0 homeomorphic to Y such that $\phi| Y_0$ is a homeomorphism. Since Y is countable, it is totally disconnected (Kuratowski [**9**, page 286, Theorem 1]) and so the same argument as in case 1 shows there is a retraction map $r : X \to Y$. By (iii), r is a map of X onto Y which has an averaging operator and so (a) implies (b). If $Y^n \neq \emptyset$ for each $n \geqq 0$, then by Corollary 4.7 of Baker [**4**], there is an onto continuous map $\psi : Y \to Y$ which has no averaging operator. Thus $\psi r : X \to Y$ is an onto map which has no averaging operator (by (ii)) and so (a) implies (c).

Lemma 1 of Pełczyński and Semadeni [**14**] shows that (a) implies (d). To prove the converse we use the assumption that X is totally disconnected. If X is uncountable and totally disconnected there is an onto map $r : X \to C$ (see case 1). For any compact metric space Y there is an onto map $\psi : C \to Y$. Thus $\psi r : X \to Y$ implies (a). Next suppose X is countable. Then $X^\alpha = \emptyset$ for some countable ordinal α and X is homeomorphic to the space of ordinal numbers less than or equal to some ordinal number γ (Mazurkiewicz and Sierpinski [**10**]). Then (d) implies that Y is homeomorphic to the space of ordinals less than or equal to some ordinal δ with $\delta \leqq \gamma$. Thus $Y \subset X$ and there is a retraction $r : X \to Y$ and the proof of the theorem is complete.

Remarks. The relationships between some of the conditions (a) through (d) of Theorem 10 for compact metric spaces X and Y seem to be either more complicated or unknown without the totally disconnected assumption.

Question. Is (a) equivalent to (b) for compact metric spaces X and Y neither of which is totally disconnected?

The natural map of the Stone-Čech compactification of the integers onto the one-point compactification shows that (a) and (b) are not equivalent for non-metric spaces (Theorem 17.7.4 of [**15**]).

Condition (a) implies condition (d) in general, but the converse is false since there is no map of the connected space $[0, 1]$ onto the Cantor set.

The answer to the question of when condition (a) implies condition (c) seems to be complicated in view of the status of the following special case of this question:

Question. Let Y be a compact metric space such that $Y^n \neq \emptyset$ for all $n \geqq 0$. What are necesssary and sufficient conditions on Y in order that there is an onto continuous map $\phi : Y \rightarrow Y$ which has no averaging operator?

Baker and Lacher (Theorems 1 and 2 of [**5**]) Baker (Theorem 4.6 of [**4**]) and Ditor (Corollary 5.8 of [**8**]) give examples of compact metric spaces Y having onto maps $\phi : Y \rightarrow Y$ which have no averaging operators. However, for the compact metric space M_2 constructed by Cook (Theorem 11 of [**6**]), every onto map $\phi : M_2 \rightarrow M_2$ has an averaging operator (since the only such map is the identity map). The necessity of the assumption that $Y^n \neq \emptyset$ for every $n \geqq 0$ is contained in Corollary 2.

Suppose Y is a compact metric space. According to Theorem 1, if $Y^n \neq \emptyset$ for each positive integer n, then there is a compact metric space X and an onto continuous map $\phi : X \rightarrow Y$ having no averaging operator. In our final application we indicate what can be said if $Y^{n+1} = \emptyset$ and $Y^n \neq \emptyset$ for some $n \geqq 0$.

PROPOSITION 11. *Suppose $n \geqq 0$ and Y is a compact metric space with $Y^n \neq \emptyset$. Then, for each $\epsilon > 0$, there is a compact metric space X and an onto continuous map $\phi : X \rightarrow Y$ with $p(\phi) \geqq 2n + 1 - \epsilon$.*

Proof. We outline the proof of this proposition. Let $\epsilon > 0$ be given and choose an integer q so that $2n/q < \epsilon$. Denote $\sigma_\phi^{(k)}(q, q, \ldots, q)$ by $\Delta_q^k(\phi)$. Then Lemma 5 remains true if $\Delta^n(\cdot)$ is replaced everywhere by $\Delta_q^n(\cdot)$. Also Lemma 6 and 7 remain true if at each point where the index set $\{1, 2\}$ is used we use instead the index set $\{1, 2, \ldots, q\}$. The restatement of Lemma 8 (which has a similar proof) in this more general setting is: Let $\{U_1, \ldots, U_q\}$ be disjoint open subsets of the compact metric space Y. For $n \geqq 0$ suppose $\bigcap_{i=1}^q \bar{U}_i = \bigcap_{i=1}^q \overline{(U_i \cap Y^n)}$. Then there is a compact metric space X and an onto continuous map $\phi : X \rightarrow Y$ with $\bigcap_{i=1}^q \bar{U}_i \subset \Delta_q^{n+1}(\phi)$. By the remark following Lemma 4, $p(\phi) \geqq 2(n + 1) + 1 - 2(n + 1)/q$. Now, if $Y^n \neq \emptyset$ then, by our new Lemma 6, the restatement of Lemma 8 is satisfied with n replaced by $n - 1$. Thus we get a compact metric space X and an onto map $\phi : X \rightarrow Y$ with $p(\phi) \geqq 2n + 1 - 2n/q \geqq 2n + 1 - \epsilon$.

Remark. The constant $2n + 1$ in this proposition is the best possible since, if Y is a compact metric space such that $Y^{n+1} = \emptyset$, then Theorem 1.9 of Baker [**3**] can be used to show that $p(\phi) \leqq 2n + 1$ for any onto continuous map $\phi : X \rightarrow Y$ where X is compact metric.

https://doi.org/10.4153/CJM-1976-087-4 Published online by Cambridge University Press

References

1. D. Amir, *Continuous function spaces with the separable projection property*, Bull. Res. Counc. of Israel, *10F* (1962), 163–164.

2. ——— *Projections onto continuous function spaces*, Proc. Amer. Math. Soc. *15* (1964), 396–402.

3. J. W. Baker, *Some uncomplemented subspaces of $C(X)$ of the type $C(X)$*, Studia Math. *36* (1970), 85–103.

4. ——— *Uncomplemented $C(X)$-subalgebras of $C(X)$*, Trans. Amer. Math. Soc. *186* (1973), 1–15.

5. J. W. Baker and R. C. Lacher, *Some mappings whose induced subalgebras are uncomplemented*, to appear, Pacific J. Math.

6. H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. *60* (1967), 241–249.

7. S. Z. Ditor, *On a lemma of Milutin concerning averaging operators in continuous function spaces*, Trans. Amer. Math. Soc. *149* (1970), 443–452.

8. ——— *Averaging operators in $C(S)$ and lower semicontinuous sections of continuous maps*, Trans. Amer. Math. Soc. *175* (1973), 195–208.

9. K. Kuratowski, *Topology I* (New York, 1966).

10. S. Mazurkiewicz and W. Sierpinski, *Contributions à la topologie des ensembles dénombrables*, Fund. Math. *1* (1920), 17–27.

11. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. *71* (1951), 151–182.

12. A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. *58* (1968).

13. ——— *On $C(S)$-subspaces of separable Banach spaces*, Studia Math. *31* (1968), 513–522.

14. A. Pełczyński and Z. Semadeni, *Spaces of continuous functions, III; Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. *18* (1959), 211–222.

15. Z. Semadeni, *Banach spaces of continuous functions*, Vol. 1 (Warsaw 1971).

Kent State University,
Kent, Ohio;
Oklahoma State University,
Stillwater, Oklahoma

https://doi.org/10.4153/CJM-1976-087-4 Published online by Cambridge University Press